Videos 37-38

Vector Bundles and Poincaré-Hopf Theorem.

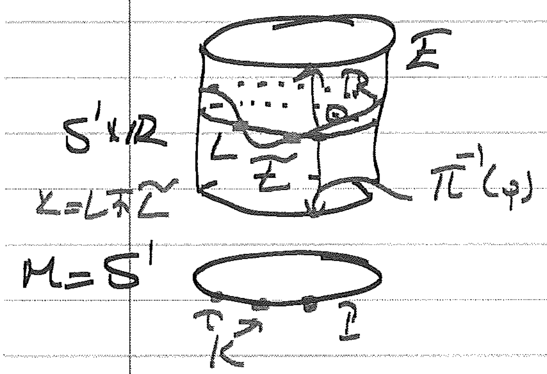
Note Title 27.04.2020

Euler Characteristic (Class) of an oriented Vector Bundle

$\mathbb{R}^k \to E^{m+k}$
$\downarrow \pi$
M^m

E is an oriented $\mathbb{R}^k$-bundle over a smooth manifold M

Let $L = s(M)$, where $s: M \to E$, $s(p) = (p, 0)$, the zero section. $\pi \circ s = Id_M$, $(\pi \circ s(p) = \pi(p,0) = p)$
$L \subseteq E$ is a closed submanifold diffeomorphic to M.



Let $K = L \pitchfork \tilde{L}$ be the transverse self intersection of L. Then K is a submanifold in E of dimension
$\dim(L) + \dim(\tilde{L}) - \dim E = m + m - (m+k)$
$= m - k.$

Clearly, $K \subseteq L$ is a submanifold of L. Since $s: M \to L$ is a diffeomorphism by identifying M with L we may regard K as a submanifold of M.

Assuming M is oriented also, $K \subseteq M^m$ is an oriented submanifold of dimension $m-k$. Then $[\eta_K] \in H^k_{DR}(M)$ the Poincaré dual of K will be called the Euler class of the vector bundle.

<u>Notation</u>: $e(E) \doteq [\eta_K]$.

<u>Remark</u>: $\mathbb{R}^m \to E \to M^m$, where both M and the bundle are oriented. Then the Euler class $e(E) \in H^m_{DR}(M) \simeq \mathbb{R}$, if we further assume

that M is compact. The real number

$\int_M e(E) \in \mathbb{R}$ is called the Euler number of the vector bundle.

$\dim K = m+m-(m+m)=0. \Rightarrow [\eta_K] \in H^m_{DR}(M).$

$Int(K,M) = \int_M \eta_K = \int_M e(M)$ the oriented (signed) sum of points in K.

This integer is also called the Euler number of the bundle $E \to M$.

2) If $k=m=1 \pmod 2$ then the self intersection

$Int(L,L) = L \pitchfork \tilde{L}$ is zero.

3) If a bundle $E \to M$ has a section $s: M \to E$ so that $s(p)$ is never zero, then $s(M)$ never intersects the zero section. Hence, $e(M)=0$.

__Examples__ 1) $M = T^2 = \underset{\theta_1}{S^1} \times \underset{\theta_2}{S^1}$, $E = T_* T^2 \to T^2$

$X(\theta_1,\theta_2) = \frac{\partial}{\partial \theta_1}$ is a nowhere zero section of T^2.

Hence, $e(E)=0$.

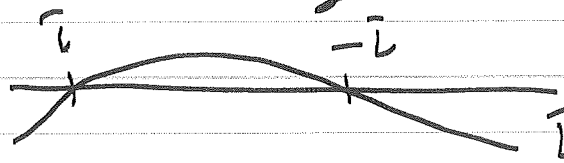
__Definition:__ For a manifold M the Euler number of its tangent bundle T_*M is called also the number number of M and we sometimes denote it by $e(M)$.

$e(M) = e(T_*M)$.

So, $e(T^2)=0$. Similarly, $e(T^n)=0$ for any n.

2) $\mathbb{CP}^1 = S^2$, $T_x\mathbb{CP}^1 = T_x\mathbb{C}\times T_x\mathbb{C} \Big/ \begin{matrix}(z,w)\sim\left(\frac{1}{z}, -\frac{w}{z^2}\right) \\ (z\neq 0)\end{matrix}$

$s_1(z) = -s_2(z) = \dfrac{1+z^2}{2}$.



$L \subseteq T_x\mathbb{CP}^1$

$\tilde{L} \subseteq T_x\mathbb{CP}^1$ complex submanifold

Hence, $\{i,-i\}$ is a complex submanifold of $\mathbb{CP}^1$. Therefore, both i and $-i$ have $+1$ -orientation.

$e(\mathbb{CP}^1) = 1+1 = 2$.

<u>Remark</u>: For the manifolds S^1 and S^2 the Euler characteristics and Euler numbers agree. This is <u>not</u> a coincidence. Indeed, it is the content of so called Poincaré-Hopf Theorem.

Now consider the complex line bundle $\mathcal{O}(k)\to\mathbb{CP}^1$

$$\begin{array}{l}\mathcal{O}(k) = \mathbb{C}\times\mathbb{C} \sqcup \mathbb{C}\times\mathbb{C} \Big/ \begin{matrix}(z,w)\sim\left(\frac{1}{z}, \frac{1}{z^k}w\right) \\ z\neq 0\end{matrix} \\ \quad\downarrow \\ \quad\mathbb{C}\end{array}$$

$s_1(z) = \dfrac{1+z^k}{2} = -s_2(z)$ is a section of $\mathcal{O}(k)$.

This section has k-complex zeros. Hence,

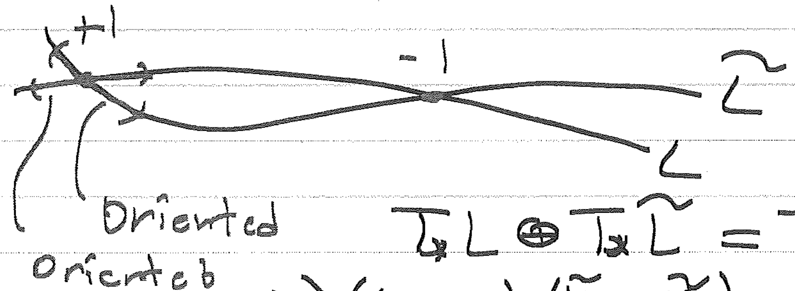
$$e(\mathcal{O}(k)) = k.$$

3) $L^{\ell} \subseteq M^{2\ell}$, L non-orientable, M oriented

L: compact submanifold.

As an example, take $L = \mathbb{RP}^2$ in $\mathbb{CP}^2 = M$.

We may define the integer self intersection of L with itself in M.



$$T_x L \oplus T_x \tilde{L} = T_x M$$

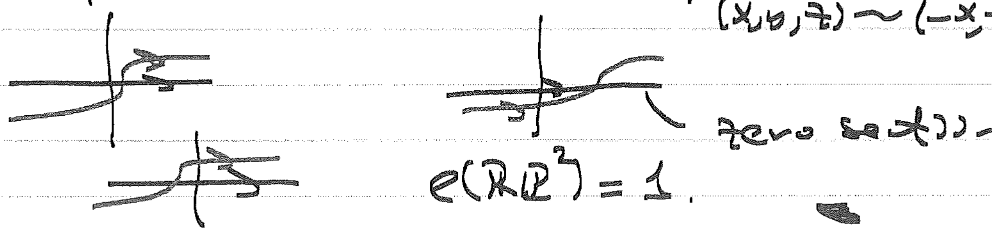
$\supset (v_1, \dots, v_\ell), (\tilde{v}_1, \dots, \tilde{v}_\ell) \quad (u_1, \dots, u_{2\ell})$

$\supset (-v_1, \dots, v_\ell)(-\tilde{v}_1, \dots, \tilde{v}_\ell)$

Hence, we may define the integer self intersection of a non-orientable compact submanifold L^{ℓ} in an oriented manifold $M^{2\ell}$.

<u>Proposition</u>: The self intersection of $\mathbb{RP}^2$ in its tangent bundle (oriented suitably) is equal one; $e(\mathbb{RP}^2) = 1$.

