# FINITE ELEMENT MODEL WITH SEMI-RIGID CONNECTIONS FOR VIBRATION ASSESSMENT OF STEEL MOMENT RESISTING FRAMED STRUCTURES

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**Abstract**: In this paper, a mixed formulation frame finite element with semi-rigid connections is developed. Consistent mass matrix of the element is obtained such that determination of vibration frequencies of members with varying geometry and material distribution as well as the presence of semi-rigid connections at any section on the element is accurately captured without the need to specify displacement shapes. An accurate shear correction coefficient for wide flange sections is taken into account in order to get closer match with exact solutions. Numerical examples on a portal frame and a multi-story steel moment resisting frame verify the accuracy of proposed element with and without semi-rigid connections.

## 1. Introduction

Semi-rigidity at connection regions of steel structures greatly influences the vibration characteristics of steel moment resisting framed structures. Finite element models should address modelling of the mass and stiffness matrices for both beam and column members in order to accurately capture transverse shear deformations and rotary inertia along a member's length, as well as the partial fixity introduced by the presence of these connections. Researchers have studied dynamic behavior of steel framed structures and suggested to consider the effect of semi-rigid connections in the last two decades. Chui and Chan [1] and Nader and Astaneh-Asl [2] conducted tests on flexible jointed steel frames accompanied with numerical analyses, and found out the importance of taking into account joint flexibility in structural models. Besides consideration of semi-rigidity at connection region, it is also important to take into account possible inelastic behavior and nonlinear geometric effects on frame members [3-5] in carrying out dynamic analysis. The literature contains significant amount of research work on the investigation of the dynamic behavior of steel framed structures with semi-rigid connections through the use of finite element method [6-10]. Available design codes also try to provide the influence of semi-rigid response for steel structures under dynamic actions, and research studies try to evaluate the code suggestions on this front. In one of such studies, Sophianapoulos [11] compared Eurocode 3 approach with closed form solutions, where the results from both approaches showed that in the fundamental modes of vibration responses are very close to each other; however, in the higher modes, the influence of the flexible joints became apparent and results differed from each other.

The amount of the studies on trying to get better predictions on capturing vibration characteristics of steel framed structures with semi-rigid connections clearly reflects the importance of coming up with more accurate and robust modeling approaches. To this end, a mixed formulation frame finite element with semi-rigid connections is developed in this study, where the element formulation bases on the use of three-field Hu-Washizu-Barr principle. Consistent mass matrix of the element is obtained such that determination of vibration frequencies of members with varying geometry and material distribution as well as the presence of semi-rigid connections at any section on the element is accurately captured without any need for specification of different displacement shape functions for each individual case. The element response does not necessitate further discretization due to the presence of semi-rigid connections. An accurate shear correction coefficient for wide flange sections is taken into account in order to get closer match with exact solutions. A commercial program, SAP2000 [12], is used to validate the results of the proposed model and the previous benchmark solutions in [9].

## 2. Frame Element Formulation

### 2.1 Kinematic Relations

Displacements on a material point on the section of a beam that deforms in *xy*-plane can be obtained by calculating Timoshenko beam theory as follows;

$$\begin{Bmatrix} u_x(x,y) \\ u_y(x,y) \end{Bmatrix} = \begin{Bmatrix} u(x) - y\theta(x) \\ v(x) \end{Bmatrix} \tag{1}$$

where $u_x(x,y)$ and $u_y(x,y)$ are the displacements in $x$ and $y$ directions, respectively of any point in the section. $u(x)$ is the displacement of the point $(x,0)$ along $x$-axis. $v(x)$ is the transverse deflections of the point $(x,0)$ from $x$-axis in $y$ direction. $\theta(x)$ is the small rotation of the beam cross section around $z$-axis.

The non-zero strain components $\boldsymbol{\varepsilon}$ include the normal strain in the x direction and shear strain with xy component, where these are calculated from section deformations as follows;

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_{xx} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} u'(x) - y\theta'(x) \\ -\theta(x) + v'(x) \end{Bmatrix} = \begin{Bmatrix} \varepsilon_a(x) - y\kappa(x) \\ \gamma(x) \end{Bmatrix} = \mathbf{a}_s(y,z)\,\mathbf{e}(x) \tag{2}$$

where $\mathbf{e}(x)$ is the section deformation vector given as follows;

$$\mathbf{e}(x) = \begin{bmatrix} \varepsilon_a(x) & \gamma(x) & \kappa(x) \end{bmatrix}^{\mathrm{T}} \tag{3}$$

