SUPPLEMENTARY PROBLEMS

1. Consider the function $f(x) = (x^2 + 1)^{-1}$ on the interval [-5, 5].
   (a) Let p be the interpolating polynomial of degree 20 interpolating the function $f(x)$ at 21 equally spaced nodes $x_i = -5 + (i/2)$, where $i = 0, 1, \ldots, 20$. Plot $f(x)$ and $p(x)$ by sampling at 100 equally spaced points. Observe the large discrepancy between $f(x)$ and $p(x)$ (Runge phenomena). Use **DividedDiff.m**.
   (b) Perform the experiment in the preceding computer problem, using interpolation at 21 Chebyshev nodes, $x_i = 5\cos(i\pi/20)$, where $i = 0, 1, \ldots, 20$. Use **DividedDiff.m**.
   (c) A remedy for the polynomial interpolation of large data sets is to consider piecewise polynomial interpolation, an example of which is the natural cubic spline interpolation. Now, construct natural cubic spline interpolant $S(x)$ that interpolates $f(x)$ at the 21 equally spaced nodes as in (a) on the interval [-5, 5]. Plot $f(x)$ and $S(x)$ by sampling at 100 equally spaced points. Use **NatCubicSpl.m**.
   (d) Draw your conclusions out of these numerical experiments.
   Note : Use the input lines for **(a)** `X = -5+((0:20)/2);` or `X = linspace[-5,5,21];` and for **(b)** `X = 5*cos((0:20)*pi/20);` and `Y = 1./(X.^2+1);` in **DividedDiff.m** and **NatCubicSpl.m**.

2. Kinematic viscosity of water, ν, is related to temperature in the following manner:

$T,\ ^{o}C$	0	4	8	12	16	20
$\nu,\ \times 10^{-2}\ cm^2/s$	1.7923	1.5676	1.3874	1.2396	1.1168	1.0105

   (a) Use Newton form of polynomial interpolation to predict ν at $T = 7.5^{o}C$ in an adaptive manner and estimate the error. You may use **DividedDiff.m**.
   Note : An adaptive manner would be to start with the closest two data points to $T = 7.5^{o}C$, namely 4 and 8, and fit a first order polynomial, $p_1(T)$, to this data. Then, new data points are added one-by-one in the order determined by the closeness to $T = 7.5^{o}C$. This procedure continues until the generated sequence of predictions, $p_1(7.5)$, $p_2(7.5)$, $p_3(7.5)$, $\ldots$, show the least change.
   Ans. : $\{p_i(7.5)\}_{i=1}^{5} = \{1.4099, 1.4082, 1.4079, 1.4080, 1.4080\}$ and error $\simeq |p_{i+1}(7.5) - p_i(7.5)|$
   (b) Suggest and implement an easy way of incorporating an additional data $\{T = 24^{o}C,\quad \nu = 0.9186 \times 10^{-2}\}$ into the interpolation process above.
   Hint : Think a way of relating the polynomial $p(x)$ that already interpolates the table to $q(x)$ that is supposed to interpolate the table plus the added data point in Newton form.
   Ans. : $q(x) = p(x) + c(x-0)\cdots(x-20)$

3. Inverse Interpolation : Suppose that $y = f(x)$ has an inverse $x = f^{-1}(y)$ on $[a, b]$, under certain conditions and f has one zero r in $[a, b]$. Construct the interpolating polynomial $x = p_n(y)$ for f^{-1} over the table

y	y_0	y_1	$\cdots$	y_n

x	x_0	x_1	…	x_n

Since $y_k = f(x_k)$ and $0 = f(r)$, it follows that $f^{-1}(y_k) = x_k$ and $r = f^{-1}(0)$. The approximation to the root is then given by $r \approx p_n(0)$. This procedure of approximating the root $r = f^{-1}(0)$ by using interpolation is called inverse interpolation.

