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Nonlinear Analysis Of A Three Dimensional Mixed Formulation Frame Finite Element Based On Hu-Washizu Functional

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Abstract

An analytical study is carried out to develop a three dimensional mixed formulation frame finite element. The aim of this study is to form three dimensional mixed formulation frame finite element by using Hu-Washizu variational that takes into account one or more combination of the interaction of the axial load, shear, moment and especially torsion by utilizing three-field of displacement, strain and stress. The displacement field used for derivation of strains is adapted from extension of Timoshenko beam theory to three dimensions. Integration of stress-strain relations along limited number of predetermined control sections is the key to carry out nonlinear analysis. Shape functions that satisfy the equilibrium and discontinuous strains are used for stress resultants in the finite element approximation. This nature of the element eliminates necessity of displacement components along the control sections of the beam element except at the nodes. In this study, three dimensional mixed element that is developed is used in nonlinear analysis of a cantilever steel member with circular cross section and the effect of interaction of axial force, shear, bending moment and torsion is demonstrated.

Keywords:*Finite element, mixed formulation, Hu-Washizu variational, frame element.*

1 Introduction

Finite element method (FEM) offers an approximate solution for real physical problems even for differential equations of which are so difficult or impossible to solve. The basic theory of FEM originates from virtual displacements and/or minimum potential energy of finite elements that are formed by discretizing the actual domain of the real case with an assumed and idealized counterpart.

Finite elements can be divided into two groups called “displacement based” and “mixed” finite element depending on the type of the fields that are utilized in determination of the shape functions necessary for analysis.

Displacement based frame finite elements use assumed shape functions for the interpolation of the displacement field along an element length. Element equilibrium is always satisfied at the nodes in the analysis of systems utilizing displacement based elements; however differential equilibrium cannot be satisfied at all times because of neglecting the continuity condition in differential equations. Therefore, section forces cannot be determined with acceptable accuracy unless increasing the number of nodes or the order of shape functions. Because of this fact, displacement based elements may be insufficient for cases like diaphragm or shear locking or in analysis of incompressible media. In order to overcome this deficiency recent studies have focused on mixed finite elements.

Mixed finite elements use integration of stress-strain relations of control sections that extend along member. The difference between displacement based and mixed finite elements is that the latter utilize additional independent fields in the variational formulation of the element. For a frame finite element under small deformations, the solution of the differential equilibrium equations provide the exact shape functions for the interpolation of the force field along the element length, thus obviating the need for displacement interpolation along an element.

Mixed elements are superior to the displacement based elements in cases like shear and membrane locking and also in capturing the nonlinear response. Despite their complex formulation that requires the storage of additional fields in an element, mixed elements can ensure the same level of robustness as observed in displacement based elements by employing lesser number of elements and reaching the solution more quickly in the analysis (Saritas and Soydas 2012).

There are numerous studies regarding mixed frame element formulations and its applications. Spacone et al. (1996) developed mixed formulation beam-column element that ensures moment and axial force equilibrium for nonlinear static and dynamic analysis of RC frame systems by neglecting shear and bond slip. Neuenhofer and Filippou (1997) offered a more efficient state determination algorithm by comparing displacement based and mixed formulation elements with inelastic material model in frame systems. De Souza (2000), in his PhD study, enhanced Neuenhofer and Filippou's elastic mixed finite element by considering inelastic large displacements in the analysis of frame systems. Hjelmstad and Taciroğlu (2005) studied the variational principles regarding mixed elements for nonlinear analysis of frame systems. Saritas (2006), in his PhD study, presented a 2 dimensional mixed element formulation that incorporates the interaction of axial force, shear force and moment for shear critical steel and RC elements. Papachristidis et al. (2010) investigated the capacity of frame systems under high shear force by using mixed elements utilizing Timoshenko beam theory and three dimensional material model that considers the interaction of axial force, shear force, bending moment and torsion, but numerical examples do not include three dimensional behavior of the elements. In a recent study (Wackerfuss and Gruttmann, 2011), a three dimensional frame element is developed by using Hu-Washizu variational and nonlinear behavior is modeled by defining additional degrees of freedoms to three transitional and rotational degrees of freedom at each nodes of the element.

