PROBLEMS

1. Use graphic representation to determine the zeros of the following functions to one correct decimal : (a) $4\sin x + 1 - x$; (b) $1 - x - e^{-2x}$; (c) $(x+1)e^{x-1} - 1$; (d) $x^4 - 4x^3 + 2x^2 - 8$; (e) $e^x + x^2 + x$; (f) $e^x - x^2 - 2x - 2$; (g) $3x^2 + \tan x$. In order to obtain graphical solution of $f(x)$ on the interval $[a,b]$, type the following statements into the MATLAB command window:

```
ezplot('f(x)',a,b)
hold on, ezplot('0',a,b)
```

   Hint : Play with the bracket $[a,b]$.

2. The following equations all have a root in the interval $(0,1.6)$. Determine these to one correct decimal accuracy using the bisection method: (a) $x\cos x = \ln x$; (b) $2x - e^{-x} = 0$; (c) $e^{-2x} = 1 - x$. Use hand computation and the MATLAB code **bisection.m**.

   Hint : You may use `ezplot` to determine a suitable bracket within the given interval. Recall that decimal accuracy control is achieved by the use of absolute error type expression with a suitable form of TOLerance.

3. Use the bisection method to find solutions accurate to two decimals for $x^4 - 2x^3 - 4x^2 + 4x + 4 = 0$ on (a) $[-2,-1]$; (b) $[0,2]$; (c) $[2,3]$; (d) $[-1,0]$. Use hand computation and the MATLAB code **bisection.m**.

   Note : In programming this polynomial function, use nested multiplication form.

4. Recall that in the n[th] step of the bisection method the root r can be estimated as
$$|r - c_n| \le \tfrac{1}{2}(b_n - a_n) = \tfrac{1}{2}\left(\tfrac{1}{2}(b_n - a_n)\right) = \ldots = \tfrac{1}{2^{n+1}}(b_0 - a_0)$$
   where $c_n = (a_n + b_n)/2$ is the midpoint. If an error tolerance ε has been prescribed in advance, it is possible to determine the number of steps required in the bisection method. Suppose that we want $|r - c_n| < \varepsilon$, then it is necessary to solve the following inequality for n: $\tfrac{1}{2^{n+1}}(b_0 - a_0) < \varepsilon$. By taking logarithms, one obtains
$$n > \frac{\ln(b_0 - a_0) - \ln(2\varepsilon)}{\ln 2}.$$
   If $a_0 = 0.1$ and $b_0 = 1.0$, how many steps of the bisection method are needed to determine the root r to eight decimals accuracy?

   Hint : This analysis should give a hint for a possible absolute error type expression for decimal control.

5. Try to devise a stopping criterion for the bisection method to guarantee that the root is determined with relative error at most ε. Quantify ε for t digit accuracy.

   Hint : At any given stage of bisection process, the bracket $[a,b]$ and the midpoint c are known. An available estimate of the unknown exact quantity (the root) may be used to normalize absolute error to get the corresponding relative error. Then ε becomes suitable form of TOLerance.

6. Use the bisection method to determine the point of intersection of the curves given by $y = x^3 - 2x + 1$ and $y = x^2$ to two siginificant digits. Use hand computation and the MATLAB code **bisection.m**.

   Hint : Convert into a root-finding problem and impose digit accuracy control as achieved by the use of a relative error type expression and a suitable form of TOLerance.

7. Find an approximation to $\sqrt[3]{25}$ correct to 4 siginificant digits using the MATLAB code **bisection.m**. (consider $f(x) = x^3 - 25$).

8. Review questions : (a) How many binary digits of precision are gained in each step of the bisection method and how many steps are required for each decimal digit of precision? (b) What do we mean when we say that an iteration method is quadratically convergent; (c) Under what assumptions is Newton's method quadratically convergent?

   Hint : Dig into the lecture notes.

9. Use Newton's method to determine the nonzero roots of (a) $x^3 = x + 4$; (b) $x = 1 - e^{-2x}$; (c) $x \ln x - 1 = 0$ to two siginificant digits. Use hand computation and the MATLAB code **newton.m**.

   Hint : Newton generates a sequence of improved estimates of the root. So, nothing is more natural than checking the consequtive elements of the sequence for digits agreement (rel. err. with special TOL).

10. What linear function $y = ax + b$ approximates $f(x) = \sin x$ best in the vicinity of $x = \pi/4$? How does this problem relate to Newton's method?

