REAL ALGEBRAIC DIFFERENTIAL FORMS ON REAL ALGEBRAIC VARIETIES

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Abstract. In [10] it is proved that any de Rham cohomology class on a nonsingular quasiprojective complex algebraic variety is realized by a real algebraic differential form and quoted that it is not known whether the same holds for real algebraic varieties. In this paper, we prove that certain de Rham cohomology classes on a real algebraic variety are realized by real algebraic differential forms.

1. Introduction and the results

In 1963 Grothendieck proved that any de Rham cohomology class on a smooth affine algebraic variety is represented by an (complex) algebraic differential form ([8]). By the virtue of Hodge decomposition this result cannot hold on smooth projective complex algebraic varieties. Nevertheless, in [10], it is proved that every de Rham class on a smooth complex quasiprojective variety is realized by a real algebraic differential form: Let Z be a nonsingular quasiprojective complex algebraic variety. Let $H^i_{dR}(Z)$ denote the ith de Rham cohomology group of the underlying smooth manifold of complex points of Z, which we will denote again by Z. Let $H^i_{dR}(Z)_{alg}$ be the subspace of classes in $H^i_{dR}(Z)$, which are represented by real algebraic differential forms. Then, every de Rham class is realized by a real algebraic differential form.

Although we believe that the same result holds for real algebraic varieties we have no proof of that. Nevertheless, in several cases we have positive partial results. We need some preliminaries to state our results.

By the virtue of the de Rham Theorem, we will identify the singular cohomology with real coefficients of X with the de Rham cohomology of X.

We begin with the following observation:

Proposition 1.1. *The subset $H^*_{dR}(X)_{alg}$ is a subalgebra of the cohomology algebra of X. Also, if $f : X \to Y$ is an entire rational map of nonsingular real algebraic varieties then $f^*(H^*_{dR}(Y)_{alg}) \subseteq H^*_{dR}(X)_{alg}$.*

Moreover, $H^0_{dR}(X)_{alg} = H^0_{dR}(X)$ if and only if X is connected.

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The first part of the above proposition follows from the definitions and the fact that the wedge product of real algebraic forms is also real algebraic (see the next section for the definition of entire rational maps). The second assertion follows from the fact that a locally constant entire rational map is constant.

Let R be any commutative ring with unity. For an R-orientable nonsingular compact real algebraic variety X define $KH_*(X,R)$ to be the kernel of the induced map on homology,

$$i_* : H_*(X,R) \to H_*(X_{\mathbb{C}},R)$$

where $i : X \to X_{\mathbb{C}}$ is the inclusion map into some nonsingular projective complexification. In [15] it is shown that $KH_*(X,R)$ is independent of the complexification $X \subseteq X_{\mathbb{C}}$ and thus an (entire rational) isomorphism invariant of X (see also [5]). Similarly, denote the image of the homomorphism

$$i^* : H^*(X_{\mathbb{C}},R) \to H^*(X,R)$$

by $ImH^*(X,R)$, which is also an isomorphism invariant. Indeed, one can define $ImH^i(X,R)$, even for non orientable X, provided that R is a field (cf. see [14]).

Theorem 1.2. *Let X be a compact nonsingular real algebraic variety of dimension n. Then*
i.) $H^i_{dR}(X) = H^i_{dR}(X)_{alg}$, for $i = 1$ and $i = n$;
ii.) $ImH^i_{dR}(X) \subseteq H^i_{dR}(X)_{alg}$.

Corollary 1.3. *Suppose that X is a compact nonsingular real algebraic surface. Then $H^i_{dR}(X) = H^i_{dR}(X)_{alg}$, for $i \geq 1$.*

Part (ii) of the above theorem is a consequence of the following theorem of the first named author.

Theorem 1.4 ([10])**.** *Every de Rham cohomology class on a nonsingular quasiprojective complex algebraic variety is realized by a real algebraic differential form. In other words, assuming the above notation, $H^i_{dR}(Z)_{alg} = H^i_{dR}(Z)$.*

Using the above theorem we can rephrase the statement $H^i_{dR}(X)_{alg} = H^i_{dR}(X)$ as follows:

Proposition 1.5. *Let X be a nonsingular real algebraic variety. Then $H^i_{dR}(X)_{alg} = H^i_{dR}(X)$ if and only if there is a quasiprojective complexification $i : X \to X_{\mathbb{X}}$ such that the induced map on cohomology ${}^*i : H^i(X_{\mathbb{C}},\mathbb{R}) \to H^i(X,\mathbb{R})$ is onto, or equivalently, the map on homology $i_* : H^i(X,\mathbb{R}) \to H^i(X_{\mathbb{C}},\mathbb{R})$ is injective.*

The 'if' part follows from the above theorem and the universal coefficient theorem. For the other part, note that any real algebraic differential form on X extends to a holomorphic form on some quasiprojective complexification. Since the de Rham cohomology algebra of X is finitely generated one can obtain the desired complexification by taking the intersection of the finitely many quasiprojective complexifications corresponding to these generators.

