PROBLEMS

1. Determine whether the three vectors in $\mathbf{R}^3$ are linearly dependent or independent: $3i+6j-k$, $8i+2j-4k$, $i-j+k$.

2. Let S be the set of all vectors parallel to the hyper-plane $4x+2y-z+3r=0$ in $\mathbf{R}^4$.

   (a) Show that S is a subspace, (b) Determine a basis for S, (c) Find its dimension.

   Hint: $S=\{\mathbf{u}=(x,y,z,r)\in\mathbf{R}^4 \mid 4x+2y-z+3r=0\}$.

3. Let $\mathbf{A}=\begin{bmatrix}1 & 2\\ 3 & 2\end{bmatrix}$ and $\mathbf{B}=\begin{bmatrix}2 & -1\\ -3 & 4\end{bmatrix}$. Show that $\mathbf{AB}\neq\mathbf{BA}$.

4. Show that the jth column of the matrix product $\mathbf{AB}$ is equal to the matrix product $\mathbf{AB}_j$, where $\mathbf{B}_j$ is the jth column of $\mathbf{B}$.

5. Show that if $\mathbf{AX}=\mathbf{B}$ has more than one solution, then it has infinitely many solutions. (Hint: If $\mathbf{X}_1$ and $\mathbf{X}_2$ are solutions, consider $\mathbf{X}_3=\alpha\mathbf{X}_1+\beta\mathbf{X}_2$ where $\alpha+\beta=1$.)

6. Let $\mathbf{A}$ be a fixed $m\times n$ matrix and define W to be the solution space of the homogeneous system $\mathbf{AX}=\mathbf{0}$, i.e. $W=\left\{\mathbf{X}\in\mathbf{R}^n \;\middle|\; \mathbf{AX}=\mathbf{0}\right\}$. Show that W is a subspace of $\mathbf{R}^n$ and write down its dimension. **Hint**: See dimension of the Solution Space.

7. Given the matrix $\mathbf{A}=\begin{pmatrix}1 & 2 & 2 & 3\\ 2 & 4 & 1 & 3\\ 3 & 6 & 1 & 4\end{pmatrix}$, determine: (a) a basis for its column space, (b) a basis for its row space of $\mathbf{A}$, (c) the rank of $\mathbf{A}$. **Hint**: Reduce (a) $\mathbf{A}^T$ and (b) $\mathbf{A}$.

8. In the following linear system, determine all values of a for which the resulting linear system has: (a) No solution, (b) Infinitely many solutions, (c) A unique solution.

$$\begin{aligned} x_1 + x_2 - x_3 &= 2\\ x_1 + 2x_2 + x_3 &= 3 \quad (\textbf{Ans}:\ -2;\ 2;\ \neq\pm 2)\\ x_1 + x_2 + (a^2-5)x_3 &= a \end{aligned}$$

9. Find all values of a for which the following linear systems have solutions.

   (i) $\begin{aligned} x+2y+z&=a^2\\ x+y+3z&=a\\ 3x+4y+7z&=8\end{aligned}$ (**Ans**: -4 or 2)   (ii) $\begin{aligned} x+2y+z&=a^2\\ x+y+3z&=a\\ 3x+4y+8z&=8\end{aligned}$ (**Ans**: any real number)

10. Find all values of a for which the following homogeneous system has nontrivial solutions:

   (i) $\begin{aligned} (1-a)x+z&=0\\ ay+z&=0\\ y-az&=0\end{aligned}$ (**Ans**: 1)

11. Consider the two matrices $\mathbf{A}=\begin{bmatrix}1 & 2 & 3\\ 0 & 1 & 2\\ 1 & 0 & 3\end{bmatrix}$ and $\mathbf{B}=\begin{bmatrix}10\\ 4\\ 6\end{bmatrix}$.

   (a) Find the reduced form $\mathbf{A}_r$ and produce a matrix $\mathbf{\Omega}$ such that $\mathbf{\Omega A}=\mathbf{A}_r$.

(b) Use $\mathbf{\Omega}$ computed above to solve $\mathbf{AX}=\mathbf{B}$ for $\mathbf{X}$. **Hint**: Use the fact that $\mathbf{A}_r=\mathbf{I}_3$.