In Equation (3), $\varepsilon_a(x)$ is the axial strain of the reference axis, $\gamma(x)$ is the shear deformation along $y$-axis and $\kappa$ is the curvature about $z$-axis. Section deformations can be easily calculated from section reference displacements as clearly visible from a one to one comparison of the terms of Equation (2). Furthermore, section compatibility matrix, $\mathbf{a}_s(y,z)$ introduced in Equation (2) is written as follows;

$$\mathbf{a}_s(y,z)=\begin{bmatrix}1 & 0 & -y\\ 0 & 1 & 0\end{bmatrix} \tag{4}$$

## 2.2 Basic System without Rigid Body Modes and Force Interpolation Functions

Element formulation is proposed in *xy*-plane, where the formulation considers two end nodes and relies on a transformation from complete system to basic system. In the whole structure, the element has 3 degrees of freedom (dof) per node, resulting in 6 dofs, where the nodes are placed at element ends. The complete system is proposed such that the axis of the element is aligned with horizontal *x*-axis. The basic system is prescribed for the purpose of removing rigid body modes of motion, and the basic system is chosen as the cantilever beam as shown in Figure 1, where the fixed and free ends are the left and right ends, respectively. The transformation matrix, **a** for an element with length L is used to relate element end forces in complete system to basic element forces as follows;

$$\mathbf{p}=\mathbf{a}^{\mathrm{T}}\mathbf{q};\quad \text{where}\quad \mathbf{a}=\begin{bmatrix}-1 & 0 & 0 & 1 & 0 & 0\\ 0 & -1 & -L & 0 & 1 & 0\\ 0 & 0 & -1 & 0 & 0 & 1\end{bmatrix} \tag{5}$$

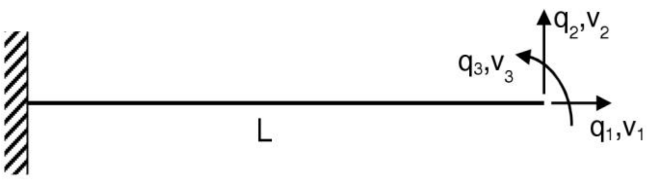



**Fig. 1: Cantilever basic system forces and deformations**

It is also possible to relate basic element deformation vector v to displacements in complete system by separating 3 rigid body modes and keeping only the basic deformation modes for the element. By this way, it is feasible to derive flexibility matrix that would have been impossible to get in the complete system because of the singularity caused by rigid body modes. Basic element deformations **v** can be calculated from nodal displacements **u** in complete system as follows;

$$\mathbf{v}=\mathbf{a}\mathbf{u} \tag{6}$$

Basic element forces at free end, **q** are shown in Figure 1 and given in Equation (5). These forces can be related to internal section forces, $\mathbf{s}(x)$ by using the force interpolation matrix $\mathbf{b}(x,L)$ for the cantilever beam configuration as follows;

$$\mathbf{s}(x)=\left[N(x)\quad V(x)\quad M(x)\right]^{\mathrm{T}}=\mathbf{b}(x,L)\mathbf{q}+\mathbf{s}_{\boldsymbol{p}}(x)$$
$$\mathbf{b}(x,L)=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & (L-x) & 1\end{bmatrix}\quad\text{and}\quad \mathbf{s}_{\boldsymbol{p}}(x)=\begin{bmatrix}L-x & 0\\ 0 & L-x\\ 0 & (L-x)^2/2\end{bmatrix}\begin{Bmatrix}w_x\\ w_y\end{Bmatrix} \tag{7}$$

By using Equation (7), it is possible to attain exact equilibrium between the forces at free end of the element and forces at any section that is x units away from the fixed end. Section forces are axial force $N(x)$, shear force in *y* direction $V(x)$, and moment about *z*-axis $M(x)$. In above equation $\mathbf{s}_{\boldsymbol{p}}(x)$ is the particular solution for uniformly distributed loads in the axial and transverse directions, i.e. $w_x$ and $w_y$, respectively. By the way, with this approach, it is easy to calculate the particular solution under arbitrary inter element loads that are concentrated or distributed.