Use inverse interpolation to find an approximation to the solution of $x - \exp(-x) = 0$ using the data

x	0.3	0.4	0.5	0.6
$\exp(-x)$	0.740818	0.670320	0.606531	0.548812

Ans. : $r = f^{-1}(0) \approx p_3(0) = 0.567142$ where $y = f(x) = x - \exp(-x)$ and

$x = f^{-1}(y) \approx p_3(y) = 0.3 + (y - f(.3))(0.5865 + (y - f(.4))(.0719 + 0.0025(y - f(.5))))$

4. The following table of values of a function $f(x)$ is given:

x	0.6	0.8	0.9	1.0	1.1	1.2	1.4
$f(x)$	1.820365	1.501258	1.327313	1.143957	.951849	.752084	.335920

Compute $f'(1.0)$ as accurate as possible using Central Difference CD(h) formula and repeated Richardson extrapolation.

Ans. : $CD(.4) = -1.85556$, $CD(.2) = -1.87294$, $CD(.1) = -1.87732$, and Richardson extrapolation $O(h^4) : q^2 CD(h) - CD(qh) / (q^2 - 1)$ and $O(h^6) : q^4 CD(h) - CD(qh) / (q^4 - 1)$ where $q = .5$.

5. Compute the integral $\int_0^4 f(x)\,dx$ where $f(x)$ is defined by the following table:

x	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
$f(x)$	-4271	-2522	-499	1795	4358	7187	10279	13633	17247

(a) using Trapezoid Rule $T(h)$ with $h = 4.0, 2.0, 1.0, 0.5$ and repeated Richardson extrapolation as accurate as possible.

Ans. : $T(4) = 25952$, $T(2) = 21692$, $T(1) = 20626$, $T(.5) = 20360$ and Richardson extrapolation $O(h^r) : q^k T(h) - T(qh) / (q^k - 1)$ where $q = .5$ and $k = 2, 4, 6$ while $r = 4, 6, 8$.

(b) using Simpson's rule $S(h)$ with $h = 4.0, 2.0, 1.0$.

Ans. : $S(4) = 20272$, $S(2) = 20271$, $S(1) = 20271$.

(c) Compare the results in (a) and (b). Do you observe any similarities? Explain.

Ans. : $S(h) \equiv T(h) + \text{Richardson}(O(h^4))$.

6. Consider the polynomial interpolation problem over the net $\{x_1, x_2, x_3\}$. Explain why

(a) $L_1(x) + L_2(x) + L_3(x) = 1$,

(b) $x_1 L_1(x) + x_2 L_2(x) + x_3 L_3(x) = x$,

(c) $x_1^2 L_1(x) + x_2^2 L_2(x) + x_3^2 L_3(x) = x^2$ for all x, while

(d) $x_1^3 L_1(x) + x_2^3 L_2(x) + x_3^3 L_3(x) \neq x^3$

where $L_1(x), L_2(x), L_3(x)$ are the Lagrange interpolants, $L_i(x) = \prod_{j=1;\, j\neq i}^{3} \frac{(x-x_i)}{(x_j - x_i)}$.

Ans. : By uniqueness of the polynomial p(x) interpolating f(x), $f(x) \equiv p(x)$ for f a polynomial of degree ≤ 2 sampled at three points. So choose, $f(x) = 1, x, x^2, x^3$ in (a), (b), (c) and (d), respectively.

7. Derive the 2nd order Forward Difference formula

$$f'(x) \approx \tfrac{1}{2h}[-f(x+2h) + 4f(x+h) - 3f(x)]$$

by using suitable three-point Lagrange polynomial interpolation of $f(x)$:

$$f(x) \approx p_2(x) = f(x_1)\, L_1(x) + f(x_2)\, L_2(x) + f(x_3)\, L_3(x)$$

with $f'(x) \approx p_2'(x)$ and show that its error term is of the form $\frac{1}{3}h^2 f'''(\zeta)$.

