PROBLEMS

1. Let $\mathbf{F}$ be any non-zero vector. Determine a scalar t such that $\|t\mathbf{F}\| = 1$.

2. Let $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$ be nonzero vectors, each orthogonal to the other two. Let $\mathbf{A}$ be any vector. Find the scalars α, β, γ such that $\mathbf{A} = \alpha\mathbf{F} + \beta\mathbf{G} + \gamma\mathbf{H}$. **Hint**: Consider $\mathbf{A}\cdot\mathbf{F}$, $\mathbf{A}\cdot\mathbf{G}$, $\mathbf{A}\cdot\mathbf{H}$.

3. Determine whether the three points $P(-1, 1, 6)$, $Q(2, 0, 1)$, $R(3, 0, 0)$ are collinear, i.e. all three lie on a straight line. **Ans.**: No, **Hint**: $\mathbf{u}\times\mathbf{v} = \mathbf{0} \quad \Leftrightarrow \quad \mathbf{u} \,||\, \mathbf{v}$.

4. Show that for $r \neq 0$ where $\mathbf{r} = (x, y, z)$ and $r = \|\mathbf{r}\|$:

(a) $\nabla(\frac{1}{r^n}) = -\frac{n}{r^{n+2}}\mathbf{r}$; **(b)** $\nabla\cdot(\frac{1}{r^n}\mathbf{r}) = \frac{(3-n)}{r^n}$; **(c)** $\nabla\times(\frac{1}{r^n}\mathbf{r}) = \mathbf{0}$.

Hint : Some useful formulas of vector analysis

(i) $\nabla(fg) = f\nabla(g) + g\nabla(f)$, (ii) $\nabla(f/g) = \left(g\nabla(f) - f\nabla(g)\right)/g^2$ at points where $g(x) \neq 0$

(iii) $\nabla\cdot(f\,\mathbf{F}) = f\,\nabla\cdot(\mathbf{F}) + \mathbf{F}\cdot\nabla(f)$, (iv) $\nabla\times(f\,\mathbf{F}) = f\,\nabla\times(\mathbf{F}) + \nabla(f)\times\mathbf{F}$

5. Let the curve C have the parametric equations: $x = \sin t$, $y = \cos t$, $z = 45t$ for $0 \le t \le 2\pi$.

(a) Write the position vector $\mathbf{R}(t)$ and tangent vector for C,

(b) Find a length function $s(t)$ for this curve, i.e. $s(t) = \int_0^t ds$

(c) Write the position vector as a function of the arclength s, $\mathbf{R}(s)$

(d) Verify that the resulting position vector $\mathbf{R}(s)$ has a derivative of length 1.

6. Construct a vector field whose streamlines are straight lines.

7. Suppose $\nabla\phi = \mathbf{i} + \mathbf{k}$. What can be said about level surfaces of the scalar field ϕ? Show that the streamlines of the vector field $\nabla\phi$ are orthogonal to the level surfaces of ϕ. **Hint**: $\mathbf{N} = \nabla\varphi$ is constant.

8. Suppose a particle following the path $\mathbf{R}(t) = (t^2, t^3 - 4t, 0)$ flies off on a tangent at $t = 2$. Compute the position of the particle at $t = 3$. **Ans.**: (8,8,0)

9. What is the distance from the point Q(9,4,5) to the line x = 1 + t, y = 1 + 2t, z = 3 + 2t?

(a) Use vector operations, i.e. consider the distance as the component of the vector $\mathbf{u} = PQ$ perpendicular to the line where P is any point on the line. **Ans.**: $\sqrt{41}$

(b) Use calculus, i,e, minimize $(x-9)^2 + (y-4)^2 + (z-5)^2$.

(c) Which point A(x,y,z) on the line is closest to the point Q? **Ans.**: A(3,5,7)

10. **(a)** Find an equation for the line $\mathbf{R}(t)$ through P(0,2, 1) and Q(1,3,3).

(b) What is an equation for the line segment between P and Q (not beyond)?

(c) What is an equation for the line in terms of x, y, z without the parameter t?

(d) Which point on the line is closest to the origin? **Ans.**: A(-2/3,4/3,-1/3)

(e) Where does the line meet the plane $x + y + z = 11$? **Ans.**: (2,4,5)

(f) What line goes through A(3, 1, 1) perpendicular to the plane $x - y - z = 1$?

11. Find parametric equations for the line starting from P(1,2,4) and passing through Q(5,5,4). Change the equations so the speed is 10. Change the start to Q. **Hint**: Use a new parameter τ that speed $= \|d\mathbf{R}/d\tau\| = 10$.

12. (a) Change t so that the speed along the helix $\mathbf{R}(t) = (\cos t, \sin t, t)$ is 1 instead of $\sqrt{2}$. Call the new parameter s. **Hint**: Recall that for s arclength parameter, speed $= \|d\mathbf{R}/ds\| = 1$.

(b) Find parametric equations to go around the unit circle with speed e^t starting from $x = 1$, $y = 0$. When is the circle completed? **Hint**: Use speed $= \|d\mathbf{R}/dt\| = e^t$.

13. The surface of a lake is represented by a region D in the xy-plane such that the depth under the point (x, y) is $f(x,y)=300-2x^2-3y^2$. In what direction should a swimmer at $P(4,9)$ swim in order for the depth of the water to decrease most rapidly? **Ans.**: $\mathbf{u}=(8,27,0)/\|(8,27,0)\|$.

14. Consider the temperature field $\phi(x,y,z)=\sqrt{x^2+y^2+z^2}$ and a point $P(1,\sqrt{2},1)$:

(a) Determine at this point the maximum and minimum rate of change of temperature.

(b) Determine the equation of the level surface ($\phi(x,y,z)=c$) that passes through P and specified by ϕ.