### 2.3 Variational Base and Finite Element Formulation of the Element

Variational form of the element is written by considering independent element nodal displacements **u**, element basic forces **q**, and section deformations **e** by using three-fields Hu-Washizu functional and implemented as part of beam finite elements by Taylor et al. [13] and Saritas and Filippou [14]. Extension to dynamic case is achieved through introduction of inertial forces $\mathbf{m\ddot{u}}$ acting at nodes by considering D'Alembert's principle to get the following variational form of the element

$$\begin{aligned}\delta\Pi_{\mathrm{HW}} &= \int_0^L \delta\mathbf{e}^{\mathrm{T}}\left(\hat{\mathbf{s}}(\mathbf{e}(x)) - \mathbf{b}(x,L)\mathbf{q} - \mathbf{s}_{\boldsymbol{p}}(x)\right)dx - \delta\mathbf{q}^{\mathrm{T}}\int_0^L \mathbf{b}^{\mathrm{T}}(x,L)\mathbf{e}(x)\,dx + \delta\mathbf{q}^{\mathrm{T}}\mathbf{a}_g\mathbf{u} \\ &+ \delta\mathbf{u}^{\mathrm{T}}\mathbf{a}_g{}^{\mathrm{T}}\mathbf{q} + \delta\mathbf{u}^{\mathrm{T}}\mathbf{m\ddot{u}} - \delta\mathbf{u}^{\mathrm{T}}\mathbf{p}_{app} = 0\end{aligned} \tag{8}$$

Above equation can also be obtained by considering the general Hu-Washizu variational form with extension to dynamic case by Barr [15]. Equation (8) should hold for arbitrary $\delta\mathbf{u}$, $\delta\mathbf{q}$ and $\delta\mathbf{e}$, thus the following three equations should be satisfied in order for the Hu-Washizu-Barr variational to be zero.

$$\mathbf{m\ddot{u}} + \mathbf{p} \equiv \mathbf{p}_{app}; \quad \text{where} \quad \mathbf{p} = \mathbf{a}_{\mathrm{g}}{}^{\mathrm{T}}\mathbf{q} \tag{9}$$

$$\mathbf{v} \equiv \int_0^L \mathbf{b}^{\mathrm{T}}(x,L)\,\mathbf{e}(x)\,dx; \quad \text{where} \quad \mathbf{v} = \mathbf{a}_{\mathrm{g}}\mathbf{u} \tag{10}$$

$$\hat{\mathbf{s}}(\mathbf{e}(x)) \equiv \mathbf{b}(x,L)\mathbf{q} + \mathbf{s}_{\boldsymbol{p}}(x) \tag{11}$$

Equation (9) is the equation of motion that holds for linear or nonlinear material response, and this equation can be collected for each element to get structure's equation of motion. A numerical time integration scheme can be employed to get a solution. Consequence of viscous damping can be simply achieved by adding $\mathbf{c\dot{u}}$ to the left hand side of the equation, where c is the damping matrix. It is also possible to determine resisting forces **p** not only in terms of displacements **u** but also as a function of velocities $\dot{\mathbf{u}}$ through the use of a material model that considers time-dependent effects, such as viscoelastic or viscoplastic material models.
For linear elastic material response, section deformations can be calculated as $\mathbf{e}=\mathbf{k}_{\mathrm{s}}^{-1}\hat{\mathbf{s}}$ to obtain the section deformations from section forces through the use of section stiffness matrix $\mathbf{k}_{\mathrm{s}}$. Substitution of section deformations **e** to Equation (10) gives:

$$\mathbf{a}_{\mathrm{g}}\mathbf{u} = \mathbf{v} = \mathbf{f}\;\mathbf{q}; \quad \text{where} \quad \mathbf{f} = \int_0^L \mathbf{b}^{\mathrm{T}}(x,L)\mathbf{f}_s\,(x)\,\mathbf{b}(x,L)\,dx \tag{12}$$

In above equation **f** is the flexibility matrix of the element in the basic system. $\mathbf{f}_{\mathrm{s}}$ is the section flexibility matrix that can be calculated from the inversion of the section stiffness matrix $\mathbf{k}_{\mathrm{s}}$. Further substitution of above equation for linear elastic response in Equation (9) results in

$$\mathbf{m\ddot{u}} + \mathbf{ku} = \mathbf{p}_{app}; \quad \text{where} \quad \mathbf{k} = \mathbf{a}^{\mathrm{T}}\mathbf{f}^{-1}\mathbf{a} \tag{13}$$

where **k** is the 6×6 element stiffness matrix in the complete system. At this point in the element formulation, presence of semi-rigid connections will be introduced through the following extended version of above equation for the calculation of element end deformations:

$$\begin{aligned}&\mathbf{v} = \mathbf{v}_{\mathrm{Frame}} + \mathbf{v}_{\mathrm{Con}}; \quad \text{where} \quad \mathbf{v}_{\mathrm{Frame}} = \int_L \mathbf{b}^{\mathrm{T}}(x)\mathbf{e}(x)\,dx; \quad \mathbf{v}_{\mathrm{Con}} = \sum_{i=1}^{nSC}\mathbf{b}^{\mathrm{T}}(x_i)\boldsymbol{\Delta}_{SC,i} \\ &\text{and} \quad \boldsymbol{\Delta}_{SC} = \left[\delta_{SC}^{axial} \quad \theta_{SC} \quad \delta_{SC}^{shear}\right]^{\mathrm{T}}\end{aligned} \tag{14}$$

The first integral along the length of the frame element can be numerically calculated by using a quadrature rule to capture spread of inelastic behavior and $n_{SC}$ is the total number of semi-rigid connections discreetly located along element length; $\Delta_{SC}$ is the vector of semi-rigid connection deformations. Introduction of semi-rigid connections along element length in Figure 1 does not alter the force field under small deformations. Element flexibility matrix is similarly discretized as follows:

$$\mathbf{f} = \mathbf{f}_{\text{Frame}} + \mathbf{f}_{\text{Con}}; \quad \text{where} \quad \mathbf{f}_{\text{Frame}} = \int_L \mathbf{b}^{\text{T}}(x)\mathbf{f}_s(x)\mathbf{b}(x)dx; \quad \text{and} \quad \mathbf{f}_{\text{Con}} = \sum_{i=1}^{nSC} \mathbf{b}^{\text{T}}(x_i)\mathbf{f}_{SC,i}\mathbf{b}(x_i) \quad (15)$$

As a remark, Equations (10) and (11) are related to the element state determination, i.e. these equations can be solved independent of Equation (9), and then the solution can be condensed out into Equation (9) such that the equations of motion can be assembled for all elements. This process was demonstrated above for the linear elastic case. In general, state determination of the element requires an iterative solution in the case of nonlinear behavior, where Equations (9) to (11) are needed to be solved. Detailed derivation of this element and its validation towards carrying out nonlinear static analysis of steel framed structures with semi-rigid connections was published in Saritas and Koseoglu [16].

## 2.4 Section Response

Section response can be obtained by the basic assumption that plane sections before deformation remain plane after deformation along the length of the beam by the use of following section compatibility matrix as $\mathbf{a}_s$ given in Equation (2), where the section compatibility matrix now contains the shear correction factor κs as follows

$$\mathbf{a}_s = \mathbf{a}_s(y) = \begin{bmatrix} 1 & 0 & -y \\ 0 & \kappa_s & 0 \end{bmatrix} \quad (16)$$

Shear correction factor $\kappa_s$ is taken as the inverse of the form factor suggested by Charney et al. [17] for an I-section:

$$\kappa_s = 1/\kappa; \quad \text{where} \quad \kappa = 0.85 + 2.32\frac{b_f t_f}{d\, t_w} \quad (17)$$

The section forces are obtained by integration of the stresses that satisfy the material constitutive relations $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{\varepsilon})$ according to

$$\mathbf{s} = \int_A \mathbf{a}_s^{\text{T}} \boldsymbol{\sigma}\, dA; \quad \text{where} \quad \boldsymbol{\sigma} = \begin{pmatrix} \sigma_{xx} \\ \sigma_{xy} \end{pmatrix} \quad (18)$$

The derivative of section forces from (18) with respect to the section deformations results in the section tangent stiffness matrix

$$\mathbf{k}_s = \frac{\partial \mathbf{s}}{\partial \mathbf{e}} = \int_A \mathbf{a}_s^{\text{T}} \frac{\partial \boldsymbol{\sigma}(\boldsymbol{\varepsilon})}{\partial \mathbf{e}}\, dA = \int_A \mathbf{a}_s^{\text{T}} \mathbf{k}_m \mathbf{a}_s\, dA \quad (19)$$

The material tangent modulus km is obtained from the stress-strain relation according to $\mathbf{k}_m = \partial\boldsymbol{\sigma}(\boldsymbol{\varepsilon})/\partial\boldsymbol{\varepsilon}$. Gauss-quadrature, the midpoint or the trapezoidal rule can be used for the numerical evaluation of the integrals in (18) and (19).