In this study, a three dimensional (3d) mixed finite element is developed by using Hu-Washizu variational, as well. The effect of interaction between axial force, shear, bending moment and torsion is demonstrated by carrying out nonlinear analysis of a cantilever steel member having circular cross section.

2 Three Dimensional Mixed Formulation Beam Element

2.1 Hu-Washizu Functional

Hu-Washizu functional is written in terms of three independent fields: stress field $\boldsymbol{\sigma}$, strain field $\boldsymbol{\varepsilon}$, and displacement field, $\mathbf{u}$ as follows:

$$\Pi_{\text{HW}}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \mathbf{u}) = \int_{\Omega} W(\boldsymbol{\varepsilon}) d\Omega + \int_{\Omega} \boldsymbol{\sigma}^{\text{T}} [\boldsymbol{\varepsilon}^{\mathbf{u}} - \boldsymbol{\varepsilon}] d\Omega + \Pi_{\text{ext}} \tag{1}$$

$W(\boldsymbol{\varepsilon})$ is the strain energy function from which stresses are derived as;

$$\hat{\boldsymbol{\sigma}}(\boldsymbol{\varepsilon}) = \frac{\partial W(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} \tag{2}$$

$\boldsymbol{\varepsilon}^{\mathbf{u}}$ is the strain vector that is compatible with the displacements $\mathbf{u}$. Π_{ext} denotes the potential energy of the external loading due to body forces and displacement and traction boundary conditions such that;

$$\Pi_{\text{ext}} = -\int_{\Omega} \mathbf{u}^{\text{T}} \mathbf{b} d\Omega - \int_{\Gamma_{\text{t}}} \mathbf{u}^{\text{T}} \mathbf{t}^{*} d\Gamma - \int_{\Gamma_{\text{u}}} \mathbf{t}^{\text{T}} [\mathbf{u} - \mathbf{u}^{*}] \, d\Gamma \tag{3}$$

where $\mathbf{b}$ denotes stresses caused by body forces and $\mathbf{t} = \boldsymbol{\sigma}.\mathbf{n}$ is the dot product of the stress vector with the outward normal $\mathbf{n}$ to the boundary. The imposed values are indicated by superscript asterisks. It is assumed that the external loading is conservative so that the work depends only on the final displacement values $\mathbf{u}$. Domain of the body and traction and displacement boundaries are Ω , Γ_{t} and Γ_{u} , respectively.

2.2 Kinematic Relations for the 3d Timoshenko Beam Element

Three field variational principle that is used in Equation (1) and (3) enables specification of $\boldsymbol{\varepsilon}$ and $\mathbf{u}$ independently, thus this allows section kinematic relations to be defined independent of the assumed displacement field for the beam. Timoshenko beam theory can be adapted for a three dimensional geometry as follows;

$$\mathbf{u}=\begin{Bmatrix} u_x(x,y,z) \\ u_y(x,y,z) \\ u_z(x,y,z) \end{Bmatrix}=\begin{Bmatrix} u(x)-y\theta_z(x)+z\theta_y(x) \\ v(x)-z\theta_x(x) \\ w(x)+y\theta_x(x) \end{Bmatrix} \tag{4}$$

In a right-handed local coordinate system, $u(x)$ is the displacement of the point $(x,0,0)$ along x-axis and $v(x)$ and $w(x)$ are the transverse deflections of the point $(x,0,0)$ from x-axis in y and z directions, respectively. $\theta_x(x)$, $\theta_y(x)$ and $\theta_z(x)$ are the rotations of the beam cross section around three orthogonal axes x, y and z, respectively, where $\theta_x(x)$ is the twisting angle. Therefore the vector of displacements at a section of the beam can be given as;

$$\mathbf{u}_\mathrm{s}=[u(x)\quad v(x)\quad w(x)\quad \theta_x(x)\quad \theta_y(x)\quad \theta_z(x)]^\mathrm{T} \tag{5}$$