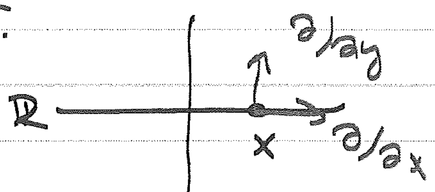
<u>Proof</u>: $S^2 \xrightarrow{\pi} \mathbb{RP}^2 = S^2 / (x,y,z) \sim (-x,-y,-z)$



$e(\mathbb{RP}^2) = 1$.

4) Let's compute the self intersection of $\mathbb{RP}^2$ in $\mathbb{CP}^2$.

$z_1 = x_1 + i y_1$, $z_2 = x_2 + i y_2$ local chart on $\mathbb{CP}^2$.
x_1, y_1 local chart on $\mathbb{RP}^2$.



$p \in \mathbb{RP}^2 \subseteq \mathbb{CP}^2$

$T_p\mathbb{RP}^2 = \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right\rangle$, $T_p\mathbb{CP}^2 = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2} \right)$.

$T_*\mathbb{RP}^2 \longrightarrow \nu_{\mathbb{RP}^2}$: normal bundle of $\mathbb{RP}^2$ in $\mathbb{CP}^2$.

$$\left(p, v = a\frac{\partial}{\partial x_1} + b\frac{\partial}{\partial x_2}\right) \longmapsto \left(p, iv = a\frac{\partial}{\partial y_1} + b\frac{\partial}{\partial y_2}\right).$$

$$\begin{aligned} T_p\mathbb{CP}^2 &= \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2} \right\rangle \\ &= -\left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2} \right\rangle \\ &= -\,T_p\mathbb{RP}^2 \oplus \nu_{\mathbb{RP}^2} \\ &= T_p\mathbb{RP}^2 \oplus (-\nu_{\mathbb{RP}^2}) \end{aligned}$$

So the isomorphism $T_*\mathbb{RP}^2 \longrightarrow \nu_{\mathbb{RP}^2}$ given by multiplication with i reverses the orientation. In other words, the positive orientation on $\nu_{\mathbb{RP}^2}$ is given by $\left(-\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}\right)$.

Hence, the $e(-\nu_{\mathbb{RP}^2}) = -e(\mathbb{RP}^2) = 1$. In other words, the self intersection of $\mathbb{RP}^2$ in

$\mathbb{CP}^2$ is -1.

5) Let's compute the self intersection of $\mathbb{RP}^2$ in $\mathbb{CP}^2$ directly.

$\phi_t : \mathbb{CP}^2 \longrightarrow \mathbb{CP}^2$, $[z_0 : z_1 : z_2] \longmapsto [z_0 : e^{it} z_1 : e^{2it} z_2]$.

$\mathbb{RP}^2 = \{[z_0 : z_1 : z_2] \in \mathbb{CP}^2 \mid \mathrm{Im}(z_i) = y_i = 0,\ i = 0, 1, 2\}$.

$\phi_t(\mathbb{RP}^2) = \mathbb{RP}^2_t = \{[x_0 : e^{it} x_1 : e^{2it} x_2] \mid x_i \in \mathbb{R}\}$

$\mathbb{RP}^2 \cap \mathbb{RP}^2_t = \{[1:0:0], [0:1:0], [0:0:1]\}$.

$\mathbb{RP}^2$ $\mathbb{RP}^2_t$ for $t > 0$ and small.

Let's compute the sign of intersection at each point:

1) $[1:0:0]$ $U_0 = \{z_0 \neq 0\}$ local charts in U_0 for $\mathbb{RP}^2$ and $\mathbb{RP}^2_t$ are given by

$\left(\frac{x_1}{x_0}, \frac{x_2}{x_0}\right)$ and $\left(\frac{e^{it} x_1}{x_0}, \frac{e^{2it} x_2}{x_0}\right)$

Putting these coordinates side by side as

$\left(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{e^{it} x_1}{x_0}, \frac{e^{2it} x_2}{x_0}\right)$. Compare this with the complex orientations of $\mathbb{CP}^2$ using the chart $(z_1/z_0, z_2/z_0)$, which is

$\left(x_1/x_0, \frac{e^{it} x_1}{x_0}, \frac{x_2}{x_0}, e^{2it} x_2/x_0\right)$. Hence, the orientations do not match at this point. Hence, the sign of intersection at $[1:0:0]$ is -1.

2) $[0:1:0]$ $\quad U_1=\{z_1\neq 0\}$ $\quad \left(\frac{z_0}{z_1}, \frac{z_2}{z_1}\right)$

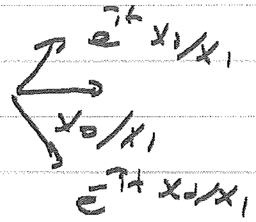
$\mathbb{RP}^2 \qquad \mathbb{RP}^2_t$

$\left(x_0/x_1, \frac{x_2}{x_1}\right) \qquad \left(e^{-it}x_0/x_1, e^{it}x_2/x_1\right)$

$\Rightarrow \left(\frac{x_0}{x_1}, \frac{x_2}{x_1}, e^{-it}x_0/x_1, e^{it}x_2/x_1\right)$

Now, the orientation for $\mathbb{CP}^2$ in this coordinate is given by

$\left(\frac{x_0}{x_1}, e^{-it}\frac{x_0}{x_1}, \frac{x_2}{x_1}, e^{it}\frac{x_2}{x_1}\right)$



Hence, the sign of the intersection $[0:1:0]$ is $+1$.

3) $[0:0:1]$ $\quad U_2=\{z_2\neq 0\}$ $\quad \left(\frac{z_0}{z_2}, \frac{z_1}{z_2}\right)$

$\mathbb{RP}^2 \qquad \mathbb{RP}^2_t$

$\left(\frac{x_0}{x_2}, \frac{x_1}{x_2}\right) \qquad \left(e^{-2it}\frac{x_0}{x_2}, e^{-it}\frac{x_1}{x_2}\right)$

$\Rightarrow \left(\frac{x_0}{x_2}, \frac{x_1}{x_2}, e^{-2it}\frac{x_0}{x_2}, e^{-it}\frac{x_1}{x_2}\right)$.

However, the complex orientation at this point is given by

$\left(\frac{x_0}{x_2}, e^{it}\frac{x_0}{x_2}, \frac{x_1}{x_2}, e^{it}\frac{x_1}{x_2}\right)$

Hence, the sign of intersection is -1.

So the total intersection number is
$-1+1-1=-1$.

<u>Remark:</u> $e(\mathbb{RP}^2)=1$ i.e., $\mathrm{Int}(\mathbb{RP}^2, \mathbb{RP}^2)=1$ in $T_*\mathbb{RP}^2$.
Also, considered as a submanifold of $\mathbb{CP}^2$, $\mathbb{RP}^2\subseteq\mathbb{CP}^2$ has $\mathrm{Int}(\mathbb{RP}^2,\mathbb{RP}^2)=-1$, i.e. $e(\nu_{\mathbb{RP}^2})=-1$.

<u>Lemma:</u> Let M^n be a smooth manifold. Then T_*M is orientable and has a canonical orientation.

<u>Proof:</u> Let $U\subseteq\mathbb{R}^n$ be an open subset, with coordinates $x_1,\dots,x_n$. Then, $T_*U=U\times\mathbb{R}^n$ has coordinates $x_1,\dots,x_n,y_1,\dots,y_n$, where $y_i: T_*U\to\mathbb{R}$, $y_i\left(\sum_j a_j\frac{\partial}{\partial x_j}(p)\right)\doteq a_j$.

$x_i: U\to\mathbb{R}$, $x_i(p)=x_i(p_1,\dots,p_n)=p_i$, $i=1,\dots,n$.
Let $V\subseteq\mathbb{R}^n$ be another open subset with coordinates $\tilde{x}_1,\dots,\tilde{x}_n$ and $F: U\to V$ a diffeomorphism.

Let $(\tilde{x}_1,\dots,\tilde{x}_n)=(f_1(x_1,\dots,x_n),\dots,f_n(x_1,\dots,x_n))=F(x_1,\dots,x_n)$.