## 2.5 Force Based Consistent Mass Matrix

The derivation of the consistent mass matrix requires the determination of the section mass matrix, where the mass is considered like a distributed load along the length of the beam in cantilever basic system. The section mass matrix is easily computed by following equation through the use of section compatibility matrix as given in Equation(4):

$$\mathbf{m}_s(x) = \int_A \mathbf{a}_s^{\text{T}} \rho(x,y)\, \mathbf{a}_s\, dA; \quad (20)$$

Mass matrix of the force-based element, which will be used in Equation (9), is written in a 6×6 dimension by the method discussed in depth by [18], i.e. in the complete system with 3 degrees of freedom per node, as follows:

$$\mathbf{m} = \begin{bmatrix} \mathbf{m}_{00} & \mathbf{m}_{0L} \\ \mathbf{m}_{L0} & \mathbf{m}_{LL} \end{bmatrix} \tag{21}$$

where the components of element mass matrix are calculated from sub-matrices

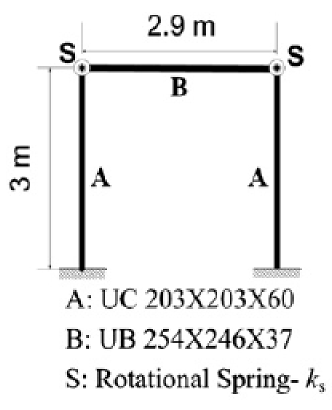
$$\begin{aligned}
\mathbf{m}_{LL} &= \mathbf{f}^{-1}\int_0^L \mathbf{b}^{\mathrm{T}}(x,L)\mathbf{k}_s^{-1}(x)\left(\int_x^L \mathbf{b}^{\mathrm{T}}(x,\xi)\mathbf{m}_s(\xi)\mathbf{f}_p(\xi)\mathbf{f}^{-1}d\xi\right)dx \\
\mathbf{m}_{L0} &= \mathbf{f}^{-1}\int_0^L \mathbf{b}^{\mathrm{T}}(x,L)\mathbf{k}_s^{-1}(x)\left(\int_x^L \mathbf{b}^{\mathrm{T}}(x,\xi)\mathbf{m}_s(\xi)\left(\mathbf{b}^{\mathrm{T}}(0,\xi)-\mathbf{f}_p(\xi)\mathbf{f}^{-1}\mathbf{b}^{\mathrm{T}}(0,L)\right)d\xi\right)dx \\
\mathbf{m}_{0L} &= \mathbf{m}_{L0} = -\mathbf{b}(0,L)\mathbf{m}_{LL} + \int_0^L \mathbf{b}(0,x)\mathbf{m}_s(x)\mathbf{f}_p(x)\mathbf{f}^{-1}dx \\
\mathbf{m}_{00} &= -\mathbf{b}(0,L)\mathbf{m}_{L0} + \int_0^L \mathbf{b}(0,x)\mathbf{m}_s(x)\left(\mathbf{b}^{\mathrm{T}}(0,x)-\mathbf{f}_p(x)\mathbf{f}^{-1}\mathbf{b}^{\mathrm{T}}(0,L)\right)dx
\end{aligned} \tag{22}$$

In above equations, element flexibility matrix **f** is obtained as given in Equation (12). The partial flexibility matrix $\mathbf{f}_{\mathrm{p}}$ is calculated as follows:

$$\mathbf{f}_p(x) = \int_0^x \mathbf{b}^{\mathrm{T}}(\xi,x)\mathbf{k}_s^{-1}(x)\mathbf{b}(\xi,x)\,d\xi \tag{23}$$

## 3. Numerical Examples

The first example is a portal frame retrieved from [9], where semi-rigid connections are defined at both ends of the beams in this structure. British UB254x146x37 section is defined to the beam and UC203x203x60 sections are defined for the columns. The length of the beam is 2.9 m and the height of the columns is 3.0 m (see Figure 2). The results are compared with the SAP2000 analyses results for 1 and 32 elements of each member of the portal frame. This comparison is conducted to verify the accuracy of the SAP2000 model with increased number of the elements. Since SAP2000 uses lumped mass approach, increasing mesh size tends to give a closer match with consistent mass matrix results as evident in Figure 2.