12. Let $\mathbf{u}_1=\begin{bmatrix}2\\1\\2\end{bmatrix}$, $\mathbf{u}_2=\begin{bmatrix}1\\0\\1\end{bmatrix}$, $\mathbf{u}_3=\begin{bmatrix}2\\1\\0\end{bmatrix}$, $\mathbf{u}_4=\begin{bmatrix}1\\0\\-1\end{bmatrix}$ be given in $\mathbf{R}^3$. Determine whether if the vector $\mathbf{u}=\begin{bmatrix}0\\-2\\4\end{bmatrix}$ belongs to the subspace spanned by $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3,\mathbf{u}_4\}$, i.e. determine whether if scalars c_1,c_2,c_3,c_4 exist such that $\mathbf{u}=c_1\mathbf{u}_1+c_2\mathbf{u}_2+c_3\mathbf{u}_3+c_4\mathbf{u}_4$.

Hint: Construct a system $\mathbf{AX}=\mathbf{B}$ and show solution exists.

13. Let $\mathbf{u}_1=\begin{bmatrix}1\\2\\1\end{bmatrix}$, $\mathbf{u}_2=\begin{bmatrix}1\\0\\2\end{bmatrix}$, $\mathbf{u}_3=\begin{bmatrix}1\\1\\0\end{bmatrix}$. Find out whether $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$ spans $\mathbf{R}^3$, i.e. any $\mathbf{u}\in\mathbf{R}^3$ can be written as $\mathbf{u}=c_1\mathbf{u}_1+c_2\mathbf{u}_2+c_3\mathbf{u}_3$ for some scalars c_1,c_2,c_3.

Hint : **(i)** Solve a non-homogeneous system for any pick $\mathbf{u}=[a\ \ b\ \ c]^T\in\mathbf{R}^3$, or **(ii)** show whether $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$ is a linearly independent set, so conclude whether it is a basis for $\mathbf{R}^3$ or not, i.e. find $\dim\{\text{span}\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}\}=3=\dim\{\mathbf{R}^3\}$ or not.

14. Let $\mathbf{u}_1=\begin{bmatrix}1\\1\\0\\0\end{bmatrix}$, $\mathbf{u}_2=\begin{bmatrix}1\\0\\1\\0\end{bmatrix}$, $\mathbf{u}_3=\begin{bmatrix}0\\1\\1\\0\end{bmatrix}$, $\mathbf{u}_4=\begin{bmatrix}2\\1\\1\\0\end{bmatrix}$ be given in $\mathbf{R}^4$. Find a basis for the subspace S spanned by $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3,\mathbf{u}_4\}$, i.e. $S=\{\mathbf{u}\in\mathbf{R}^4 \mid \mathbf{u}=c_1\mathbf{u}_1+c_2\mathbf{u}_2+c_3\mathbf{u}_3+c_4\mathbf{u}_4\}$.

Hint: Form a matrix **A** and reduce.

15. For what values of a is the vector $\begin{bmatrix}a^2\\-3a\\-2\end{bmatrix}$ in span$\left\{\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix},\begin{bmatrix}1\\3\\4\end{bmatrix}\right\}$? (**Ans**: 1 or 2)

16. Suppose **A** is an $m\times n$ matrix and define W to be the subset of all $m\times 1$ matrices **B** in $\mathbf{R}^m$ for which the linear system $\mathbf{AX}=\mathbf{B}$ has a solution: (a) Is W a subspace of $\mathbf{R}^m$? (b) What is the relationship between W and the column space of **A**?

Hint: Recall, $\dim\{\text{col. space}\{\mathbf{A}\}\}=\dim\{\text{col. space}\{\mathbf{A}|\mathbf{B}\}\}$ for existence of solution to $\mathbf{AX}=\mathbf{B}$.

17. Let **A** be a fixed $m\times n$ matrix of rank r. Under what conditions on those numbers m, n, and r, does: (a) **A** has an inverse? (b) $\mathbf{AX}=\mathbf{B}$ has infinitely many solutions for every **B** in $\mathbf{R}^m$? (c) $\mathbf{AX}=\mathbf{0}$ has $\mathbf{X}=\mathbf{0}$ (the trivial solution) as the only solution?

18. For a nonsingular square matrix **A**, show that $[\mathbf{A}|\mathbf{A}^2]_r=[\mathbf{I}|\mathbf{A}]$. **Hint**: $[\mathbf{A}|\mathbf{I}]_r=[\mathbf{I}|\mathbf{A}^{-1}]$.