On the other hand, we know that if $\mathcal{B}=\left\{\frac{\partial}{\partial x_1},\dots,\frac{\partial}{x_n}\right\}$ and $\mathcal{B}'=\left\{\frac{\partial}{\partial\tilde{x}_1},\dots,\frac{\partial}{\partial\tilde{x}_n}\right\}$ are bases for T_pU and T_qV, $q=F(p)$, then $A(p)=(DF_p)_{\mathcal{B}}^{\mathcal{B}'}=\left(\frac{\partial f_i}{\partial x_j}(p)\right)$.

Let $\tilde{y}_1,\dots,\tilde{y}_n$ be coordinates on T_*V given by

$\tilde{y}_i\left(\sum_j b_j\frac{\partial}{\partial\tilde{x}_j}\right)=b_j$.

Hence, the diffeomorphism $\varphi=(F,DF): T_*U\to T_*V$ between the total spaces of tangent bundles is given in coordinates as

$$\varphi(x_1,\dots,x_n,y_1,\dots,y_n) = \left(F(x_1,\dots,x_n),\ A(p)(y_1,\dots,y_n)\right).$$

So its Jacobian matrix has the form

$$D\varphi_{p,v} = \begin{bmatrix} DF_p & 0 \\ * & A(p) \end{bmatrix} = \begin{bmatrix} A(p) & 0 \\ * & A(p) \end{bmatrix},$$ which has

determinant $\det(D\varphi_{p,v}) = \det A(p)\,\det(A(p))$

$$= (\det A(p))^2 > 0.$$

This finishes the proof. ◀

<u>Example:</u> $F(x_1,x_2) = (f_1(x_1,x_2),\ f_2(x_1,x_2))$, $F:\mathbb{R}^2 \to \mathbb{R}^2$.

$$A = DF = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}. \text{ So } \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix} = \begin{pmatrix} y_1 \frac{\partial f_1}{\partial x_1} + y_2 \frac{\partial f_1}{\partial x_2} \\ y_1 \frac{\partial f_2}{\partial x_1} + y_2 \frac{\partial f_2}{\partial x_2} \end{pmatrix}$$

$$\varphi(x_1,x_2,y_1,y_2) = \left(f_1,\ f_2,\ y_1 \frac{\partial f_1}{\partial x_1} + y_2 \frac{\partial f_1}{\partial x_2},\ y_1 \frac{\partial f_2}{\partial x_1} + y_2 \frac{\partial f_2}{\partial x_2}\right).$$

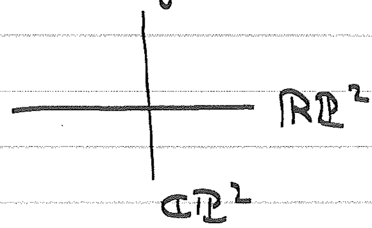
So

$$D\varphi = \begin{bmatrix} \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} & \begin{matrix} 0 & 0 \\ & \end{matrix} \\ \begin{bmatrix} y_1 \frac{\partial^2 f_1}{\partial x_1^2} + y_2 \frac{\partial^2 f_1}{\partial x_1 \partial x_2} & y_1 \frac{\partial^2 f_1}{\partial x_1 \partial x_2} + y_2 \frac{\partial^2 f_1}{\partial x_2^2} \end{bmatrix} & \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \\ \begin{bmatrix} y_1 \frac{\partial^2 f_2}{\partial x_1^2} + y_2 \frac{\partial^2 f_2}{\partial x_1 \partial x_2} & y_1 \frac{\partial^2 f_2}{\partial x_1 \partial x_2} + y_2 \frac{\partial^2 f_2}{\partial x_2^2} \end{bmatrix} & \end{bmatrix}$$

where $[\quad] = \left(\mathrm{Hess}(f_1)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right)^T$ and $[\quad] = \left(\mathrm{Hess}(f_2)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right)^T$.

For the self intersection of $\mathbb{RP}^2$ in $\mathbb{CP}^2$ we note the following fact that the tangent bundle of $\mathbb{CP}^2$ restricted to $\mathbb{RP}^2$ is the complexification of the tangent bundle of $\mathbb{RP}^2$:

$$T_*\mathbb{CP}^2\big|_{\mathbb{RP}^2} \simeq T_*\mathbb{RP}^2 \otimes_{\mathbb{R}} \mathbb{C}.$$



Now let's consider the computations

$e(\mathbb{RP}^2) = \mathrm{Int}(\mathbb{RP}^2, \mathbb{RP}^2)$ in $T_*\mathbb{RP}^2$ and
$e(\nu_{\mathbb{RP}^2}) = \mathrm{Int}(\mathbb{RP}^2, \mathbb{RP}^2)$ in $\mathbb{CP}^2$.

If x_1, x_2 are coordinates on $\mathbb{RP}^2$ then the orientation on $T_*\mathbb{RP}^2$ is "given basically"

$$\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \qquad y_i\left(\frac{\partial}{\partial x_j}\right) = \delta_{ij}.$$

On the other hand, the coordinates on $\mathbb{CP}^2$ is given by $z_1 = x_1 + i y_1$, $z_2 = x_2 + i y_2$ and thus the complex orientation on the tangent bundle is given by

$$\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1} = i\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2} = i\frac{\partial}{\partial x_2}$$

Note that the two orientations do <u>not</u> match!

<u>Remark:</u> If X^n is a submanifold of a complex manifold M^{2n} ($\dim_{\mathbb{C}} M = n$) so that

$T_*X \otimes_{\mathbb{R}} \mathbb{C} \simeq T_*M\big|_X$ then the complex orientation on T_*M restricted to X is given by $\frac{\partial}{\partial x_1}, \overset{\partial/\partial y_1}{i\frac{\partial}{\partial x_1}}, \frac{\partial}{\partial x_2}, \overset{\partial/\partial y_2}{i\frac{\partial}{\partial x_2}}, \dots, \frac{\partial}{\partial x_n}, \overset{\partial/\partial y_n}{i\frac{\partial}{\partial x_n}}$, whereas the orientation on T_*X is given by $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}$.

Hence, the two orientations differ by

$1+2+\dots+(n-1) = \frac{n(n-1)}{2}$ transpositions. Hence,

$e(X) = (-1)^{n(n-1)/2}\, e(\nu_X)$ <u>or</u> equivalently,

$$\mathrm{Int}_{T_*X}(X,X) = (-1)^{n(n-1)/2}\, \mathrm{Int}_M(X,X).$$

Gysin Exact Sequence

$\pi: E^{n+r} \to M^n$ smooth oriented vector bundle over M. Put a metric on E and let $P \to M$ be the unit sphere bundle of E.

$$\pi^{-1}(p) \cong E_p \simeq \mathbb{R}^r \supseteq S^{r-1} \text{ the unit sphere.}$$

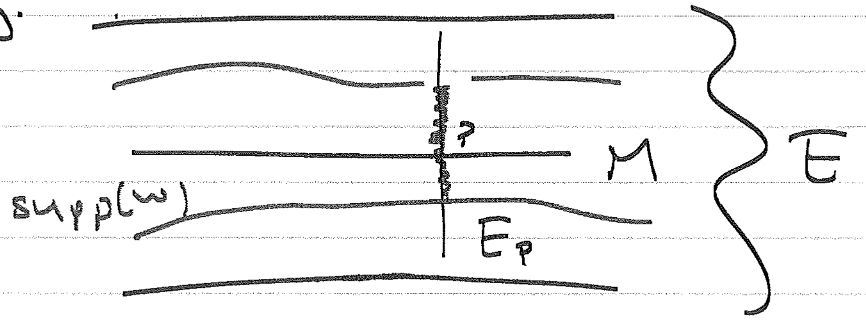
Theorem: Assume the above setup. Then we have an exact sequence of the form

$$\cdots \xrightarrow{\pi^*} H^{i-1}_{DR}(P) \xrightarrow{\int_{S^{r-1}}} H^{i-r}_{DR}(M) \xrightarrow{\wedge e(E)} H^i_{DR}(M) \xrightarrow{\pi^*} H^i_{DR}(P) \xrightarrow{\int_{S^{r-1}}} \cdots,$$

here $\int_{S^{r-1}}$ represents integration along fibers and $e(E)$ is the Euler class of $\pi: E \to M$.