Strains that are compatible with the displacement field $\mathbf{u}$ can be derived from Equation (4) for small strains as follows;

$$\begin{aligned} \varepsilon_{xx}^u &= \frac{du_x(x,y,z)}{dx}=u'(x)-y\theta_z{}'(x)+z\theta_y{}'(x) \\ \gamma_{xy}^u &= \frac{du_x(x,y,z)}{dy}+\frac{du_y(x,y,z)}{dx}=-\theta_z(x)+v'(x)-z\theta_x{}'(x) \\ \gamma_{xz}^u &= \frac{du_x(x,y,z)}{dz}+\frac{du_z(x,y,z)}{dx}=\theta_y(x)+w'(x)+y\theta_x{}'(x) \end{aligned} \tag{6}$$

It should be noted that ε_{yy}^u , ε_{zz}^u and γ_{yz}^u are equal to zero as a result of the derivation.

The strain fields for the beam are selected independently from those in Equation (6) as follows;

$$\boldsymbol{\varepsilon}=\begin{Bmatrix} \varepsilon_{xx} \\ \gamma_{xy} \\ \gamma_{xz} \end{Bmatrix}=\begin{Bmatrix} \varepsilon_a(x)-y\kappa_z(x)+z\kappa_y(x) \\ \gamma_y(x)-z\varphi(x) \\ \gamma_z(x)+y\varphi(x) \end{Bmatrix} \tag{7}$$

$\varepsilon_a(x)$, $\kappa_y(x)$ and $\kappa_z(x)$ are the axial strain and curvature around y and z axes, respectively. $\gamma_y(x)$and$\gamma_z(x)$ are the shear distortions of the section in y and z directions, respectively. $\varphi(x)$is the angle of twist of the cross section. Hence section deformations, $\boldsymbol{e}(x)$ are given as;

$$\boldsymbol{e}(x)=[\varepsilon_a(x)\quad \kappa_y(x)\quad \kappa_z(x)\quad \gamma_y(x)\quad \gamma_z(x)\quad \varphi(x)]^\mathrm{T} \tag{8}$$

2.3 Variation of Hu-Washizu Functional

The variation of the Equation (1) gives;

$$\delta\Pi_\mathrm{HW}=\int_\Omega \hat{\boldsymbol{\sigma}}(\boldsymbol{\varepsilon})\boldsymbol{\delta\varepsilon}d\Omega+\int_\Omega \delta\boldsymbol{\sigma}^\mathrm{T}\,[\boldsymbol{\varepsilon}^\mathbf{u}-\boldsymbol{\varepsilon}]d\Omega+\int_\Omega \boldsymbol{\sigma}^\mathrm{T}\,[\delta\boldsymbol{\varepsilon}^\mathbf{u}-\delta\boldsymbol{\varepsilon}]d\Omega+\delta\Pi_\mathrm{ext} \tag{9}$$

The functional in Equation (9) can be generalized for an inelastic material by assuming that $\hat{\boldsymbol{\sigma}}(\boldsymbol{\varepsilon})$ describes an inelastic material although the variation in Equation (2) is based on a strain-energy function that is in accordance with Cauchy elastic material model. Equations (6) and (7) can be substituted into Equation (9) by noting that for 3d beam element $\sigma_{yy}=\sigma_{zz}=\hat{\sigma}_{yy}=\hat{\sigma}_{zz}=\hat{\sigma}_{yz}=\gamma_{yz}=0$. Furthermore, section stress resultants for a 3d beam element can be defined as in Equation (10) and section stress resultants can be computed by taking integration over the section area A as presented in the next equations.

$$\boldsymbol{s}(\mathrm{x}) = [N(x) \quad M_z(x) \quad M_y(x) \quad V_y(x) \quad V_z(x) \quad T(x)]^{\mathrm{T}} \tag{10}$$

$$N = \int_A \sigma_{xx} dA \quad \text{(Axial force)} \qquad M_y = \int_A z\sigma_{xx} dA \quad \text{(Moment about y-axis)} \qquad M_z = \int_A -y\sigma_{xx} dA \quad \text{(Moment about z-axis)} \tag{11}$$

$$V_y = \int_A \sigma_{xy} dA \quad \text{(Shear force in y direction)} \qquad V_z = \int_A \sigma_{xz} dA \quad \text{(Shear force in z direction)} \qquad T = \int_A \left(-z\sigma_{xy} + y\sigma_{xz}\right) dA \quad \text{(Torsion around x-axis)}$$

