SUPPLEMENTARY PROBLEMS

1. We can obtain qualitative information about the solutions to first order differential equation by drawing its direction (flow) field. Use the qualitative approach with the following equations and plot the given solution over its direction field:

   (a) $y' = e^y$, $y(0) = 1$, $0 \le t \le 0.20$, whose exact solution is $y(t) = \ln(e/(1 - et))$.

   (b) $y' = \dfrac{1}{t^2} - \dfrac{y}{t} - y^2$, $y(1) = -1$, $1 \le t \le 2$, whose exact solution is $y(t) = -1/t$.

   (c) $y' = \dfrac{2}{t} y + t^2 e^t$, $y(1) = 0$, $1 \le t \le 2$, whose exact solution is $y(t) = t^2 (e^t - e)$.

   (d) $u' = 3u + 2v$, $u(0) = 0$; $v' = 4u + v$, $v(0) = 1$, $0 \le t \le 1$, whose exact solution is
   $u(t) = \frac{1}{3}(e^{5t} - e^{-t})$, $v(t) = \frac{1}{3}(e^{5t} + 2e^{-t})$.

   (e) $u' = -4u - 2v + \cos t + 4\sin t$, $u(0) = 0$; $v' = 3u + v - 3\sin t$, $v(0) = -1$, $0 \le t \le 2$,
   whose exact solution is $u(t) = 2e^{-t} - 2e^{-2t} + \sin t$, $v(t) = -3e^{-t} + 2e^{-2t}$.

   <u>Note</u> : The following sequence of commands draws a direction field and the solution for the system:

   ```
   (a) [T, Y] = meshgrid(0:0.02:0.20, 1:0.02:1.8);
       FY = exp(Y); % direction field
       L = sqrt(1 + FY.^2); % arrow length
       t = linspace(0,0.2,100);
       YE = log(exp(1)./(1-exp(1)*t));
       quiver(T, Y, 1./L, FY./L, 0.5)
       axis equal tight, hold on
       plot(t,YE)
       xlabel 't'
       ylabel 'y'
       title 'Vector field and the solution for the equation (a)'

   (d) [U, V] = meshgrid(0:5:50, 0:5:50);
       FU = 3*U + 2*V;
       FV = 4*U + V;
       t = linspace(0,1,100);
       UE = (1/3)*(exp(5*t)-exp(-t));
       VE = (1/3)*(exp(5*t)+2*exp(-t));
       quiver(U, V, FU, FV, 0.5)
       axis equal tight, hold on
       plot(UE,VE)
       xlabel 'u'
       ylabel 'v'
       title 'Vector field and the solution for the system (d)'
   ```

2. Rewrite the system
$$y'' = t^2 - y' - z^2,$$
$$z'' = t + z' + y^3,$$

with initial conditions $y(0)=0$, $y'(0)=1$, $z(0)=1$, $z'(0)=0$, as an initial value problem for a system of first-order equations.

Ans.: $\frac{d\mathbf{X}}{dt} = \frac{d}{dt}\begin{bmatrix} y & y' & z & z' \end{bmatrix} = \begin{bmatrix} y' & t^2 - y' - z^2 & z' & t + z' + y^3 \end{bmatrix}$ subject to $\mathbf{X}(0) = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}$.

3. Determine a Taylor expansion for the solution of the equation $y' = y^2$, $y(0) = 1$. Use this approximation to compute y for $t = 0.2$ and $t = 1.2$ to correct four decimals. Compare with the exact solution, and explain why the second case ($t = 1.2$) was unsuccessful.

Ans.: $\frac{dy}{dt} = y^2; \frac{d^2y}{dt^2} = 2yy' = 2y^3; \frac{d^3y}{dt^3} = 6y^2y' = 6y^4; ...; \frac{d^ny}{dt^n} = n!y^{n+1}$ thus $y(0+h) \approx \sum_{n=0}^{N} h^n = \frac{1-h^{N+1}}{1-h}$ convergent for $h < 1$ as $N \to \infty$.