Proof uses vertically compactly supported cohomology.

$$\begin{array}{c} \mathbb{R}^r \to E \\ \downarrow \pi \\ M \end{array}$$



$$\xrightarrow{d} \Omega^k_{vc}(E) \xrightarrow{d} \Omega^{k+1}_{vc}(E) \xrightarrow{d} \cdots$$

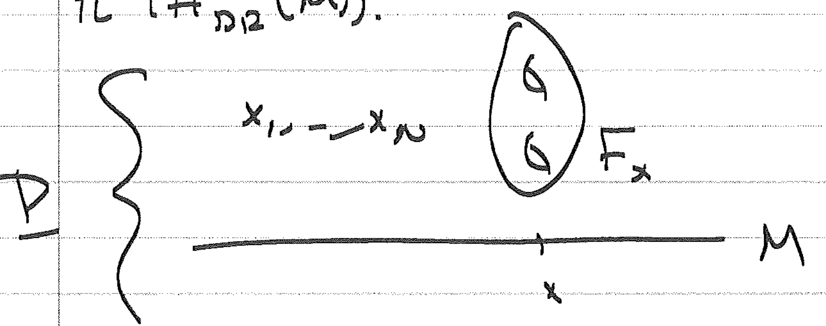
$$H^k_{vc}(E) \doteq \frac{\ker(d: \Omega^k_{vc}(E) \to \Omega^{k+1}_{vc}(E))}{\mathrm{Im}(d: \Omega^{k-1}_{vc}(E) \to \Omega^k_{vc}(E))}.$$

$$H^{r+k}_{vc}(E) \xrightarrow{\int_{\mathbb{R}^k}} H^k_{DR}(M)$$

Leray-Hirsch and Künneth Theorems

Theorem (Leray-Hirsch)

Let $\pi: P \to M$ be a fiber bundle, whose fibers are diffeomorphic to the manifold F. Assume that a subset $\{x_1, \ldots, x_N\}$ of $H^*_{DR}(P)$ exists so that their restriction to each $H^*_{DR}(F_x)$, $F_x = P^{-1}(x)$, is a basis for $H^*_{DR}(F_x)$. Then, $H^*_{DR}(P)$ is a free module with basis $\{x_1, \ldots, x_N\}$ over the subalgebra $\pi^*(H^*_{DR}(M))$.



Proof was very similar ideas to that of the proof of Poincaré Duality

Special Case: $P_M = M \times N \to M$, where M and N are smooth manifolds. $P_N : M \times N \to N$

The map $P_N^* : H^*_{DR}(N) \to H^*_{DR}(M \times N)$ is an injective map, and we may consider $H^*_{DR}(N)$ as a subalgebra of $H^*_{DR}(M \times N)$. Pick a basis $\{x_1, \ldots, x_N\}$ of $H^*_{DR}(N)$. Then $\{x_1, \ldots, x_N\}$ satisfies the condition of Leray-Hirsch Theorem for the fiber bundle

$$P_M : M \times N \to M.$$

So by the Leray-Hirsch $H^*_{DR}(M\times N)$ is a free module over $H^*_{DR}(M)$ with basis $\{x_1, \ldots, x_N\}$, which is an $\mathbb{R}$-basis for the vector space $H^*_{DR}(N)$. Hence, we have

<u>Theorem</u> (Künneth Formulas)

$$H^*_{DR}(M\times N) = H^*_{DR}(M) \otimes_{\mathbb{R}} H^*_{DR}(N) \quad \text{and}$$

$$H^k_{DR}(M\times N) = \bigoplus_{i+j=k} H^i_{DR}(M) \otimes H^j_{DR}(N).$$

<u>Definition:</u> <u>Poincaré</u> Series of a smooth manifold whose cohomology is finite dimensional is defined to the series

$$P_M(t) \doteq \sum_{k=0}^{\infty} b_k(M)\, t^k.$$

Clearly, if $\dim M = m$, then $P_M(t)$ is a polynomial of degree at most m.

<u>Corollary:</u> If M and N are smooth manifolds whose Poincaré Series are defined then

$$P_{M\times N}(t) = P_M(t)\, P_N(t).$$

<u>Ex</u> $H^k_{DR}(S^1) = \begin{cases} \mathbb{R} & \text{if } k=0,1 \\ 0 & \text{otherwise} \end{cases}$

$P_{S^1}(t) = 1+t$. Hence, $P_{\underbrace{S^1\times\cdots\times S^1}_{n}}(t) = (1+t)^n$

__Theorem__ (Poincaré–Hopf)

For any compact orientable manifold M, the Euler number of M is equal to the Euler characteristic of M.

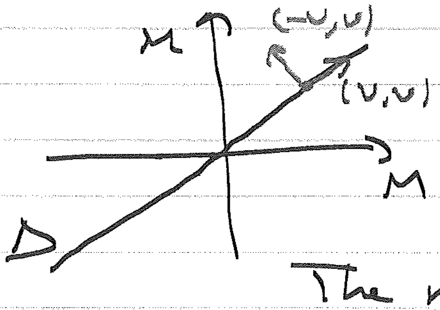
__Proof__ Must show: $\chi(M) = \int_M e(M)$.

First consider the map $f: M \to M \times M$ given by $f(x) = (x,x)$, for all $x \in M$. Let Δ denote $f(M)$:

$$\Delta = f(M) = \{(x,x) \mid x \in M\}.$$

Clearly, $f: M \to \Delta$ is a diffeomorphism. Moreover, if $\nu(\Delta)$ is the normal bundle of Δ in $M \times M$ then

$$T_*(M \times M) = T_*\Delta \oplus \nu(\Delta)$$



$$T_*\Delta = \{(v,v) \mid v \in T_*M\}$$

$$\nu(\Delta) = \{(-v,v) \mid v \in T_*M\}$$

The normal bundle $\nu(\Delta)$ is oriented via the equation

$$T_*(M \times M) = T_*\Delta \oplus \nu(\Delta).$$

So if $T_x M$ has oriented basis $\{v_1, \dots, v_n\}$.

The $T_{(p,p)}\Delta$ is oriented by the basis $\{(v_1,v_1),\dots,(v_n,v_n)\}$ and $\nu(\Delta)$ is oriented by $\{(-v_1,v_1),\dots,(-v_n,v_n)\}$ and $T_{(p,p)}M\times M$ oriented by $\{(v_1,0),\dots,(v_n,0),(0,v_1),\dots,(0,v_n)\}$.

<u>Claim:</u> The orientations of $T_{(p,p)}M\times M$ induced by $\{(v_1,0),\dots,(v_n,0),(0,v_1),\dots,(0,v_n)\}$ and $\{(v_1,v_1),\dots,(v_n,v_n),(-v_1,v_1),\dots,(-v_n,v_n)\}$ are the same.

<u>Ex</u> $M=\mathbb{R}^2$, $T_pM=\{(1,0),(0,1)\}$

$T_{(p,p)}\Delta M=\{(1,0,1,0),(0,1,0,1)\}$ and

$\nu(M)=\{(-1,0,1,0),(0,-1,0,1)\}$.

$$\begin{bmatrix}1&0&1&0\\0&1&0&1\\-1&0&1&0\\0&-1&0&1\end{bmatrix}\to\begin{bmatrix}1&0&1&0\\0&1&0&1\\0&0&2&0\\0&0&0&2\end{bmatrix}\to\begin{bmatrix}1&0&1&0\\0&1&0&1\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$$\to\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$ which is the basis for $T_x(M\times M)$.

$\{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}$.

Claim: The map $T_*M \longrightarrow \nu(\Delta)$ given by

$(p,v) \longmapsto ((p,p),(-v,v))$ is a vector bundle isomorphism as oriented bundles. ——

Therefore the self intersection of M in T_*M is the same the self intersection of the diagonal in $M\times M$.

$$e(M) = \underset{T_*M}{\mathrm{Int}}(M_0, M_0) = \underset{M\times M}{\mathrm{Int}}(\Delta,\Delta),$$ where

M_0 is the zero section of M in T_*M.