Remark 1.6. Any regular map on a real algebraic variety extends to some nonsingular complexification possibly after some blowing ups. However, a closed real algebraic differential form on a real algebraic varieties may not extend to any smooth projective complexification even as a closed smooth form. Indeed, if a closed smooth p-form ω on X extends to some smooth projective complexification as a closed smooth form then the cohomology class $[\omega]$ must lie in $ImH^p_{dR}(X)$.

For example consider the variety $S^1 \times \cdots \times S^1$. Since S^1 bounds in its complexification $ImH^p_{dR}(X) = 0$, $p > 0$. However, by Theorem 1.2 (i) any de Rham class on X is represented by a real algebraic differential form.

Example 1.7. Let S^n be the unit sphere in $\mathbb{R}^{n+1}$. For odd n it is known that S^n bounds in its complexification and therefore $ImH^n_{dR}(S^n) = 0$ ([15]). However, the volume form of X

$$\omega = \frac{1}{2^n\ \pi} \sum_{i=1}^{n+1} (-1)^{i-1}\ x_i\ dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \cdots \wedge dx_{n+1}$$

is clearly real algebraic. Hence, $H^n_{dR}(S^n) = H^n_{dR}(S^n)_{alg}$.

Indeed the above holds for any homogeneous space: It is a classical result that any compact Lie group has the structure of a real linear algebraic group and this structure is unique. Moreover, Dovermann and Masuda proved in [7] that any homogeneous space G/H has the structure of a nonsingular real algebraic G-variety, and this structure is unique. In [12] Kulkarni proved that the inclusion map of the real algebraic variety into some quasiprojective complexification $G/H \to (G/H)_{\mathbb{C}}$ is a homotopy equivalence. So, by Theorem 1.4 we obtain

Corollary 1.8. *If $X = G/H$ is a homogeneous space as above then, for all $i \geq 1$, we have $H^i_{dR}(X) = H^i_{dR}(X)_{alg}$.*

For product varieties we have the following obvious result.

Proposition 1.9. *Let X and Y be nonsingular real algebraic varieties. Then,*

$$\oplus_{i+j=k} H^i_{dR}(X)_{alg} \otimes H^j_{dR}(Y)_{alg} \subseteq H^k_{dR}(X \times Y)_{alg}.$$

Remark 1.10. Let X be a compact nonsingular real algebraic variety with more than one topological component. Then Proposition 1.1 implies that $H^0_{dR}(X)_{alg} \neq H^0_{dR}(X)$ and thus

$$\oplus_{i+j=1} H^i_{dR}(X)_{alg} \otimes H^j_{dR}(S^1)_{alg} \subsetneqq H^1_{dR}(X \times S^1) = H^1_{dR}(X \times S^1)_{alg},$$

where the last equality follows from Theorem 1.2. Hence, a real algebraic de Rham class representing a product class may not be written as a product of algebraic de Rham classes. Indeed, we can construct real algebraic forms explicitly representing the classes in $H^0_{dR}(X) \otimes H^1_{dR}(S^1)$ (see the proof of the next proposition). Moreover, the method of constructions works not only S^1 but for any odd dimensional standard sphere S^d. Namely, we have the following result.

Proposition 1.11. *Let X be any compact nonsingular real algebraic variety and d a positive odd integer. Then*

$$H^0_{dR}(X) \otimes H^d_{dR}(S^d) \subseteq H^d_{dR}(X \times S^d)_{alg}.$$

Next theorem is more technical and has more topological aspects.

Theorem 1.12. *Now, let $X \subseteq \mathbb{R}^N$ be a compact n-dimensional nonsingular real algebraic variety, $a \in H^k(X, \mathbb{Z})$ and k is odd. Let α be the image of the class a under the Alexander duality*

$$H^k(X, \mathbb{Z}) \xrightarrow{\sim} \bar{H}_{N-k-1}(\mathbb{R}^N - X, \mathbb{Z}).$$

If α is represented by a nonsingular algebraic variety L, which is also a complete intersection, as above, then the class a is represented by a real algebraic differential form.

We finish this section with a result on algebraic realization of smooth closed manifolds. It is known that any smooth closed manifold M has an algebraic model X such that any continuous complex vector bundle over X is algebraic. It follows then $H^{2i}(X, \mathbb{Q}) = H^{2i}_{\mathbb{C}-alg}(X, \mathbb{Q}) = ImH^{2i}(X, \mathbb{Q})$ (cf. see [13]). Hence, by Theorem 1.2 we have the following theorem.

Theorem 1.13. *Let M be a smooth connected closed manifold. Then M has an algebraic model X with $H^i_{dR}(X) = H^i_{dR}(X)_{alg}$, for $i = 1,\ \dim(M)$ or any even integer.*

2. Proofs

All real algebraic varieties under consideration in this report are nonsingular. It is well known that real projective varieties are affine (Proposition 2.4.1 of [1] or Theorem 3.4.4 of [3]). Moreover, compact affine real algebraic varieties are projective (Corollary 2.5.14 of [1]) and therefore, we will not distinguish between real compact affine varieties and real projective varieties.