4. In a computation with Euler's method, the following results were obtained with various step sizes:

h	0.05	0.1	0.2
$y(x^*,h)$	1.22726	1.22595	1.22345

where $y(x^*,h)$ denotes the numerical value of the solution $y = y(x)$ at x^* as a result of the use of Euler's method with step length h. Note that
$$y(x^*,h) = y(x^*) + c_1 h + c_2 h^2 + c_3 h^3 + O(h^4).$$
Compute a better value by repeated Richardson extrapolation.

Ans.: $O(h^2): (qy(x^*,h) - y(x^*,h))/(q-1)$ where $q = .5$.

5. Consider the initial value problem $y' = y - 3e^{-2t}$, $y(0) = 1$. The exact solution is $y(t) = e^{-2t}$. Find the numerical solution on $t \in [0,12]$ using

   (a) Euler's method (T1) with $h = 0.5$ to get $\{ta_k, ya_k\}$,

   (b) Euler's method with $h = 0.05$ to get $\{tb_k, yb_k\}$,

   (c) Taylor method of order 2 (T2) with $h = 0.5$ to get $\{tc_k, yc_k\}$,

   (d) and plot them together with the exact solutions.

   This exercise shows that while local error is controlled by using smaller h or a higher order solver, the global error (defined as the cumulative error committed in taking the steps from the initial to the terminal point) can not be controlled completely by the numerical method because it depends also on the differential equation whether it is stable or unstable.

   The solution to the differential equation with initial condition $y(0) = 1 + \varepsilon$ is $y(t) = e^{-2t} + \varepsilon e^t$. Note that as t increases, the solutions with initial values slightly different from 1 separate very sharply from the solution with initial value exactly 1. This is an unstable differential equation with sensitive dependence on the initial conditions.

   Note : You may use the following sequence of commands:
   ```
   plot(ta,ya,'r'), hold on
   plot(tb,yb,'k')
   plot(tc,yc,'y')
   ```

```
ezplot('exp(-2*t)',[0 12]), hold off
axis([0 12 -1 1])
xlabel 't'
ylabel 'y'
title 'Exact and Approximate Solutions'
```

Ans.: $T1: y_{k+1} = y_k + h(y_k - 3\exp(-2t_k))$, $T2: y_{k+1} = y_k + h(y_k - 3\exp(-2t_k)) + \frac{1}{2}h^2(y_k + 3\exp(-2t_k))$.

6. Solve

$$\frac{dy}{dt} = 3y + 3t \quad \text{on} \quad (0, 0.2) \quad \text{with}$$

$$y(0) = 1$$

   using

   (a) Euler's method with $h = 0.05$.

   (b) Runge-Kutta-4 with $h = 0.05$.

   (c) Compare your results of part(a) and (b) with the exact solution

   $$y(t) = \frac{4}{3}\exp(3t) - t - \frac{1}{3}.$$

7. Solve the following Boundary Value Problem

$$\frac{d^2 y}{dx^2} + x\frac{dy}{dx} - 3y = 4.2x \quad \text{on} \quad (0,1) \quad \text{with}$$

$$y(0) = 0, \quad y(1) = 1.9$$

   using

   (a) Shooting method with Euler scheme. You may assume the initial slopes to be $z_1 = 0.8$ and $z_2 = 1$ with $h = 0.25$.

   (b) Finite Difference method with $h = 0.25$.

   (c) Compare your results graphically with the exact solution $y = x^3 + 0.9x$.

8. Compound A diffuses through a 4-cm-long tube and reacts as it diffuses. The equation governing diffusion with reaction is

$$D\frac{d^2 A}{dx^2} - kA = 0.$$

   At one end of the tube, there is a large source of A at a concentration of A_0 M (i.e. $A(0) = A_0 = 0.1$). At the other end of the tube there is an adsorbent material that quickly absorbs any A, making the concentration 0 M (i.e. $A(4) = 0.0$). If $D = 1 \times 10^{-6}$ cm^2/s and $k = 4 \times 10^{-6}$ s^{-1}, what is the concentration of A as a function of distance in the tube?

   (a) Apply the linear shooting method manually for, say, $N = 10$ subintervals as follows:

   (i) Convert the second order ODE above to a first order system by letting $U = A$ and $V = A'$.

(ii) Numerically integrate the resulting system using (**I**) Runge-Kutta-4 (**RK4.m**), (**II**) Taylor-2 (**T2.m**) methods for $U_1(0) = A_0$, $V_1(0) = 0.0$ to get $U_1(x)$ and for $U_2(0) = A_0$, $V_2(0) = 1.0$ to get $U_2(x)$.