Let $\omega \in H^n_{DR}(M\times M)$ be Poincaré dual ($n = \dim M$)

of the submanifold Δ of $M\times M$.

Then $e(M) = \underset{M\times M}{\mathrm{Int}}(\Delta,\Delta) = \int_\Delta \omega$.

So, we need to show that $\int_\Delta \omega = \chi(M)$.

Let $\{a_i\}$ be an $\mathbb{R}$-basis of the vector space $H^*_{DR}(M) = \bigoplus_{k=0}^{n} H^k_{DR}(M)$. Now by Poincaré duality there is another basis, say $\{b_i\}$ of $H^*_{DR}(M)$ so that

$$\int_M a_i \wedge b_j = \delta_{ij}.$$

Let $\pi_i : M\times M \to M$, $i=1,2$, be the projection maps onto the first and second factor.

$$\pi_1(p,q) = p, \quad \pi_2(p,q)=q, \quad (p,q)\in M\times M.$$

<u>Claim</u>: $\omega = \sum_{i=1}^{n} (-1)^{\deg(a_i)} \pi_1^*(a_i)\wedge \pi_2^*(b_i)$

(This result is called Diagonal Approximation.)

<u>Proof</u>: By the Künneth Theorem the cohomology $H^*_{DR}(M\times M)$ is generated by the set

$$\{ \pi_1^*(a_i)\wedge \pi_2^*(b_j) \mid \forall i,j \}.$$

Hence, $\omega = \sum c_{ij}\, \pi_1^*(a_i)\wedge \pi_2^*(b_j)$, for some $c_{ij}\in \mathbb{R}$.

Let $f : M \to M\times M$ denote the diagonal map $f(p) = (p,p)$. Then

$$\int_\Delta \pi_1^*(b_\ell)\wedge \pi_2^*(a_k) = \int_M f^*(\pi_1^*(b_\ell)\wedge \pi_2^*(a_k))$$

$$\left.\begin{array}{l}\pi_1\circ f = id_M\\ \pi_2\circ f = id_M\end{array}\right) \Rightarrow (\pi_i\circ f)^* = f^*\circ \pi_i^* = id_{H^*_{DR}(M)}$$

$$= \int_M b_\ell \wedge a_k$$

$$= (-1)^{\deg(a_k)\deg(b_\ell)} \int_M a_k\wedge b_\ell$$

$$= (-1)^{\deg(a_k)\deg(b_\ell)}\, \delta_{k\ell}.$$

On the other hand, since ω is the Poincaré dual of the submanifold Δ in $M \times M$ we have

$$\int_\Delta \pi_1^*(b_\ell) \wedge \pi_2^*(a_k) = \int_{M\times M} \pi_1^*(b_\ell) \wedge \pi_2^*(a_k) \wedge \omega$$

$$= \sum c_{ij} \int_{M\times M} \pi_1^*(b_\ell) \wedge \pi_2^*(a_k) \wedge \pi_1^*(a_i) \wedge \pi_2^*(b_j)$$

$$= \sum c_{ij} (-1)^{\deg(a_i)(\deg(a_k)+\deg(b_\ell))}$$

$$\int_{M\times M} \pi_1^*(a_i) \wedge \pi_1^*(b_\ell) \wedge \pi_2^*(a_k) \wedge \pi_2^*(b_j)$$

$$= \sum c_{ij} (-1)^{\deg(a_i)(\deg(a_k)+\deg(b_\ell))}$$

$$\int_{M\times M} \pi_1^*(a_i \wedge b_\ell) \wedge \pi_2^*(a_k \wedge b_j)$$

$$= \sum c_{ij} (-1)^{\deg(a_i)(\deg(a_k)+\deg(b_\ell))} \delta_{i\ell}\delta_{kj}$$

$$= c_{\ell k} (-1)^{\deg(\ell)(\deg(a_k)+\deg(b_\ell))}$$

Comparing the two results we obtain

$$c_{ij} = (-1)^{\deg(a_i)} \delta_{ij}.$$

This finishes the proof of the claim. $\blacksquare$

Finishing the proof of the theorem:

$$e(M) = \int_M e(T_*M) = \int_\Delta \omega$$

$$= \int_{\Delta} \sum_i (-1)^{\deg(a_i)} \pi_1^*(a_i) \wedge \pi_2^*(b_i)$$

$$= \sum_i (-1)^{\deg(a_i)} \int_{\Delta} \pi_1^*(a_i) \wedge \pi_2^*(b_i)$$

$$= \sum_i (-1)^{\deg(a_i)} \int_M f^*(\pi_1^*(a_i) \wedge \pi_2^*(b_i))$$

$$= \sum_i (-1)^{\deg(a_i)} \int_M a_i \wedge b_i$$

$$= \sum_i (-1)^{\deg(a_i)} \delta_{ii} \qquad H^*_{DR}(M) = \bigoplus_{k=0}^{n} H^k_{DR}(M)$$

$$= \sum_i (-1)^{\deg(a_i)}$$

$$= \sum_i (-1)^k b_k(M)$$

$$= \chi(M).$$

This finishes the proof of the theorem. ∎

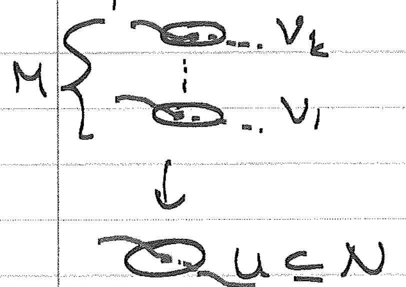
Videos 41-42

Note Title 4.05.2020

<u>Corollary</u>: Let $f: M \to N$ be a covering space of compact manifolds of degree k. Then $\chi(M) = k\,\chi(N)$, provided that M and N are oriented and f is orientation preserving.

<u>Proof</u>: By Poincaré–Hopf Theorem it is enough to show that $e(M) = k\, e(N)$.

$f: M \to N$, $f^{-1}(p) = \{q_1, \dots, q_k\}$.



$f: V_i \to U$ diffeomorphism.

$$\begin{tikzcd} f^*(T_*N) \arrow[r, "1"] \arrow[d, "2"'] & M \arrow[d, "f"] \\ T_*N \arrow[r, "\pi"'] & N \end{tikzcd}$$

$$f^*(T_*N) = \{(p, w) \in M \times T_*N \mid f(p) = \pi(w)\}$$

$$(p, w) \xmapsto{1} p, \quad (p, w) \xmapsto{2} w$$

<u>Claim</u>: The map $\phi: T_*M \to f^*(T_*N)$ defined by

$$\phi(p, v) = (p, Df_p(v)), \quad v \in T_pM,$$ is a

vector bundle isomorphism.

Now let $s: N \to T_*N$ be a section transverse to the zero section. Then the Euler number $e(N)$ is the signed count of s.

Then $\tilde{s}: M \to f^*(T_*N) \cong T_*M$, $\tilde{s}(p) = (p, s(f(p)))$

is a section of $f^*(T_x N)$. Since f is a local diffeomorphism for every zero of s, $\tilde{s}$ has exactly k zeros with the same sign. Then

$e(M) = k\, e(N)$, and this finishes the proof.

Lefschetz Fixed Point Theorem:

Let $f: M \to M$ be a smooth map of a compact oriented manifold. Let Γ_f be the graph of f in $M \times M$:

$$\Gamma_f = \{(p, f(p)) \in M \times M \mid p \in M\}.$$

$$\Delta = \{(p, p) \in M \times M \mid p \in M\}.$$

$\Delta \cap \Gamma_f$ the set of fixed points of f.

$\Lambda_f = \Gamma_f \pitchfork \Delta$ is a finite integer.

Theorem: $\Lambda_f = \sum_{k=0}^{n} (-1)^k \operatorname{Tr}(f^*: H^k_{DR}(M) \to H^k_{DR}(M))$

Remark: If f is identity. Then $\Lambda_f = \Gamma_f \pitchfork \Delta = \Delta \pitchfork \Delta$

$= e(M)$ and $\Lambda_f = \sum_{k=0}^{n} (-1)^k \underbrace{\operatorname{Tr}(\mathrm{Id}: H^k_{DR}(M) \to H^k_{DR}(M))}_{\dim H^k_{DR}(M) = b_k}$

$= \chi(M)$.

