PROBLEMS

1. **(a)** Give a parametrization of the surface of the ellipsoid $\frac{1}{a^2}x^2 + \frac{1}{b^2}y^2 + \frac{1}{c^2}z^2 = 1$ analogous to spherical coordinates. **Hint**: Spherical representation satisfies $\frac{1}{\rho^2}x(\theta,\phi)^2 + \frac{1}{\rho^2}y(\theta,\phi)^2 + \frac{1}{\rho^2}z(\theta,\phi)^2 = 1$.

   (b) Set up an integral to compute the surface area of the ellipsoid.

2. Let $f(x)$ be a positive C^1 function of $x \in [a,b]$.

   (a) Find a parametrization of the surface in R^3 obtained by rotating the graph of f around the x-axis. **Ans**: $\mathbf{\Phi}(x,\theta) = (x, \quad f(x)\cos\theta, \quad f(x)\sin\theta)$, $a \le x \le b$, $0 \le \theta < 2\pi$.

   (b) What is the area of this surface? **Ans**: $2\pi\int_a^b f\sqrt{1+(f')^2}dx$.

3. For which numbers a and b is $\mathbf{F} = axy\mathbf{i} + (x^2 + by)\mathbf{j}$ is a gradient field? **Hint**: $\nabla \times \nabla\phi = \mathbf{0}$.

4. A wire of constant density ρ lies on the semicircle $x^2 + y^2 = a^2$, $y \ge 0$. Find its mass M and the center of mass $(\bar{x}, \bar{y})$. **Ans**: $M = \rho a\pi$ and $(\bar{x} = 0, \bar{y} = 2a/\pi)$.

5. Find $\int \mathbf{F} \cdot d\mathbf{R}$ along the space curve $x = t$, $y = t^2$, $z = t^3$, $0 \le t \le 1$: (a) $\mathbf{F} = \text{grad}(xy + xz)$ (b) $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + z\mathbf{k}$. **Ans**: (a) 2, (b) 1/6.

6. (a) Find the unit tangent vector $\mathbf{T}$ and the speed ds/dt along the path $\mathbf{R} = 2t\mathbf{i} + t^2\mathbf{j}$. **Hint**: $\mathbf{T} = d\mathbf{R}/dt / \|d\mathbf{R}/dt\|$.

   (b) For $\mathbf{F} = 3x\mathbf{i} + 4\mathbf{j}$, find $\mathbf{F} \cdot \mathbf{T}ds$ using (a) and $\mathbf{F} \cdot d\mathbf{R}$ directly. **Hint**: $ds = \|d\mathbf{R}/dt\| dt$.

   (c) What is the work from $(2,1)$ to $(4,4)$? **Ans**: 30

7. Find a field $\mathbf{F}$ whose work around the unit square ($y = 0$, $x = 1$, $y = 1$, $x = 0$) equals 4. **Hint**: Use Green's Thm. $\text{work} = \oint_C F \cdot d\mathbf{R} = \iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dD$.

8. Compute $\int \mathbf{F} \cdot d\mathbf{R}$ along the straight line $\mathbf{R} = t\mathbf{i} + t\mathbf{j}$ and the parabola $\mathbf{R} = t\mathbf{i} + t^2\mathbf{j}$ from $(0,0)$ to $(1,1)$. If $\mathbf{F}$ is a gradient field ($\mathbf{F} = \nabla\phi$), use its potential ϕ:

   (a) $\mathbf{F} = \mathbf{i} - 2\mathbf{j}$, **Ans**: -1, (b) $\mathbf{F} = 2xy^2\mathbf{i} + 2yx^2\mathbf{j}$, **Ans**: 1, (c) $\mathbf{F} = x^2y\mathbf{i} + xy^2\mathbf{j}$. **Ans**: $1/2$ & $17/35$

9. Around the unit circle, find $\oint ds$, $\oint dx$ and $\oint x ds$. **Ans**: 2π; 0; 0

10. Compute both sides of Green's Theorem:

    (a) $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$, D : upper half of the disk $x^2 + y^2 \le 1$. **Ans**: 0

    (b) $\mathbf{F} = x^2\mathbf{i} + xy\mathbf{j}$, C : square with sides $y = 0$, $x = 1$, $y = 1$, $x = 0$. **Ans**: 1/2

11. Show that $\oint (x^2y + 2x)dy + xy^2dx$ depends only on the area of the region D bounded by C.

12. Find the area inside the hypocycloid $x = \cos^3 t$, $y = \sin^3 t$. **Ans**: $3\pi/8$

13. Which differentials are exact, i.e. $= d\phi$ for some scalar field $\phi = \phi(x,y)$: (a) $ydx - xdy$, (b) $x^2dx + y^2dy$, (c) $y^2dx + x^2dy$?