    Hint : Does tangent-line ring a bell?

11. Verify that when Newton's method is used to compute $\sqrt{R}$ (by solving the equation $x^2 = R$), the sequence of iterates is defined by

$$x_{n+1} = \tfrac{1}{2}\left(x_n + \frac{R}{x_n} \right)$$

    and show that

$$x_{n+1}^2 - R = \left(\frac{x_n^2 - R}{2x_n} \right)^2 .$$

    Interpret this equation in terms of quadratic convergence.

    Hint : Can you find $e_n = x_n - \sqrt{R}$ and $e_{n+1} = x_{n+1} - \sqrt{R}$ in the last expression? How are they related?

12. Use the secant method to determine the roots of the following equations to two correct decimals : (a) $2x = e^{-x}$; (b) $\tan x + \cosh x = 0$. Use hand computation and the MATLAB code **secant.m**.

    Hint : `ezplot` to the resque to get a bracket or suitable initial guesses. How to impose decimal control?

13. One wants to solve the equation $x + \ln x = 0$, whose root is $r \approx 0.5$, by iteration, an done chooses among the following iteration formulas: (i) $x_{n+1} = -\ln x_n$; (ii) $x_{n+1} = e^{-x_n}$; (iii) $x_{n+1} = \frac{1}{2}(x_n + e^{-x_n})$. (a) Which of the formulas can be used? (b) Which formula should be used? (c) Give an even better formula.

Hint : For (a), numerical experiments with $x_0 = 0.5$ or the condition $|\phi'(r \approx 0.5)| = K < 1$ may help. For (b), think of the relation between $0 \le K < 1$ and the speed of convergence. For (c), does Newton ring a bell?

14. The equation $x^2 - a = 0$ (for the square root $r = \sqrt{a}$) can be written equivalently in the form $x = \phi(x)$ in many different ways: (i) $\phi(x) = \frac{1}{2}(x + \frac{a}{x})$; (ii) $\phi(x) = a/x$; (iii) $\phi(x) = 2x - \frac{a}{x}$. Discuss the convergence (or nonconvergence) behavior of the iteration $x_{n+1} = \phi(x_n)$, $n = 0, 1, 2, \ldots$, for each of these three iteration functions. In case of convergence, which one is the fastest convergent?

Hint : Again, the condition $\left|\phi'(r = \sqrt{a})\right| = K < 1$ may help.

15. One wants to use the iteration formula $x_{n+1} = 2^{x_n - 1}$ to solve the equation $2x = 2^x$. Investigate if and to what the iteration sequence converges for various choices of x_0.

Hint : This calls for numerical experiments and graphical techniques (see lecture notes).

16. Determine parameters a, b, and c so that the order of the iterative method

$$x_{n+1} = a\,x_n + b\frac{s}{x_n^2} + c\frac{s^2}{x_n^5}$$

for $s^{1/3}$ becomes as high as possible. For this choice of a, b, and c, indicate how the error in x_{n-1} ($e_{n-1} = |x_{n-1} - r|$) depends on the error in x_n ($e_n = |x_n - r|$).

Hint : Well, three unknowns a, b and c require three equations for unique determination. One equation, of course, comes from the fixed point check $\phi(s^{1/3}) = s^{1/3}$. What about the other two? They may be something to do with the speed (order) of convergence.

17. **(i)** Show that each of the following functions has a fixed point at r precisely when $f(r) = 0$, where $f(x) = x^4 + 2x^2 - x - 3$: **(a)** $\phi_1(x) = (3 + x - 2x^2)^{1/4}$; **(b)** $\phi_2(x) = (x + 3 - x^4)^{1/2}/\sqrt{2}$; **(c)** $\phi_3(x) = (x+3)^{1/2}/(x^2+2)^{1/2}$; **(d)** $\phi_4(x) = (3x^4 + 2x^2 + 3)/(4x^3 + 4x - 1)$. **(ii)** Perform four iterations, by letting $x_0 = 1$ and $x_{n+1} = \phi(x_n)$, $n = 0, 1, 2, 3$ on each of the functions ϕ_i. **(iii)** Which function do you think gives the best approximation to the solution?

Hint : Explore the ways the equation $f(x) = 0$ can be put into the forms $x = \phi(x)$.

18. Use a fixed-point iteration method to determine a solution accurate to two decimals for $2\sin\pi x + x = 0$ on $[1, 2]$. Use $x_0 = 1$.