So the theorem reduces to the Poincaré-Hopf of $f = \mathrm{id}_M$.

Proof of the Theorem: Recall the form $\omega = \sum_i (-1)^{\deg(a_i)} \pi_1^*(a_i) \wedge \pi_2(b_i)$, from the proof of Poincaré-Hopf. Let $\phi: M \to M \times M$ be given by $\phi(p) = (p, f(p))$, which is clearly a diffeomorphism from M to its image Γ_f.

Then $\Gamma_f \pitchfork \Delta = (-1)^{\dim M} \Delta \pitchfork \Gamma_f$

$$= (-1)^{\dim M} \int_{\Gamma_f} \omega$$

$$= (-1)^{\dim M} \int_M \phi^*(\omega), \quad \phi: M \to \Gamma_f \text{ is a diffeomorphism}$$

Note that $f^*(b_i) = \sum_j \lambda_{ij} b_j$ for some $\lambda_{ij} \in \mathbb{R}$, ($\{b_i\}$ is a basis for $H^*_{DR}(M)$)

So, $\sum_k (-1)^k \operatorname{Tr}(f^*(H^k_{DR}(M) \to H^k_{DR}(M)) = \sum_i (-1)^{\deg(b_i)} \lambda_{ii}$.

Now let's continue the computation we started above

$$\Gamma_f \pitchfork \Delta = (-1)^{\dim M} \int_M \phi^*(\omega)$$

$$= \sum_i (-1)^{\dim M - \deg(a_i)} \int_M \phi^*(\pi_1^*(a_i) \wedge \pi_2^*(b_i))$$

$$= \sum_i (-1)^{\deg(b_i)} \int_M \phi^*(\pi_1^*(a_i)) \wedge \phi^*(\pi_2^*(b_i))$$

$$= \sum_i (-1)^{\deg(b_i)} \int_M 1_M^*(a_i) \wedge f^*(b_i)$$

$$= \sum_i (-1)^{\deg(b_i)} \int_M a_i \wedge \Big(\sum_j \lambda_{ij}\, b_j\Big)$$

$$= \sum_{i,j} (-1)^{\deg(b_i)} \int_M \lambda_{ij}\, a_i \wedge b_j$$

$$= \sum_{i,j} (-1)^{\deg(b_i)} \lambda_{ij}\, \delta_{ij}$$

$$= \sum_i (-1)^{\deg(b_i)} \lambda_{ii}\,,$$ which

finishes the proof. ∎

<u>Example</u>: Let M be a smooth manifold which compact and orientable. If $\chi(M) \neq 0$ and $f: M \to M$ is any smooth map homotopic to the identity, then f has a fixed point.

<u>Proof</u>: $f \underset{\text{homotopy}}{\simeq} id \Rightarrow f^* = id_{H^*_{DR}(M)}$

$\Lambda_f = \chi(M) \neq 0 \Rightarrow f$ has a fixed point.

∎

<u>Remark</u>: $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x+1$, has <u>no</u> fixed points, even though, $\chi(\mathbb{R}) = b_0(\mathbb{R}) - b_1(\mathbb{R}) = 1 - 0 = 1 \neq 0$.

Note that $f \simeq id$ via $f_t = x+t$, $t \in [0,1]$.

$f_0 = id$, $f_1 = f$.

Riemann-Hurwitz Theorem:

$f : \Sigma_1 \to \Sigma_2$ a holomorphic map between compact Riemann Surfaces. The set of critical points of f, say $C = \{p \in \Sigma_1 \mid f'(p) = 0\}$. Clearly, C is closed because

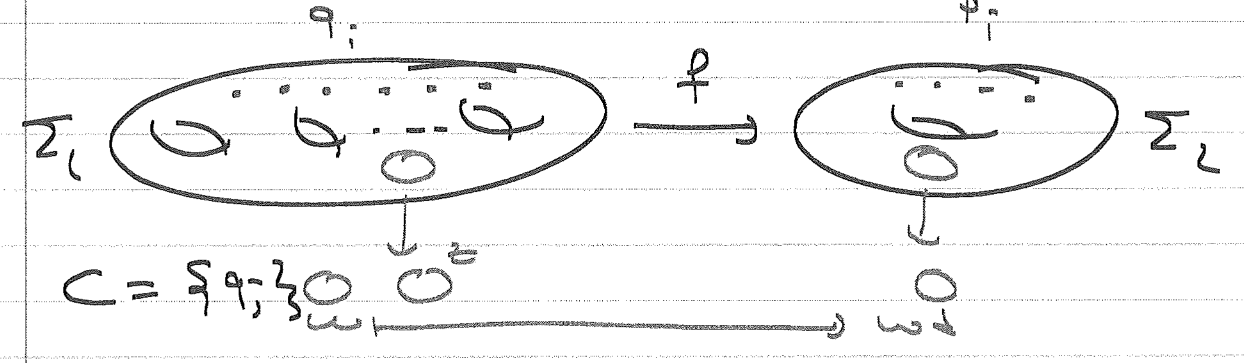
$$C = g^{-1}(0), \quad g = f'.$$

Since g is also holomorphic C cannot have an accumulation point. Since Σ_1 is compact and C is closed C should be a finite set. Hence, $f : \Sigma_1 \setminus C \to \Sigma_2 \setminus f(C)$ has no critical values.

On the other hand, any holomorphic map is open and thus $f : \Sigma_1 \to \Sigma_2$ is onto because,

$f(\Sigma_1)$ is both open and closed in Σ_2 (assuming both Σ_1 and Σ_2 connected).

Let $N = \deg(f)$. So $N = f^{-1}(q)$ for any regular value $q \in \Sigma_2 \setminus f(C)$, because since f is holomorphic every zero comes with sign $+1$ (f is orientation preserving).



For any $p \in \Sigma_1$ choose local coordinates at $p \in \Sigma_1$ and $q = f(p) \in \Sigma_2$ so that around p f is given by a power series as

$$f(z) = a_d z^d + a_{d+1} z^{d+1} + \cdots + a_n z^n + \cdots$$

$d \in \mathbb{N}$, $a_n \in \mathbb{C}$, $a_d \neq 0$.

$$f(z) = z^d h(z), \quad h(z) = a_d + a_{d+1} z + \cdots\cdots$$

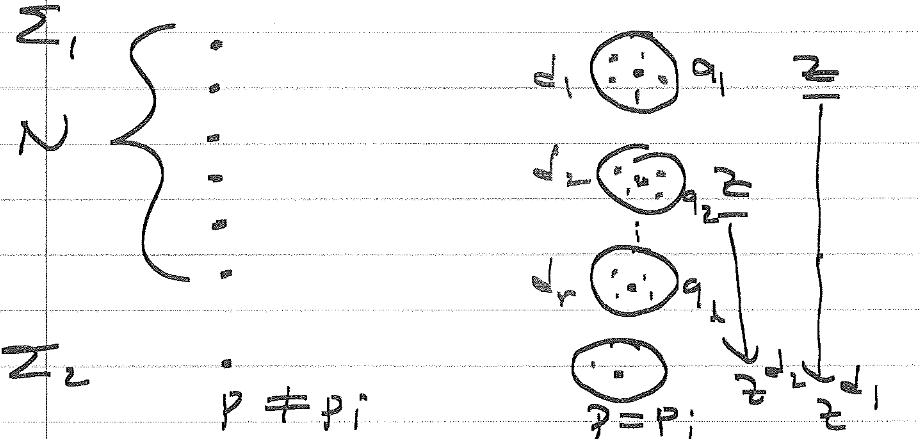
$$h(0) = a_d \neq 0, \quad f(z) = z^d e^{g(z)}, \text{ where } h(z) = e^{g(z)}$$