Equation (11) is substituted into Equation (9) and the integration is carried out along the length, L of the beam, we get the following form for the variational functional;

$$\begin{aligned}
\delta\Pi_{\mathrm{HW}} &= \int_0^L \left\{\delta\varepsilon_a(x)\left(\widehat{N} - N\right) + \delta\kappa_z(x)\left(\widehat{M}_z - M_z\right) + \delta\kappa_y(x)\left(\widehat{M}_y - M_y\right)\right\} d\mathrm{x} \\
&+ \int_0^L \left\{\delta\gamma_y(x)\left(\widehat{V}_y - V_y\right) + \delta\gamma_z(x)\left(\widehat{V}_z - V_z\right) + \delta\varphi(x)\left(\widehat{T} - T\right)\right\} d\mathrm{x} \\
&+ \int_0^L \left\{\delta N\left(u'(x) - \varepsilon_a(x)\right) + \delta M_z\left(\theta_z{}'(x) - \kappa_z(x)\right) + \delta M_y\left(\theta_y{}'(x) - \kappa_y(x)\right)\right\} d\mathrm{x} \\
&+ \int_0^L \Big\{\delta V_y\left(v'(x) - \theta_z(x) - \gamma_y(x)\right) + \delta V_z\left(w'(x) + \theta_y(x) - \gamma_z(x)\right) \\
&\qquad\qquad + \delta T(\theta_x{}'(x) - \varphi(x))\Big\} d\mathrm{x} \\
&+ \int_0^L \left\{\delta u'(x) N + \delta\theta_z{}'(x) M_z + \delta\theta_y{}'(x) M_y\right\} d\mathrm{x} \\
&+ \int_0^L \left\{\left(\delta v'(x) - \delta\theta_z(x)\right) V_y + \left(\delta w'(x) + \delta\theta_y(x)\right) V_z + \delta\theta_x{}'(x) T\right\} d\mathrm{x} \\
&+ \delta\Pi_{\mathrm{ext}}
\end{aligned} \tag{12}$$

where $\widehat{N}$, $\widehat{M}_y$, $\widehat{M}_z$,$\widehat{T}$, $\widehat{V}_y$ and $\widehat{V}_z$ are similar to the expressions in Equation (11) and they are the section stress resultants such that the stress terms are interchanged with the stresses, $\hat{\sigma}_{xx}$, $\hat{\sigma}_{xy}$ and $\hat{\sigma}_{xz}$ that are satisfying the material constitutive relations. The formulation enables the utilization of constitutive relation $\hat{\boldsymbol{\sigma}} \equiv \hat{\boldsymbol{\sigma}}(\boldsymbol{\varepsilon})$ in Equation (9) for any type of material. Π_{ext} can be defined as the variation of the work-conjugate of the displacement and stress resultant fields since it is the variation of external potential energy. In real structural engineering problems moment tractions around y and z axes are negligible, therefore $\overline{m}_y(x) = \overline{m}_z(x) = 0$. If it is assumed that the body forces are zero so that they are eliminated from the variation of external potential, then;

$$\delta\Pi_{\mathrm{ext}} = -\int_0^L \left\{\bar{n}(x)\delta u(x) + \bar{q}_y(x)\delta v(x) + \bar{q}_z(x)\delta w(x) + \overline{m}_x(x)\delta\theta_x(x)\right\} d\mathrm{x} - \delta\Pi_{\mathrm{bc}} \tag{13}$$

$\bar{n}(x)$ and $\bar{q}_y(x)$, $\bar{q}_z(x)$, are distributions of longitudinal tractions and transverse tractions along x, y and z axes, respectively. $\delta\Pi_{\mathrm{bc}}$ is the variation of the energy due to boundary conditions.