For real algebraic varieties $X \subseteq \mathbb{R}^r$ and $Y \subseteq \mathbb{R}^s$ a map $F : X \to Y$ is said to be entire rational if there exist $f_i, g_i \in \mathbb{R}[x_1, \ldots, x_r]$, $i = 1, \ldots, s$, such that each g_i vanishes nowhere on X and $F = (f_1/g_1, \ldots, f_s/g_s)$. We say X and Y are isomorphic if there are entire rational maps $F : X \to Y$ and $G : Y \to X$ such that $F \circ G = id_Y$ and $G \circ F = id_X$. Isomorphic algebraic varieties will be regarded the same. We refer the reader for the basic definitions and facts about real algebraic geometry to [1, 3].

Proof of Theorem 1.2. (i.) The case $i = 1$: Since $H^i(X, \mathbb{R}) = H^i(X, \mathbb{Z}) \otimes \mathbb{R}$ it suffices to show that each integer class is realized by a real algebraic differential form. Let $a \in H^1(X, \mathbb{Z})$. Since S^1 is a $K(\mathbb{Z}, 1)$ there exists a smooth map $f : X \to S^1$ such that $a = f^*([u])$, where we can take u to be the real algebraic 1-form on $S^1 \subseteq \mathbb{R}^2$

$$u = \frac{x\ dy - y\ dx}{2\pi}.$$

The cohomology class $2a$ is represented by $g^*[u]$, where $g(x) = (f(x))^2$, and the mod 2 reduction of $2a$ is clearly zero. Now, by a result of Ivanov ([11]) $g : X \to S^1$ is homotopic to an entire rational map. Then, Proposition 1.1 implies that $2a$ and hence a is represented by a real algebraic differential form.

The case $i = n$: The top exterior power of the cotangent bundle of X is algebraic, because the cotangent bundle is algebraic. So, any smooth volume form can be approximated by a real algebraic differential form, which will be readily closed by dimension reasons. Hence, $H^n_{dR}(X) = H^n_{dR}(X)_{alg}$.

Part (ii) follows from [10] as mentioned earlier.

□

Proof of Theorem 1.12. As before we can assume that a and α are integer classes. Let $L = Z(f_1, \ldots, f_l)$, the common zero set of f_i's and such that at each point $x \in L$ the gradients of f_i's are linearly independent. Then the differential form

$$\omega_L = \frac{1}{2^{l-1}\ \pi} \sum_{i=1}^{l} (-1)^{i-1} f_i \frac{df_1 \wedge \cdots \wedge df_{i-1} \wedge df_{i+1} \cdots \wedge df_l}{(f_1^2 + \cdots + f_l^2)^{l/2}}$$

is a closed differential form on $\mathbb{R}^N - L$ so that for any closed smooth orientable submanifold $K \subset \mathbb{R}^N - L$ of dimension $l - 1 = N - \dim(L) - 1$ the integral

$$\int_K \omega_L$$

is, up to sign, the linking number of K and L.

Note that l, the codimension of L in $\mathbb{R}^N$, is the even integer $k+1$ and thus ω_L is algebraic. Moreover, the proof of Alexander duality implies that the restriction of the de Rham class $[\omega_L]$ to X is, up to sign, is nothing but the class a (cf. see page 351 in [6]). This finishes the proof. □

Proof of Proposition 1.11. Let $X_1, \cdots, X_n$ be the topological components of X and consider the locally constant function $f_0 : X \to \mathbb{R}^{d+1}$ with $f_0(X_i) = (4i, 0, \cdots, 0)$, for $i = 1, \ldots, n$. By Weierstrass Approximation Theorem we can find a polynomial function $f : X \to \mathbb{R}^{d+1}$ such that $f(X_i) \subseteq B(4i, \frac{1}{4})$.

Define a function $\phi : X \times S^d \to \mathbb{R}^{d+1}$ by $\phi(x, v) = f(x) + v$, for any $(x, v) \in X \times S^d$, where S^d is the unit sphere in $\mathbb{R}^{d+1}$. Note that for any i, $\phi(X_i \times S^d) \subseteq B(4i, 2) - B(4i, \frac{1}{2})$ and hence for any $x \in X_i$ the image $\phi(\{x\} \times S^d)$ is a sphere containing the point $(4i + \frac{3}{8}, 0, \ldots, 0)$ inside.

Let ω_i be the angular form of $\mathbb{R}^{d+1}$ centered at $(4i+\frac{3}{8},0,\ldots,0)$, $i=1,\ldots,n$,

$$\omega_i = \frac{1}{2^d\ \pi}\ \sum_{j=1}^{d+1}(-1)^{j-1}x_j\frac{dx_1\wedge\cdots\wedge dx_{j-1}\wedge dx_{j+1}\cdots\wedge dx_{d+1}}{((x_1-4j-\frac{3}{8})^2+\cdots+x_{d+1}^2)^{(d+1)/2}}.$$

Note that these real algebraic forms ($d+1$ is an even integer) are closed and the pullbacks $\phi^*([\omega_i])$ spans $H^0_{dR}(X)\otimes H^d_{dR}(S^d)$. This finishes the proof. □

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