8. The rate of cooling of a body can be expressed as

$$\frac{dT}{dt} = -k(T - T_a)$$

where T temperature of the body, T_a temperature of the body, and k proportionally constant. Thus, this equation (called Newton's law of cooling) specifies that the rate of cooling is proportional to the difference in the temperatures of the body and of the surrounding medium. If a metal ball heated to 90^o C is dropped into water that is held constant at $T_a = 25^o C$, the temperature of the ball changes, as in

Time, min	0	5	10	15	20	25
T, $^o C$	90	49.9	33.8	28.4	26.2	25.4

Utilize numerical differentiation to determine dT/dt at each value of time. Plot dT/dt versus $T - T_a$, observe a linear trend (remove out-of-trend points, if any) and employ linear least squares fit to evaluate k.

Note : Use second-order central-difference formula ($f'(x) \approx f(x+h) - f(x-h)/(2h)$) to estimate the derivatives at the inner nodes and second-order forward (+) or backward (-) formulas ($f'(x) \approx -f(x \pm 2h) + 4f(x \pm h) - 3f(x)/(\pm 2h)$) at the outer nodes.

Ans. : $\{dT/dt\} = \{-10.4, -4.29, -2.15, -0.760, -0.300, -0.0200\}$ and $k \approx 0.17$ after removing the data at $T = 90^o C$.

9. Construct an integration rule of the form

$$\int_{-1}^{1} f(x)\,dx \approx A\, f(-\tfrac{1}{2}) + B\, f(0) + C\, f(\tfrac{1}{2})$$

that is exact for all polynomials of degree ≤ 2; that is, determine values for A, B, and C, and apply it to the computation of $\int_0^2 \exp(-x^2)\,dx$.

Hint #1: Make the relation exact for $1, x, x^2$ and find a solution of the resulting equations for A, B, and C. If it is exact for these polynomials, it is exact for all polynomials of degree ≤ 2. This method of construction of an integration rule is called the method of undetermined coefficients.

Hint #2: An integration rule for the interval $[-1,1]$ can be used on the interval $[a,b] \equiv [0,2]$ by applying a suitable linear transformation or change of variable.

Ans. : $A = C = 4/3, B = -2/3$ and $I \approx .9337$.

10. Do there exist a, b, c, and d so that the function

$$S(x) = \begin{cases} a x^3 + x^2 + c x & -1 \le x \le 0 \\ b x^3 + x^2 + d x & 0 \le x \le 1 \end{cases}$$

is a natural cubic spline that agrees with the absolute value function $|x|$ at the nodes?

Ans. : No, contradiction.

11. Determine the parameters a, b, c, d, and e so that S is a natural cubic spline:

$$S(x) = \begin{cases} a + b(x-1) + c(x-1)^2 + d(x-1)^3 & 0 \le x \le 1 \\ (x-1)^3 + e x^2 - 1 & 1 \le x \le 2 \end{cases}.$$

Ans. : $a = -4, b = -6, c = -3, d = -1, e = -3$.

12. Consider $f(x) = \exp(-x)$.

(a) Construct a natural (free) cubic spline $S(x)$ to approximate $f(x) = \exp(-x)$ by using the values given by $f(x)$ at $x = 0, 0.25, 0.75,$ and 1.0. Integrate the spline over $[0,1]$, and compare the result to $\int_0^1 \exp(-x)\,dx = (e-1)/e$. Use the derivatives of the spline to approximate $f'(0.5)$ and $f''(0.5)$. Compare the approximations to the actual values. You may use **NatCubicSpl.m** to construct $S(x)$.

Ans. : $I \approx .6320$, $f'(0.5) \approx -.6032$, $f''(0.5) \approx .7003$.

(b) Show that the four point formula

$$f'(x) \approx \tfrac{1}{12h}[f(x-2h) - 8f(x-h) + 8f(x+h) - f(x+2h)]$$

is of fourth order accurate using Taylor expansions.