2.4 Finite Element Approximation

If Equation (13) is substituted into Equation (12) and approximation of the section forces and their variation satisfy the following conditions; then some terms in Equation (12) vanish and this obviates the necessity to approximate the displacements u, v, w, θ_x , θ_y and θ_z along the beam. Displacement values at the nodes are enough for the formulation.

$$
\begin{array}{lll}
N'(x) + \bar{n}(x) = 0 & V_y'(x) + \bar{q}_y(x) = 0 & M_z'(x) + V_y(x) = 0 \\
T'(x) + \bar{m}_x(x) = 0 & V_z'(x) + \bar{q}_z(x) = 0 & M_y'(x) - V_z(x) = 0
\end{array} \tag{14}
$$

$$
\begin{array}{lll}
\delta N'(x) = 0 & \delta V_y'(x) = 0 & \delta M_z'(x) + \delta V_y(x) = 0 \\
\delta T'(x) = 0 & \delta V_z'(x) = 0 & \delta M_y'(x) - \delta V_z(x) = 0
\end{array} \tag{15}
$$

Equation (12) can be rearranged by assuming constant axial and transverse tractions along the element and ignoring $\delta\Pi_{\mathrm{bc}}$ until the assembly of the elements as follows;

$$
\begin{aligned}
\delta\Pi_{\mathrm{HW}} = \; & \int_0^L \delta\boldsymbol{e}(x)^{\mathrm{T}}\big(\hat{\boldsymbol{s}}(x) - \boldsymbol{b}(x)\boldsymbol{q} - \boldsymbol{s}_p(x)\big)dx - \delta\boldsymbol{q}^T \int_0^L \boldsymbol{b}(x)^{\mathrm{T}}\boldsymbol{e}(x)\,dx + \delta\boldsymbol{q}^{\mathrm{T}}\boldsymbol{a}_{\mathrm{g}}\boldsymbol{u}_{\mathrm{el}} \\
& + \delta\boldsymbol{u}_{\mathrm{el}}{}^{\mathrm{T}}\boldsymbol{a}_{\mathrm{g}}{}^{\mathrm{T}}\boldsymbol{q} - \delta\boldsymbol{u}_{\mathrm{el}}{}^{\mathrm{T}}\boldsymbol{a}_{\mathrm{r}}{}^{\mathrm{T}}\boldsymbol{p}_w
\end{aligned} \tag{16}
$$

where section force vector $\boldsymbol{s}(x)$, matrix of interpolation functions $\boldsymbol{b}(x)$, basic element force vector $\boldsymbol{q}$, section stress resultants due to element loading $\boldsymbol{w}$, and rigid mode of applied tractions at the nodes $\boldsymbol{p}_w$ are defined as follows;

$$
\boldsymbol{s}(x) = \boldsymbol{b}(x)\boldsymbol{q} + \boldsymbol{s}_p(x) \tag{17}
$$

$$
\boldsymbol{q} = [q_1 \quad q_2 \quad q_3 \quad q_4 \quad q_5 \quad q_6]^{\mathrm{T}} \tag{18}
$$

$$
\boldsymbol{b}(x) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{x}{L} - 1 & x/L & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{x}{L} - 1 & x/L & 0 \\
0 & -1/L & -1/L & 0 & 0 & 0 \\
0 & 0 & 0 & 1/L & 1/L & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix} \tag{19}
$$

$$
\boldsymbol{s}_p(x) = \begin{bmatrix}
L\left(1 - \frac{x}{L}\right) & 0 & 0 & 0 \\
0 & \frac{L^2}{2}\left(\left(\frac{x}{L}\right)^2 - \frac{x}{L}\right) & 0 & 0 \\
0 & 0 & \frac{L^2}{2}\left(\left(-\frac{x}{L}\right)^2 + \frac{x}{L}\right) & 0 \\
0 & \frac{L}{2}\left(1 - \frac{2x}{L}\right) & 0 & 0 \\
0 & 0 & L\left(1 - \frac{2x}{L}\right) & 0 \\
0 & 0 & 0 & L\left(1 - \frac{x}{L}\right)
\end{bmatrix}
\begin{Bmatrix} w_x \\ w_y \\ w_z \\ m_x \end{Bmatrix} \tag{20}
$$

$$
\boldsymbol{p}_w = \left(w_x L \quad w_y\frac{L}{2} \quad w_z\frac{L}{2} \quad m_x L \quad 0 \quad 0 \quad 0 \quad w_y\frac{L}{2} \quad w_z\frac{L}{2} \quad 0 \quad 0 \quad 0\right) \tag{21}
$$