14. Find the surface area $\iint dS$:

    (a) Paraboloid $z = x^2 + y^2$ below the plane $z = 4$. **Ans**: $\frac{\pi}{6}(17^{3/2} - 1)$)

    (b) Plane $z = x - y$ inside the cylinder $x^2 + y^2 = 1$. **Ans**: $\sqrt{3}\pi$

(c) Plane $2z = 3x + 4y$ above the square $0 \le x \le 1,\ 0 \le y \le 1$. **Ans**: $\sqrt{29}/2$

(d) Spherical band $x^2 + y^2 + z^2 = 1$ between $z = 0$ and $z = 1/\sqrt{2}$. **Ans**: $\sqrt{2}\pi$

15. Compute the surface integrals $\iint g(x, y, z)dS$:

(a) $g = xy$ over the triangle $x + y + z = 1,\ x, y, z \ge 0$. **Ans**: $\sqrt{3}/24$

(b) $g = x^2 + y^2$ over the top half of $x^2 + y^2 + z^2 = 1$. **Ans**: $2/3$

(c) $g = xyz$ on $x^2 + y^2 + z^2 = 1$ above $z^2 = x^2 + y^2$. **Ans**: 0

(d) $g = x$ on the cylinder $x^2 + y^2 = 4$ between $z = 0$ and $z = 3$. **Ans**: 0

16. Find the flux $\iint \mathbf{F} \cdot \mathbf{n} dS$ for $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ over the surfaces in (15). **Ans**: $1/2$, 2π, $\pi(2-\sqrt{2})$, 24π.

17. Compute the flux $\iint \mathbf{F} \cdot \mathbf{n} dS$ by the Divergence Theorem:

(a) $\mathbf{F} = x\mathbf{i} + x\mathbf{j} + x\mathbf{k}$, S : unit sphere $x^2 + y^2 + z^2 = 1$. **Ans**: $4\pi/3$

(b) $\mathbf{F} = x\mathbf{i} + 2y\mathbf{j}$, S : sides $x = 0,\ y = 0,\ z = 0,\ x + y + z = 1$. **Ans**: $1/2$

(c) $\mathbf{F} = \rho(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$, S : sphere $\rho = a$. **Ans**: $4\pi a^4$

(d) $\mathbf{F} = x^2\mathbf{i} + 8y^2\mathbf{j} + z^2\mathbf{k}$, V : unit cube. **Ans**: 10

18. Find $\iiint \operatorname{div}(x^2\mathbf{i} + y\mathbf{j} + 2\mathbf{k})dV$ in the cube $0 \le x, y, z \le a$. Also compute $\iint \mathbf{F} \cdot \mathbf{n} dS$ for all six faces and add. **Ans**: $a^4 + a^3$.

19. Integrate $\iint \mathbf{r} \cdot \mathbf{n} dS$ over the faces of the box $0 \le x \le 1$, $0 \le y \le 2$, $0 \le z \le 3$ and check by the Divergence Theorem. ($\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$). **Ans**: 18.

20. Find $\oint \mathbf{F} \cdot d\mathbf{R}$ by Stokes' Theorem: ($\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$)

(a) $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{k}$, C : circle $x^2 + z^2 = 1,\ y = 0$. **Ans**: 0

(b) $\mathbf{F} = \mathbf{i} \times \mathbf{r}$, C : circle $x^2 + z^2 = 1,\ y = 0$. **Ans**: 0

(c) $\mathbf{F} = (\mathbf{i} + \mathbf{j}) \times \mathbf{r}$, C : circle $y^2 + z^2 = 1,\ x = 0$. **Ans**: 2π

(d) $\mathbf{F} = (y\mathbf{i} - x\mathbf{j}) \times (x\mathbf{i} + y\mathbf{j})$, C : circle $x^2 + y^2 = 1,\ z = 0$. **Ans**: $\pi/2$

21. Find a potential ϕ if it exists, i.e. $\mathbf{F} = \operatorname{grad}(\phi)$: (a) $\mathbf{F} = z\mathbf{i} + \mathbf{j} + x\mathbf{k}$, (b) $\mathbf{F} = e^{x-z}\mathbf{i} - e^{x-z}\mathbf{k}$

(c) $\mathbf{F} = 2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}$, (d) $\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + (xy + z^2)\mathbf{k}$. **Hint**: $\nabla \times \nabla\phi = \mathbf{0}$.

22. Write down Green's Theorem in the xz plane from Stokes' Theorem. **Hint**: Set $\mathbf{n} = \mathbf{j}$.

23. Compute $\iint \operatorname{curl}(\mathbf{F}) \cdot \mathbf{n} dS$ over the top half of the sphere $x^2 + y^2 + z^2 = 1$ and $\oint \mathbf{F} \cdot d\mathbf{R}$ around the equator: (a) $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$, (b) $\mathbf{F} = \mathbf{r}$, (c) $\mathbf{F} = \mathbf{a} \times \mathbf{r}$, (d) $\mathbf{F} = (\mathbf{a} \times \mathbf{r}) \times \mathbf{r}$, where $\mathbf{a}$ is a constant vector and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. **Ans**: -2π, 0, $2\pi a_3$, 0.

24. The circle C in the plane $x + y + z = 6$ has radius r and center $(1, 2, 3)$. The field is $\mathbf{F} = 3z\mathbf{j} + 2y\mathbf{k}$. Compute $\oint \mathbf{F} \cdot d\mathbf{R}$ around C. **Ans**: $-\frac{1}{\sqrt{3}}\pi r^2$.

25. S is the top half of the unit sphere and $\mathbf{F} = z\mathbf{i} + x\mathbf{j} + xyz\mathbf{k}$. Find $\iint \operatorname{curl}(\mathbf{F}) \cdot \mathbf{n} dS$. **Ans**: π.