(c) Use the four point formula and the values of $f(x)$ at $x = 0, 0.25, 0.75,$ and 1.0 to approximate $f'(0.5)$. Compare this result to the exact value and to the approximation in (a).

Ans. : $f'(0.5) \approx -.6065$.

13. A process produces noisy periodic data over $[0, 2\pi]$ from which the period of the process is to be extracted. One way is to perform a Linear-Least-Squares fit to the data, over the subspace spanned by

$$\{1, \sin(x), \cos(x), \sin(2x), \cos(2x), \ldots, \sin(Nx), \cos(Nx)\},$$

that is, using the fitting function

$$f(x) = c_1 + c_2 \sin(x) + c_3 \cos(x) + \ldots + c_{n-1} \sin(Nx) + c_n \cos(Nx).$$

In order to test this procedure, let $N = 3$, generate the noisy periodic data, construct the normal equations to be solved for the expansion coefficients $\{c_k\}$ and verify that the test produces the hidden period.

Note : Here is a Matlab code segment to achieve this:

```
N=3;
X=linspace(0,2*pi,20); X=X(:); % equispaced grid of size 20 in [0,2pi]
Y=-3*sin(2*X)+4*cos(X)+0.1*rand(size(X)); % Noisy periodic data
Phi=ones(size(X)); % tab(1): Tabular form of function 1 over grid
for k=1:N
      Phi=[Phi sin(k*X) cos(k*X)]; % [tab(sin(kx)) tab(cos(kx))]
end
A=Phi'*Phi; % The coefficient matrix and
B=Phi'*Y; % the RHS vector of the Normal equations A*C=B
% Solve A*C=B for the expansion coefficients C=[Ck] where
% f(x)= C1 + C2 sin(x) + C3 cos(x) + ... + Cn-1 sin(Nx) + Cn cos(Nx)
% using lufactpiv.m, forwardsolve.m, backsolve.m.
```

14. Given the data

x	4.0	4.2	4.5	4.7	5.1	5.5	5.9	6.3	6.8	7.1
y	102.56	113.18	130.11	142.05	167.53	195.14	224.87	256.73	299.50	326.72

(a) Construct the least-squares polynomial approximation of degree one.

Ans. : $72.08x - 194.14$; $E = 18.14$

(b) Construct the least-squares polynomial approximation of degree two.

Ans. : $6.62x^2 - 1.14x + 1.24$; $E = 0.038$

(c) Construct the least-squares polynomial approximation of degree three.

Ans. : $-0.014x^3 + 6.85x^2 - 2.38x + 3.43$; $E = 0.023$

(d) Construct the least-squares approximation of the form $g(x) = b\exp(ax)$.

Ans. : $24.26\exp(.37x)$; $E = 19.93$

(e) Construct the least-squares approximation of the form $g(x) = b\,x^a$.

Ans. : $6.23x^{2.02}$; $E = 11.87$

(f) Compute the error $E = \sqrt{\sum_{k=1}^{m} (y_k - g(x_k))^2}$ in each case.

15. Trace your hand on an A4 size paper with a pencil. Then place an origin at an appropriate corner of the paper and draw the orthogonal x and y coordinate system with respect to this origin. Discretize the trace of your hand using as many points $i = 1, \ldots, M$ as you feel necessary and determine the x and y coordinates, $\{(x_i, y_i)\}$, of these points relative to the coordinate system placed on the paper. Compute the arclength data $s_{k+1} = s_k + \sqrt{(x_{k+1} - x_k)^2 + (y_{k+1} - y_k)^2}$ for $k = 1, \ldots, M-1$ with $s_1 = 0$. Construct Natural Cubic Spline, $S_x(s)$ and $S_y(s)$, interpolating each of the data sets,

$\{(s_i, x_i)\}$ and $\{(s_i, y_i)\}$, respectively. Use **NatCubicSpl.m**. Plot $S_x(ss)$ versus $S_y(ss)$ using a fine grid, say, `ss = linspace(min(s),max(s),500)`.