$$\Rightarrow f(z) = w^d, \quad w = z\, e^{g(z)/d},$$

$$w'(z) = e^{g(z)/d} + z\,(g(z)/d)'\, e^{g(z)/d}$$

$w'(0) = a_d + 0 \neq 0$. Hence, w has an holomorphic inverse so that $w = z e^{g(z)/d}$ is a holomorphic coordinate change. So if we replace w with $z\, e^{g(z)/d}$ we get

$$w \mapsto z \xrightarrow{f} w^d$$



$$f^{-1}(p) = \{q_1, \dots, q_n\}$$

d_i: Ramification index at q_i and denoted e_{q_i}

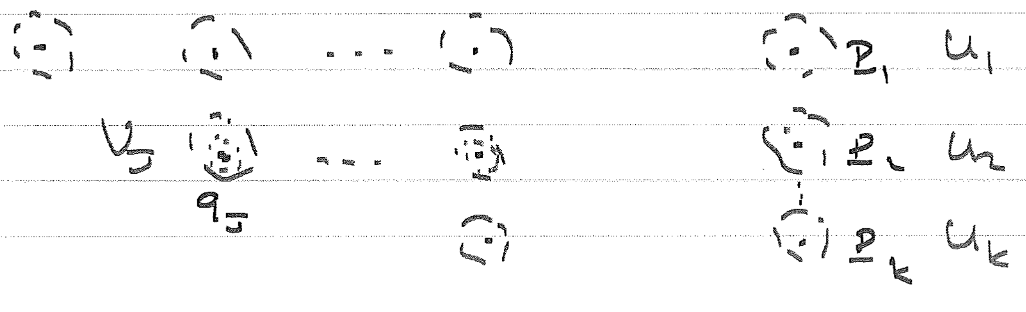
$$N = \deg f = \sum_{j=1}^{r} \deg(f)_{q_j} = \sum_{j=1}^{r} e_{q_j} = d_j$$

<u>Theorem</u> Assume the above set up. Then

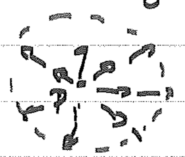
$$\chi(\Sigma_1) = \deg(f)\, \chi(\Sigma_2) - \sum_{q \in \Sigma_1} (e_q - 1),$$

where e_q is the local degree of f at q.

<u>Proof</u>: $\Sigma_1 \xrightarrow{f} \Sigma_2$



X_0 vector field on Σ_2 so that near each p_i it is given as $X(z) = z$, so that its index

U_i  is $+1$ and equals zero outside $\bigcup_{i=1}^{k} U_i$. Then perturb X_0 outside $\bigcup_{i=1}^{k} U_i$ so that it is

transverse to the zero vector field.

f is a local diffeomorphism outside $f^{-1}(R)$ so that we obtain a vector field $\tilde{X}_0$ on $\Sigma_2 - f^{-1}(R)$ s.t.

$$f(\tilde{X}_0(q)) \doteq X(f(q))$$

for all $q \in \Sigma_2 \setminus f^{-1}(R)$.

On each V_j, $f : V_j \longrightarrow f(V_j) = U_i$ is given by $z \longmapsto z^d$ for some d so that if we define $\tilde{X}(z) \doteq z/d$ on V_j we get

$$Df_z(\tilde{X}(z)) = d\, z^{d-1} \cdot \left(\frac{z}{d}\right) = z^d = X(z^d) = X(f(z)).$$

Hence, $\tilde{X}$ is well defined on all of Σ_1 and it satisfies $Df_q(\tilde{X}(q)) = X(f(q))$ at all $q \in \Sigma_1$.

The indices of zeros of $\tilde{X}$ at the points of $f^{-1}(R)$ are all $+1$, since $\tilde{X}$ is given by $z \mapsto z/d$ for some d. The indices of zeros of $\tilde{X}$ at a point q of $f^{-1}(p)$, $p \notin R$, is the same of that of X at p, because $f(q) = p$ and f is a diffeomorphism near q. Note that for each $p \notin R$ $f^{-1}(p)$ has exactly N points. Thus we get

$$\chi(\Sigma_1) - |f^{-1}(R)| = N\left(\chi(\Sigma_2) - |R|\right).$$

So, $\chi(\Sigma_1) = N\chi(\Sigma_2) + |f^{-1}(R)| - N|R|$

$$\Rightarrow \chi(\Sigma_1) = N\chi(\Sigma_2) + \sum_{q_i \in f^{-1}(R)} 1 - \underbrace{\Big(\sum_j e_{q_j}\Big)}_{\substack{\shortparallel \\ N \text{ for each } p_i}} k$$

$$= N\chi(\Sigma_2) + \sum_{q_i \in f^{-1}(R)} 1 - \sum_{q_i \in f^{-1}(R)} e_{q_i}$$

$$= N\chi(\Sigma_2) + \sum_{q_i \in f^{-1}(R)} (1 - e_{q_i})$$

$$= \deg(f)\,\chi(\Sigma_2) + \sum_{q \in \Sigma_1} (1 - e_q).$$

■

Theorem (Hurwitz)

Let Σ_g, $g \geq 2$ be a compact Riemann surface of genus g. If $G \leq \mathrm{Aut}(\Sigma_g)$ is a finite subgroup of holomorphic automorphisms of Σ_g then

$$|G| \leq 84(g-1).$$

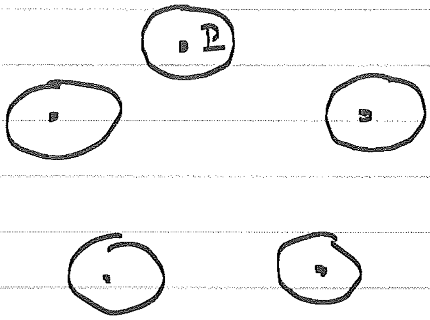
Proof: $f : \Sigma_g \to \Sigma_g$ continuous map. If f has infinitely fixed points, $\{p \in \Sigma_g \mid f(p) = p\}$, then since Σ_g is a compact space the closed set $\{p \in \Sigma_g \mid f(p) = p\}$ must have an accumulation point, say p_0. If f is analytic then the function defined locally near p_0 by $f(z) - z$ has infinitely many zeros converging to p_0 and thus $f(z) - z = 0$ on that open set.
$\Rightarrow f \equiv z$ on Σ_g. Hence, we see that any analytic function $f : \Sigma_g \to \Sigma_g$ can have

finitely many fixed points.

$G \leq \mathrm{Aut}(\Sigma_g)$ finite group. It follows that G has <u>no</u> fixed points on $\Sigma_g - C$, where C is a finite set. Hence G acts freely on $\Sigma_g \setminus C$. Moreover, G acts on C. Assume that the G-orbits of points of C are $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_n$.

$$C = \mathcal{O}_1 \cup \dots \cup \mathcal{O}_n.$$

$\mathcal{O}_i$

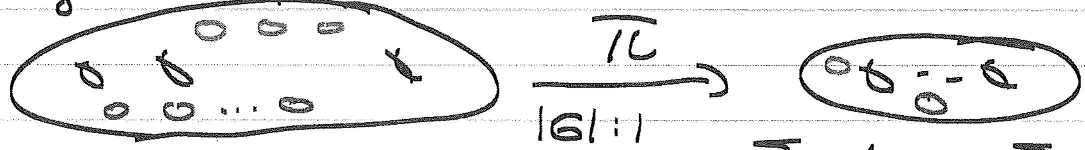


$$\mathrm{Stab}_G(p) \xrightarrow[\text{inj.}]{} GL(T_p\Sigma) = \mathbb{C} = \mathbb{C}^*$$

$\uparrow$ cyclic for all $p \in C$.

$$z \longmapsto z^d$$

Σ_g



$\Sigma_g / G \simeq \Sigma_h$

$|C| = N, \quad |\pi(C)| = n$

By the previous theorem

$$2 - 2g - N = |G|(2 - 2h - n)$$

$$\begin{aligned} 2g - 2 &= |G|(2h - 2 + n) - N \\ &= |G|(2h - 2 + n) - \sum_{i=1}^{n} |\mathcal{O}_i| \\ &= |G|(2h - 2 + n) - \sum_{i=1}^{n} \frac{|G|}{k_i}, \text{ where} \end{aligned}$$

$k_i = \mathrm{Stab}_G(p)$, $p \in Q_i$.