Transformation of element displacements to deformations is demonstrated in Figure 1. $\boldsymbol{a}$ is transformation matrix, $\boldsymbol{a}_{\mathrm{r}}$ is rotation matrix from global to local reference system, $\bar{\boldsymbol{u}}$ and $\boldsymbol{u}_{\mathrm{el}}$ are displacement degrees of freedom in local system and $\mathbf{v}$ is basic element deformation vector such that 12 element end displacement degrees of freedom are reduced to 6 by separating 6 rigid body modes and 6 deformation modes of displacement similar to the case for basic element force vector $\boldsymbol{q}$.

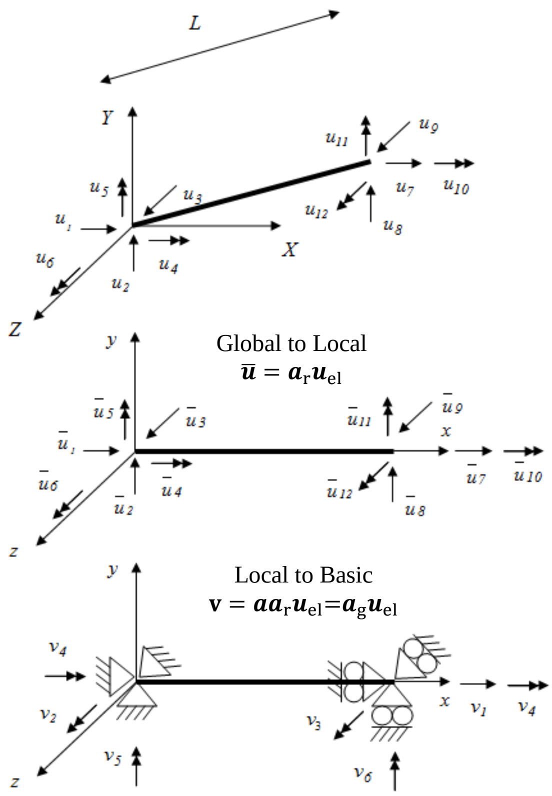
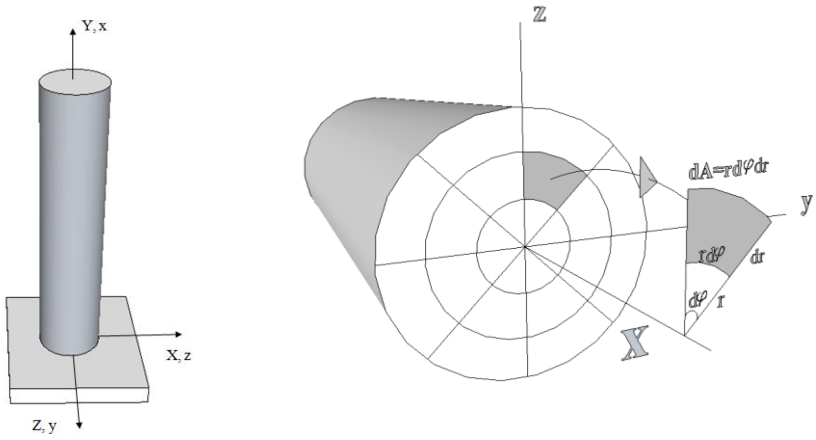


Figure 1. Transformation of element displacements to deformations.

Equation (16) is minimized by equating the expression to zero. Resulting expression will be generally nonlinear and solution of it is carried out by linearly approximating the equation by using first order Taylor series expansion.