Hint : Construct a suitable iteration function $x_{n+1} = \phi(x_n)$ to get to the root in $[1, 2]$ using $x_0 = 1$. Apply the decimal controls to the resulting sequence.

19. Solve $x^3 - x - 1 = 0$ for the root in $[1, 2]$, using fixed-point iteration method. Obtain an approximation to the root accurate to two decimals.

Hint : Construct a suitable iteration function $x_{n+1} = \phi(x_n)$ to get to the root in $[1, 2]$. Apply the decimal controls to the resulting sequence.

20. For each the following equations, determine a function ϕ and an interval $[a, b]$ on which fixed-point iteration will converge to a positive solution of the equation: (a) $3x^2 - e^x = 0$; (b) $x - \cos x = 0$. Find the solutions to four significant digits.

Hint : This calls for graphical and numerical experimentation. Do not forget to apply the digit controls to the resulting sequence.

21. Consider the simple nonlinear equation $f(x) = x^2 - 3x + 1 = 0$. Knowing that this equation has two roots $x^0 = 1.5 \pm \sqrt{1.25} \approx 2.6180 \text{ or } 0.382$, investigate the practicability of the fixed-point iteration.

a) First consider the following iterative formula : $x_{k+1} = \phi_a(x_k) = \frac{1}{3}(x_k^2 + 1)$. Noting that the first derivative of this iteration function is $\phi_a'(x) = \frac{2}{3}x$, determine which solution attracts this iteration. In addition, run the MATLAB routine **fixedpoint.m** to perform the iteration with initial points $x_0 = 0$, $x_0 = 2$, and $x_0 = 3$. What does the routine yield for each initial point?

b) Now, consider the following iterative formula: $x_{k+1} = \phi_b(x_k) = 3 - \frac{1}{x_k}$. Noting that the first derivative of this iteration function is $\phi_a'(x) = \frac{1}{x^2}$ determine which solution attracts. In addition, run the MATLAB routine **fixedpoint.m** to perform the iteration with initial points $x_0 = 0.2$, $x_0 = 1$, and $x_0 = 3$. What does the routine yield for each initial point?

Note : This exercise illustrates that the outcome of an algorithm may depend on the starting point.

22. Consider the nonlinear equation $f(x) = \tan(\pi - x) - x = 0$. Obtain graphical solution of this equation by typing the following statements into the MATLAB command window:

```
ezplot('tan(pi-x)',-pi/2,3*pi/2)
hold on, ezplot('x',-pi/2,3*pi/2)
```

to demonstrate that the equation has infinitely many roots.

a) Use the bisection method for finding the solution between 1.5 and 3 to 4-decimal accuracy. Could you get the right solution? If not, explain why you failed and suggest how to make it.

b) Show that the sign of $f(x)$ changes between $x = 1.5$ and 2.0 and also between $x = 2.0$ and 2.5. Noting this, try using the bisection method to find a solution between 1.5 and 2.0. Check the validity of the solution, that is, check if $f(x) = 0$ or not. If the solution is not good, explain the reason.

c) In order to find the solution around $x = 2.0$ by using the fixed-point iteration with the initial point $x_0 = 2.0$, use the iteration function $x = \phi_1(x) = \tan(\pi - x)$. Could you get the solution near 2? Will it be better if you start the routine with any different initial point? What is wrong?

d) Now, try with another iteration function $x = \phi_2(x) = \pi - \tan^{-1} x$. What could you get? Is it the right solution? Does this iteration function work with different initial value, like 0 or 6, which are far from the solution we want to find?

23. Recall that the secant method was devised to remove the necessity of the derivative and improve the convergence. But, it sometimes turns out to be worse that the Newton method. Apply the routines **newton.m** and **secant.m** to solve $f(x) = x^3 - x^2 - x + 1 = 0$ starting with the initial point $x_0 = -0.2$ one time and $x_0 = -0.3$ for another time.

24. (a) For what starting values will Newton's method converge for the function $f(x) = x^2 / (1 + x^2)$? (b) What happens if the Newton iteration is applied to $f(x) = \arctan x$ with $x_0 = 2$? For what starting values will Newton's method converge? (c) Starting at $x = 3$, $x < 3$, or $x > 3$, analyze what happens when Newton's method is applied to the function $f(x) = 2x^3 - 9x^2 + 12x + 15$; (d) Repeat for $f(x) = |x|$, starting with $x < 0$ or $x > 0$.

<u>Hint</u> : This calls for numerical experiments and graphical techniques (see lecture notes).