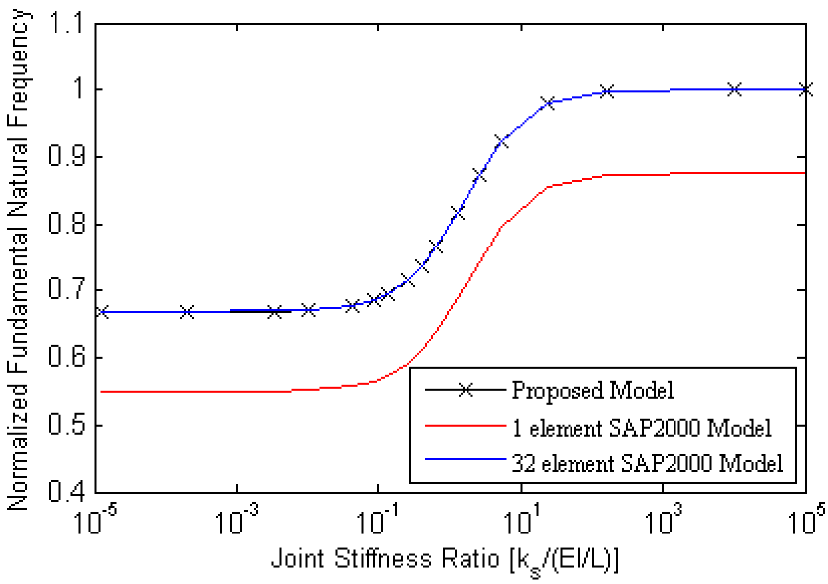




**Fig. 2: Portal Frame from the study of [9] and**
**Fundamental Natural Frequency versus Joint Stiffness Ratio for Portal Frame**

The second example is a 3 bay and 6 stories steel frame with European sections HEB260 for columns and IPE300 for beams as given in [9]. The length of the beams is 6.0 m and the height of the columns is 3.75 m. The results of the proposed model are compared with SAP2000 models and results from the study of [9]. On the contrary to the former example, SAP2000 model is prepared with four elements for each member. Since the structural system is a much larger one compared to the previous example, lumped mass approach presents better convergence with 4 elements per member for SAP2000 as compared to the results by ABAQUS model from [9] (see Fig. 3); however, the results of SAP2000 could not perfectly capture the fundamental mode of vibration due to the deficiency of SAP2000 in capturing an accurate shear correction factor for these IPE sections. It is apparent from below figure that proposed model showed very close results with ABAQUS by the use of single element discretization per member.

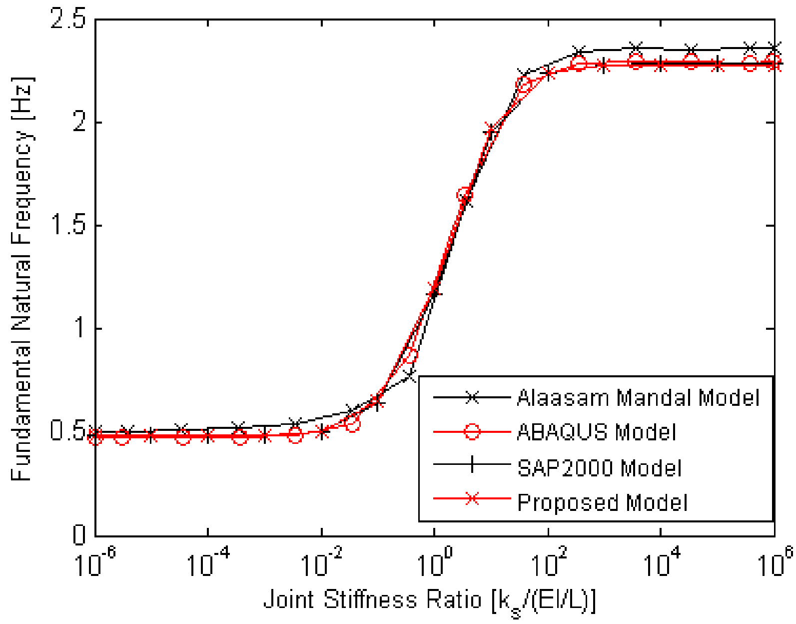



**Fig. 3: Fundamental Natural Frequency versus Joint Stiffness Ratio for 3 Bays 6 Stories Steel Framed Structure**

## 4. Conclusions

The proposed model introduces consistent mass and stiffness matrices for a frame member in the presence of semi-rigid connections without the need to further discretize the member into more elements, and this approach provides simpler solution and modeling strategy towards analysis of steel framed structures with semi-rigid connections. Validation studies clearly showed the robustness of the proposed model compared to other finite element programs widely used in research and practice.

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