3 Numerical Example

Nonlinear analysis of a cantilever steel member with circular cross-section is carried out by using a single 3d mixed finite element per span. Orientations of local and global coordinates of the member are demonstrated in Figure 2.a. Units of the parameters are left intentionally as numerical to investigate the variation of the effect of the applied loading rather than the parameters itself. The length of the member, L is assumed to be 120 units, diameter, d of the section is taken to be 18 units. The modulus of elasticity, E for the steel is 29,000 units, the yield strength of it, f_y is 36 units. Three dimensional J2 plasticity material model is used for the description of steel material response. The cross-section is divided into pieces in both radial and circumferential directions as in Figure 2.b and numerical integration is performed accordingly. The response of the beam element is modeled through aggregation of the response of various sections over the element length. Gauss quadrature is used in the determination of the location of these sections, and five sections are used in each case.



a. Orientation of axes b. Discretization of circular section (local coordinates)

Figure 2. Orientation of axes and radial and circumferential discretization of circular section.

Two different pseudo-load cases are applied to the member. In the first case, the member is loaded axially in the negative global Y-direction (Force History) and member is displaced 6 units in global X-direction (Disp. History 1) and 6 units in global Z-direction (Disp. History 2) as shown in Figure 3. In that case axial force at yield, N_y is assumed to be area of the section times yield strength of the member. Four different sub-cases are investigated such that applied axial forces are 0, 0.25, 0.50 and 0.75N_y, respectively and base shear vs. top displacements in X and Z directions for each sub-cases are plotted in Figures 4 and 5, respectively.

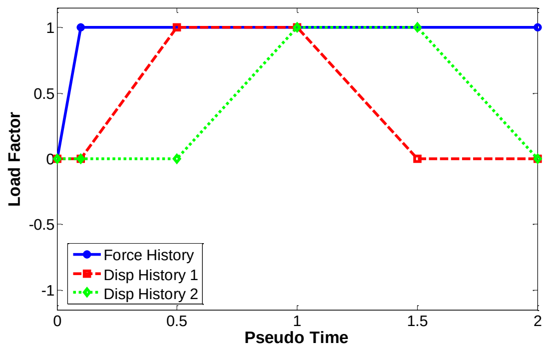


Figure 3. Force and displacement pseudo-time histories for fist and second cases

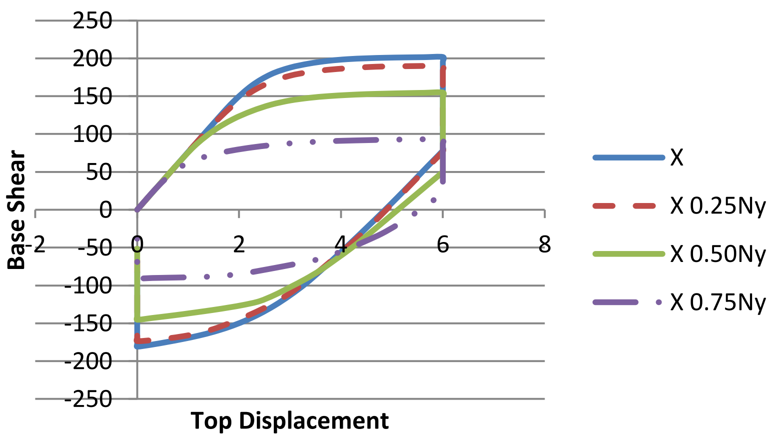


Figure 4. Base shear vs. top displacement in global X-direction for first loading case.

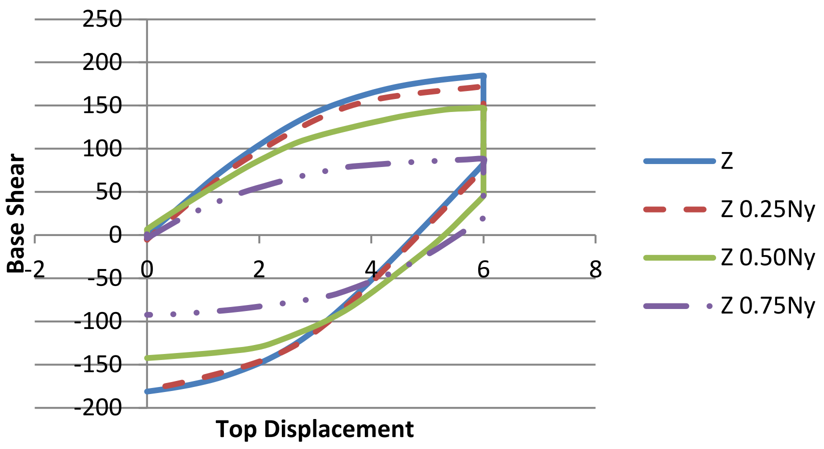


Figure 5. Base shear vs. top displacement in global Z-direction for first loading case.

