PROBLEMS

1. Let $\mathbf{A}=\begin{bmatrix}2 & 1 & 3\\ -1 & 2 & 0\\ 3 & -2 & 1\end{bmatrix}$. (a) Find $\text{adj}(\mathbf{A})$, (b) Compute $\det(\mathbf{A})$, (c) Find the inverse of **A** (d) Show that $\mathbf{A}(\text{adj}(\mathbf{A}))=(\text{adj}(\mathbf{A}))\mathbf{A}=\det(\mathbf{A})\mathbf{I}_3$, (e) Show that $\det(\text{adj}(\mathbf{A}))=(\det(\mathbf{A}))^2$.
2. Using only elementary row or elementary column operations (do not expand determinants), verify the following:

$$\text{(a)}\ \begin{vmatrix}a-b & 1 & a\\ b-c & 1 & b\\ c-a & 1 & c\end{vmatrix}=\begin{vmatrix}a & 1 & b\\ b & 1 & c\\ c & 1 & a\end{vmatrix},\qquad \text{(b)}\ \begin{vmatrix}1 & a & bc\\ 1 & b & ca\\ 1 & c & ab\end{vmatrix}=\begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix}$$

3. For each of the following matrices find, if possible, a nonsingular matrix **P** such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is diagonal, i.e. **A** is diagonalizable:

$$\text{(a)}\ \begin{bmatrix}4 & 2 & 3\\ 2 & 1 & 2\\ -1 & -2 & 0\end{bmatrix},\qquad \text{(b)}\ \begin{bmatrix}1 & 1 & 2\\ 0 & 1 & 0\\ 0 & 1 & 3\end{bmatrix},\qquad \text{(c)}\ \begin{bmatrix}1 & 2 & 3\\ 0 & 1 & 0\\ 2 & 1 & 2\end{bmatrix}$$

4. Let λ be an eigenvalue of the $n\times n$ matrix **A**. Show that the subset S of $\mathbf{R}^n$ consisting of the zero vector and of all eigenvectors of **A** associated with λ is a subspace of $\mathbf{R}^n$ where

$$S=\{\ \mathbf{E}\in\mathbf{R}^n \mid \mathbf{A}\mathbf{E}=\lambda\mathbf{E}\ \}\cup\{0\}.$$

This subspace is called the eigenspace associated with λ.
5. Show that if **A** is upper (lower) triangular matrix, then the eigenvalues of **A** are the elements on the main diagonal of **A**. <u>**Hint**</u>: Determinant of a triangular matrix is the product of its diagonal elements.
6. Let $\mathbf{A}=\begin{pmatrix}2 & 2 & 3 & 4\\ 0 & 2 & 3 & 2\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1\end{pmatrix}$.

(a) Find a basis for the eigenspace associated with the eigenvalue $\lambda=1$.

(b) Find a basis for the eigenspace associated with the eigenvalue $\lambda=2$.
7. Let $\mathbf{D}=\begin{bmatrix}2 & 0\\ 0 & -2\end{bmatrix}$. Compute $\mathbf{D}^9$.
8. Let $\mathbf{A}=\begin{bmatrix}3 & -5\\ 1 & -3\end{bmatrix}$. Compute $\mathbf{A}^9$. <u>**Hint**</u>: Find a matrix **P** that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is a diagonal matrix **D** and show $\mathbf{A}^9=\mathbf{P}\mathbf{D}^9\mathbf{P}^{-1}$.
9. The Cayley-Hamilton theorem states that a matrix satisfies its characteristic polynomial, i.e. **A** is an $n\times n$ matrix with characteristic polynomial $p_n(\lambda)=\det(\mathbf{A}-\lambda\mathbf{I}_n)$ such that $p_n(\lambda)=\lambda^n+a_1\lambda^{n-1}+\ldots+a_{n-1}\lambda+a_n$, then $\mathbf{A}^n+a_1\mathbf{A}^{n-1}+\ldots+a_{n-1}\mathbf{A}+a_n\mathbf{I}_n=\mathbf{0}$. Verify the Cayley Hamilton theorem for the following matrices:

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & -3 \end{bmatrix}$, (b) $\begin{bmatrix} 3 & 3 \\ 2 & 4 \end{bmatrix}$.

10. $\mathbf{A}$ is an $n \times n$ matrix whose charateristic polynomial is $p_n(\lambda) = \lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n$. If A is nonsingular, show that $\mathbf{A}^{-1} = \frac{1}{a_n}(\mathbf{A}^{n-1} + a_1\mathbf{A}^{n-2} + \ldots + a_{n-2}\mathbf{A} + a_{n-1}\mathbf{I}_n)$. How is the relation $a_n = \lambda_1\lambda_2 \ldots \lambda_n = \det(\mathbf{A})$ connected to the existence of $\mathbf{A}^{-1}$?

11. Verify that $\mathbf{Q} = \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}$ is an orthogonal matrix and find its inverse.

12. Find a third column so that the matrix $\mathbf{Q} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} & ? \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} & ? \\ \frac{1}{\sqrt{3}} & \frac{-3}{\sqrt{14}} & ? \end{bmatrix}$ is orthogonal.

<u>**Hint**</u>: Construct orthonormal column vectors.

13. Diagonalize each given matrix and find an orthogonal matrix $\mathbf{Q}$ that $\mathbf{Q}^T\mathbf{A}\mathbf{Q}$ is diagonal.

(a) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, (b) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix}$, (c) $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

14. Apply the Gram-Schmidt orthogonalization process to $\mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ to construct an orthonormal set of vectors $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$. Can you write the result in the form of $\mathbf{A} = \mathbf{QR}$ where $\mathbf{A} = [\mathbf{u} \quad \mathbf{v} \quad \mathbf{w}]$, $\mathbf{Q} = [\mathbf{q}_1 \quad \mathbf{q}_2 \quad \mathbf{q}_3]$ and $\mathbf{R}$ is an upper-triangular matrix?

15. Write each of the following quadratic forms in their canonical forms:

(a) $g(x, y) = -3x^2 + 5xy - 2y^2$,

(b) $g(x, y, z) = 2x^2 + 3xy - 5xz + 7yz$,

(c) $g(x, y, z) = 3x^2 + xy - 2xz + y^2 - 4yz - 2z^2$.

16. Find the general solution of the differential equations as applications of eigenproblem:

(a) $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -4 & 3 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, (b) $x''' - 3x'' - 10x' + 24x = 0$.