(iii) Compute λ from the equation $0.0 = \lambda U_1(4) + (1-\lambda)U_2(4)$.

(iv) Finally compute $A(x) = U(x) = \lambda U_1(x) + (1-\lambda)U_2(x)$.

(b) Plot and compare with the exact solution

$$A(x) = A_0 \frac{\exp(c(4-x)) - \exp(-c(4-x))}{\exp(4c) - \exp(-4c)}$$

where $c = \sqrt{k/D}$ and tabulate the absolute error for the $N = 10$ subintervals and comment on the different techniques.

```
% RK4.m
global k D
a = 0; b = 4; N = 10; h = (b - a)/N;
k = 4e-6; D = 1e-6;
U = 0.1; V = 0.0; X = 0; Y = [U; V];
A(1) = Y(1);
% Stepping by Runge-Kutta-4:
for i = 1:N
  K1 = h*F(Y,X);
  K2 = h*F(Y + 0.5*K1,X + 0.5*h);
  K3 = h*F(Y + 0.5*K2,X + 0.5*h);
  K4 = h*F(Y + K3,X + h);
  Y = Y + (1/6)*(K1 + 2*K2 + 2*K3 + K4);
  X = X + h; A(i+1) = Y(1);
end, clf, plot(0:h:4,A)
```

```
% T2.m
global k D
a = 0; b = 4; N = 10; h = (b - a)/N;
k = 4e-6; D = 1e-6;
U = 0.1; V = 1.0; X = 0; Y = [U; V];
A(1) = Y(1);
% Stepping by Taylor-2:
for i = 1:N
  Y = Y+h*F(Y,X)+(1/2)*(h^2)*DF(Y,X);
  X = X + h; A(i+1) = Y(1);
end, clf, plot(0:h:4,A)
```

```
% F.m
function F = F(Y,X)
global k D
F1 = Y(2); F2 = (k/D)*Y(1);
F = [F1; F2];
```

```
% DF.m
function DF = DF(Y,X)
global k D
DF1 = (k/D)*Y(1); DF2 = (k/D)*Y(2);
DF = [DF1; DF2];
```

9. The following <u>nonlinear</u> system of ODEs known as Lotka-Volterra equations represent the population dynamics between two species, the prey and the predator:

$$\frac{dx}{dt} = \alpha x + \beta x y$$

$$\frac{dy}{dt} = \gamma y + \delta x y$$

where $x = x(t)$ and $y = y(t)$ designate the number of prey and predator, respectively, at time t. In the prey equation, αx ($\alpha > 0$) represents natural increase in the prey population, i.e. $\alpha x = (\text{birth - natural death})x$, while $\beta x y$ ($\beta < 0$) represents that the number of times that the predator kills the prey depends on the chance probability of the two coming together, i.e. higher x and y, higher the kill rate. In the predator equation, we assume that the number of predators would decrease by natural causes if the prey were absent,

contributing a γy ($\gamma < 0$) term. However, the number of predators increases as a result of encounters with prey, contributing a $\delta x y$ ($\delta > 0$) term.

This eco-system is in a stable cycle for the parameters $\alpha = 0.25$, $\beta = -0.01$, $\gamma = -1.00$, $\delta = 0.01$ with an initial population of $x(0) = 80$ and $y(0) = 30$. Numerically integrate this system using Euler (write **Euler.m** as above) and Runge-Kutta-2 (write **RK2.m** as above) methods form $t = 0$ to $t = 15$ for the step sizes $h = 1.00, 0.50, 0.25$. Plot x versus y over the figure for the direction (flow) field below and comment on the effects of the step sizes and the different methods.

Note : The following sequence of commands draws a direction (flow) field for the system:

```
[X, Y] = meshgrid(0:10:200, 0:10:200);
alpha = 0.25; beta = -0.01; gamma = -1.00; delta = 0.01;
U = alpha*X + beta*X.*Y;
V = gamma*Y + delta*X.*Y;
quiver(X, Y, U, V, 0.5)
axis equal tight
xlabel 'x'
ylabel 'y'
title 'Vector field for the Lotka-Volterra system'
```