In the second case, torsion is applied around global Y-direction and member is displaced 6 units in global X-direction and 6 units in global Z-direction similar to the previous case. In that case torsion at yield, T_y is $(2\pi/3)(d/2)^3(0.577f_y)$ which is obtained by assuming a constant shear force that is a multiple of yield strength $(\tau_y = 0.577f_y)$ in the section at yield and taking integral of it over the area. Four different sub-cases are investigated such that applied axial forces are 0, 0.25, 0.50 and 0.75T_y, respectively and base shear vs. top displacements in X and Z directions for each sub-cases are plotted in Figures 6 and 7, respectively.

According to the Figures 4 to 7, effect of axial load on nonlinear behavior is more pronounced compared to the effect of torsion for the 3d element. 3d element is also capable of reflecting the dependency of nonlinear behavior on direction of loading.

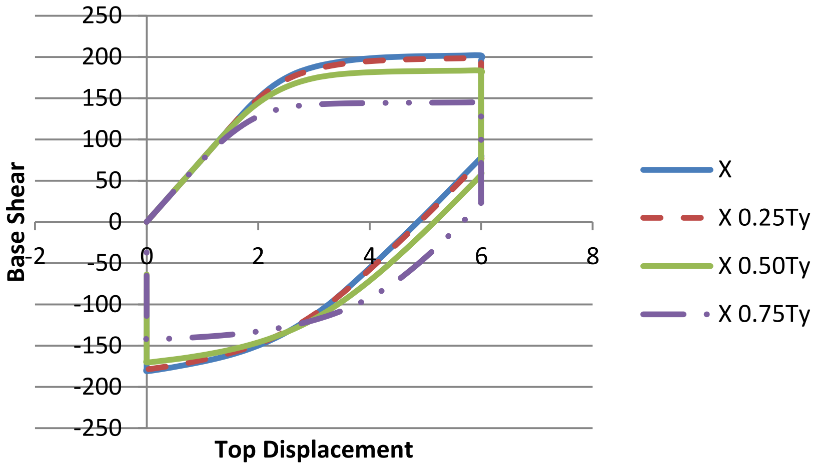


Figure 6. Base shear vs. top displacement in global X-direction for second loading case.

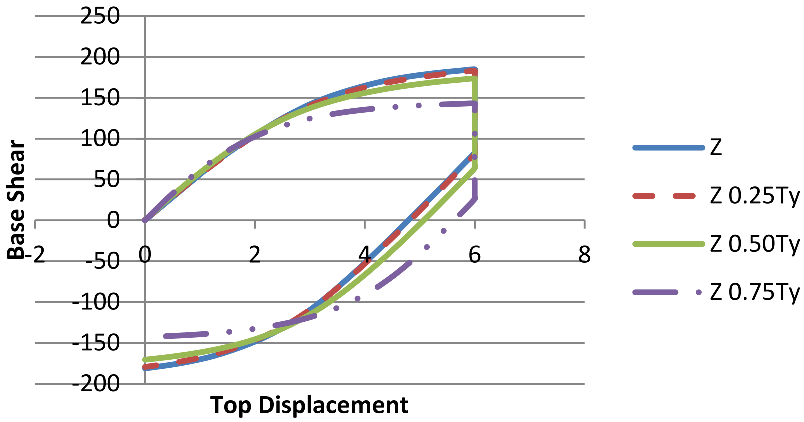


Figure 7. Base shear vs. top displacement in global Z-direction for second loading case.

4 Conclusions

In this study, a 3d mixed formulation frame finite element that is based on Hu-Washizu functional is developed, and nonlinear analysis of a cantilever steel member with circular cross section is performed. It is shown that the element is able to model the effect of interaction of axial force, shear, bending moment and torsion. 3d element can be used in cases where the complex interaction of the 3d nature of section forces becomes important in frame type elements; such as the exterior columns in buildings that are exposed to an earthquake.

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