(c) Find the equation of the tangent plane and normal line to the surface at the point P.

(d) Find the angle of intersection between the surface in (b) and the surface $z^2+x^2=2$ at P. **Ans.**: 45°

15. Find the equation of the plane, $ax+by+cz=d$, that

(a) is perpendicular to the vector $\mathbf{u}=(1,3,3)$ and passes through the point $P(6,8,9)$.

(b) passes through the points $P(1,3,8)$, $Q(3,6,9)$ and $R(1,6,0)$.

(c) is tangent to the surface $x^2+y^3-z=c$ at the point $P(9,8,6)$.

(d) contains the two vectors $\mathbf{u}=(1,1,-2)$, $\mathbf{v}=(-4,3,1)$ and the point $P(1,1,1)$.

16. Find <u>an equation</u> of the plane that contains the two vectors $\mathbf{u}=(1,1,-2)$ and $\mathbf{v}=(-4,3,1)$. Construct a vector **w** that is perpendicular to **u** and lives in the same plane with **u** and **v**. **Ans.**: $\mathbf{w}=(-1,1,0)$

17. Find the equation of the line, $\mathbf{R}(t)=(x_0+tu_1, y_0+tu_2, z_0+tu_3)$, that

(a) is passing through the point $P(5,0,4)$ and in the direction of the vector $\mathbf{u}=(1,5,1)$.

(b) is passing through the points $P(1,0,4)$ and $Q(1,0,5)$.

(c) is perpendicular to the surface $x^2-2y^3+z^3=c$ at the point $P(0,1,4)$.

18. Given the scalar field $f(x,y,z)=e^x+yz$, determine

(a) the directional derivative of f at the point $P(6,2,1)$ in the direction of the vector $\mathbf{u}=(2,1,6)$.

(b) the direction along which f is increasing the fastest at the point $P(1,1,3)$.

(c) the level surface $f(x,y,z)=c$ that contains the point $P(2,1,6)$.

(d) the direction perpendicular to the level surface in (c) at the point P.

19. Given the vector field $\mathbf{F}=x^2\mathbf{i}+7y^2\mathbf{j}-3z^2\mathbf{k}$ and the scalar field $f(x,y,z)=6x-2y+2z^2$,

(a) verify that $\nabla\cdot(\nabla\times\mathbf{F})=0$, i.e. the divergence of a curl field vanishes.

(b) verify that $\nabla\times(\nabla f)=\mathbf{0}$, i.e. curl of a gradient field vanishes.

(c) find the particular streamline to the vector field **F** at the point $P(2,1,7)$. **Hint**: Solve $dx/x^2=dy/7y^2=-dz/3z^2$, pairwise.

20. Parametric equations for a curve is given by $x=6t^2$, $y=9t^2$, $z=t^2$.

(a) Write the position vector and tangent vector for the curve.

(b) Find the length of the curve for $1\le t\le 9$. **Ans.**: $L=80\sqrt{118}$.

21. Let $f(x,y)=e^{xy}\sin(x+y)$.

(a) In what direction, starting at $P(0,\pi/2)$, is f changing the fastest? **Ans.**: $\mathbf{u}=(1,0)$

(b) In what directions starting at $P(0,\pi/2)$ is f changing at 50% of its maximum rate? **Ans.**: $\mathbf{u}=(\frac{1}{2},\pm\frac{\sqrt{3}}{2})$

(c) Let $\mathbf{c}(t)$ be a streamline of $\mathbf{F}=\text{grad}(f)$ with $\mathbf{c}(0)=(0,\pi/2)$. Calculate $\left.\frac{d}{dt}f(\mathbf{c}(t))\right|_{t=0}$. **Ans.**: $\pi^2/4$

22. Let $\mathbf{c}(t)=(x(t),y(t),z(t))$ be a path with $\|c(t)\|=\text{constant}$; i.e., the curve lies on a sphere. Show that $\mathbf{c}'(t)$ is orthogonal to $\mathbf{c}(t)$.

23. Recall that a curve $\mathbf{R}(s)$ parametrized in terms of arclength s satisfies $\|d\mathbf{R}/ds\| \equiv \|\mathbf{R}'(s)\| = 1$.

(a) Show that the unit vectors $\mathbf{v}_1(s) = \mathbf{R}'$, $\mathbf{v}_2(s) = \mathbf{R}''/\|\mathbf{R}''\|$ and $\mathbf{v}_3(s) = \mathbf{v}_1 \times \mathbf{v}_2$ form an orthonormal set of vectors, namely,

$$\mathbf{v}_n \cdot \mathbf{v}_m = \delta_{nm} \equiv \begin{cases} 1 & \text{if} \quad n = m \\ 0 & \text{if} \quad n \neq m \end{cases}$$

for $n, m = 1, 2, 3$. They are called Frenet trihedron (Frenet, 1847) of moving orthogonal vectors.

(b) Show that they satisfy the Formulas of Frenet:

$$\mathbf{v}_1' = \kappa \mathbf{v}_2$$
$$\mathbf{v}_2' = -\kappa \mathbf{v}_1 + \tau \mathbf{v}_3$$
$$\mathbf{v}_3' = -\tau \mathbf{v}_2$$

where κ is called the curvature and τ is called the torsion (or twisting number) of the curve $\mathbf{R}(s)$.

(c) Show that for a circular helix $\mathbf{R}(t) = (r\cos t,\ r\sin t,\ ct)$,

$$\kappa = \frac{r}{r^2 + c^2} \qquad \text{and} \qquad \tau = \frac{c}{r^2 + c^2}.$$

Thus verifying that τ may be taken as a measure of how much a curve deviates from a planar path, while κ measures deviation from a straight line path.