25. Using a calculator, observe the speed with which Newton's method converges in the case of $f(x) = (x^2 - 1)^m$ with $m = 1$ and 8. Use $x_0 = 1.1$.

26. In order to accelerate Newton method for multiple roots, Newton method is modified as

$$\textbf{(a)}\ x_{k+1} = x_k - M\frac{f(x_k)}{f'(x_k)} \quad \text{and } \textbf{(b)}\ x_{k+1} = x_k - \frac{u(x_k)}{u'(x_k)} \quad \text{where } u(x) = \frac{f(x)}{f'(x)}$$

with M : the order of multiplicity of the root (if known) we want to find. Based on these two ideas, modify the routine **newton.m** to solve $f(x) = (x-5)^4 = 0$.

27. Use Newton's method to solve the equation $x^2 + 2xe^x + e^{2x} = 0$ starting with $x_0 = 0$ and iterate until five-decimal accuracy is obtained. Do the results seem unusual** for Newton's method? Use a suitably modified Newton's method to redo the calculations.

**<u>Note</u>: One may use the following estimator for the order (speed) of convergence p:

$$e_{n+1} \approx c e_n^p \quad \Rightarrow \quad \frac{e_{n+1}}{e_n^p} \approx \frac{e_{n+2}}{e_{n+1}^p} \approx c \quad \Rightarrow \quad \frac{e_{n+1}}{e_{n+2}} \approx \left(\frac{e_n}{e_{n+1}}\right)^p \quad \Rightarrow \quad p \approx \frac{\ln(e_{n+1}/e_{n+2})}{\ln(e_n/e_{n+1})}$$

where $e_n \approx |x_{n+1} - x_n|$.

Hint : The root may have (unknown) multiplicity >1.

28. Apply Newton method to find the zero $x = 2$ of the function $f(x) = \text{sign}(x-2)\sqrt{|x-2|}$ for which $f(x)/f'(x) = 2(x-2)$. Could you get the right solution? If not, explain why you failed. A plot of $f(x)$ may be obtained by typing the following statements into the MATLAB command window:

```
ezplot('sign(x-2)*sqrt(abs(x-2))',0,4)
```

Similarly, investigate the behavior of the secant method on this function.

29. Here is a cubic polynomial with three closely spaced real roots.

$$p(x) = 816x^3 - 3835x^2 + 6000x - 3125$$

(a) Plot $p(x)$ for $1.43 \le x \le 1.71$. Show the location of the three roots.

(b) Starting with $x_0 = 1.5$, what does Newton's method do?

(c) Starting with $x_0 = 1$ and $x_1 = 2$, what does the secant method do?

(d) Starting with the interval $[1, 2]$, what does bisection do?

Note : In programming this polynomial function, use nested multiplication form.

30. The formula for the secant method

$$x_{n+1} = x_n - f(x_n)\left(\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right)$$

can also be written as

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}.$$

Establish this, and explain why the latter form is inferior in finite precision environment.

Hint : In the first case, x_{n+1} is improved upon x_n as much as the right-most expression permits. In the second case, x_{n+1} is computed using a convoluted expression that may cause cancellation errors.

31. Consider Newton method with Armijo rule where the Newton iteration for finding the root r, $f(r) = 0$, is modified to read

$$x_{n+1} = x_n + \lambda \frac{f(x_n)}{f'(x_n)} \quad \text{with} \quad \lambda = 2^{-m}, \quad m \ge 0 \text{ integer.}$$

It is useful in the globalization of the Newton method in those cases where the new guess $x_0 + h$ leads to an increase in the magnitude of the function $|f(x_0 + h)| > |f(x_0)|$ and thus one should try a smaller step size $x_0 + \frac{1}{2}h$. If the magnitude of f still increases, one should apply the reduction, $x_0 + \frac{1}{4}h$, $x_0 + \frac{1}{8}h, \dots$, repeatedly, until it leads to a local decrease in $|f(x)|$. Modify **Newton.m** to implement the Armijo rule along the suggested lines (or write your own code):

```
F0 = F(P0); FP0 = FP(P0); D = -F0/FP0;
                  . . .
PN = P0 + D; FN = F(PN);
      if abs(FN) < abs(F0),
      P0 = PN; . . .
      else
      D = D/2;
end
```

Test the code on the problem: Solve $f(x) = \arctan(x)$ for $r = 0$ by starting the Newton iteration at $x_0 = 10$.