$$2g-2 = |G|\left(2h-2+n-\sum_{i=1}^{n}\frac{1}{k_i}\right)$$

$$= |G|\left(2h-2+\sum_{i=1}^{n}\left(1-\frac{1}{k_i}\right)\right)$$

$$|G| = \frac{2g-2}{2h-2+\sum_{i=1}^{n}\left(1-\frac{1}{k_i}\right)}$$

To find an upper bound for $|G|$ we need to find a lower bound for the sum

$$\sum_{i=1}^{n}(1-1/k_i), \text{ when } k_i \geq 2.$$

__Claim__ The lower bound for $2h-2+\sum_{i=1}^{n}\left(1-\frac{1}{k_i}\right)$ is

$$-2+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{2}{3} = \frac{1}{6} \quad (h=0,\ n=4)$$

for $n \geq 4$ and for $0 < n \leq 3$

$$-2+\frac{1}{2}+\frac{2}{3}+\frac{6}{7} = \frac{1}{42} \quad (h=0,\ n=3)$$

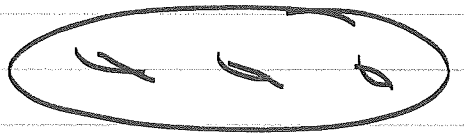
$$k_1 = \frac{1}{2} \quad k_2 = \frac{1}{3} \quad k_3 = \frac{1}{7}.$$

So by the claim $|G| \leq (2g-2)/\tfrac{1}{42} = 84(g-1)$.

__Remark:__ If we require that the number of branch points of the cover $\Sigma_g \to \Sigma_h$ is ≥ 4, then we get a better bound

$$|G| \leq (2g-2)/(1/6) = 12(g-1).$$

<u>Example</u> Klein Quartic: $x^3y + y^3z + z^3x = 0$ in $\mathbb{CP}^2$.

Σ_3: 

$G = \mathrm{Aut}(\Sigma_3)$
Holomorphic automorphisms of Σ_3.

$$|G| \leq 84(g-3) = 168.$$

$|G| = 168$ and G is the unique simple group of order 168.

Videos 43-44

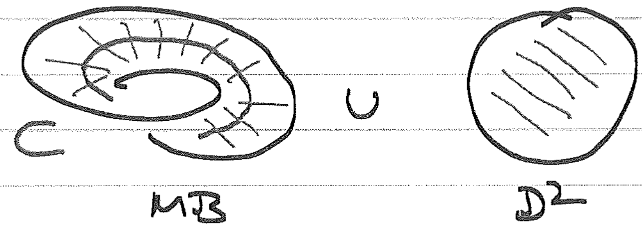
Note Title 10.05.2020

Some Applications:

1) $\mathbb{RP}^2$ does not embed into $\mathbb{R}^3$.

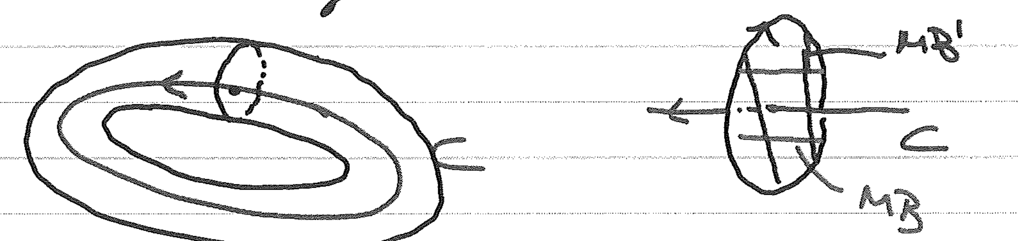
Proof: First assume that $\mathbb{RP}^2$ embeds into $\mathbb{R}^3$.

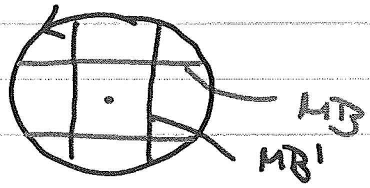
$\mathbb{RP}^2 = MB \cup_\partial D^2$



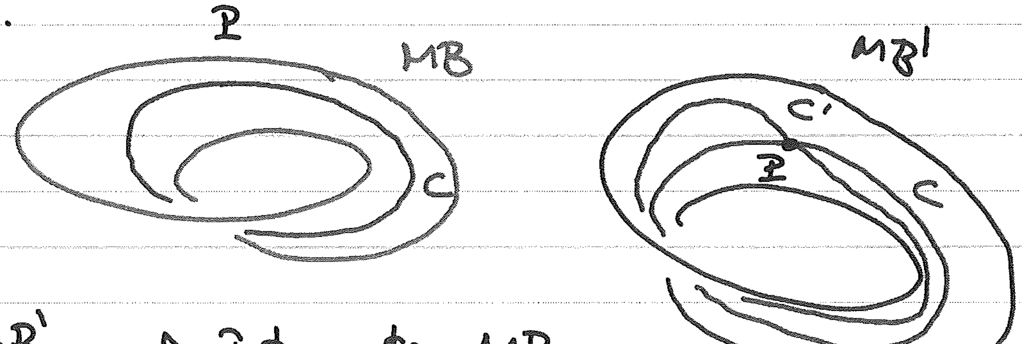
C: center of the Möbius Band.

$C \subseteq MB \subseteq \mathbb{RP}^2 \subseteq \mathbb{R}^3$, consider the tubular neighborhood ν of C in $\mathbb{R}^3$.





Since C and $\mathbb{R}^3$ are both oriented the disk bundle ν is oriented. Now rotate each disk 90° degrees using the orientation to get another copy of the Möbius band. Note the center circle do not move under rotation.



$C' \subseteq MB'$ and intersects MB at one point P, where C' is a copy of C

inside the rotated copy MB' of MB. Choosing the tubular neighborhood we see that C', which is a circle intersects MB and $\mathbb{RP}^2$ only at one point transversally. Hence the unoriented intersection of the closed manifolds $C' \cong S^1$ and $\mathbb{RP}^2$ in $\mathbb{R}^3$ is

$$\mathrm{Int}(C', \mathbb{RP}^2) = 1 \pmod 2$$

However, C' and $\mathbb{RP}^2$ are closed submanifolds of $\mathbb{R}^3$. Since $\mathbb{R}^3$ is unbounded by translating C' with vector we can make sure that C' and $\mathbb{RP}^2$ do not intersect at all. That is still a transversal intersection and thus

$$\mathrm{Int}(C', \mathbb{RP}^2) = 0 \pmod 2.$$

This is clearly a contradiction!

Hence, $\mathbb{RP}^2$ cannot be embedded inside $\mathbb{R}^3$.

Remark: 1) $\mathbb{RP}^2 \subseteq \mathbb{RP}^3$ where $\mathbb{RP}^3$ is also oriented, however and $\mathrm{Int}(C', \mathbb{RP}^2) = 1 \pmod 2$. Since $\mathbb{RP}^3$ is compact it is not possible to translate C' far enough so that C' and $\mathbb{RP}^2$ do not intersect any more.

2) $N = \mathbb{RP}^2 \times \mathbb{R}$, $\mathbb{RP}^2 \times \{p\} \longrightarrow \mathbb{RP}^2 \times \{p'\}$

$p \neq p' \Longrightarrow$ these two copies do not intersect.

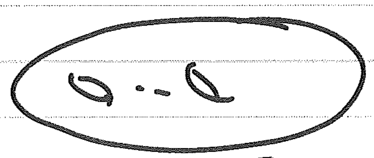
Q: Which parts of the above proof does not work in this case?

__Answer:__ The tubular neighborhood of the center circle C in $N = \mathbb{RP}^2 \times \mathbb{R}$ is not orientable. Therefore, rotating each disc $\pi/2$-radian counterclockwise is __not__ possible.

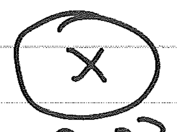
3) $\mathbb{R}^3 \subseteq S^3 = \mathbb{R}^3 \cup \{\infty\}$ and therefore $\mathbb{RP}^2$ does __not__ embed into S^3.

4) Now let N be any closed non-orientable surface. Then the above argument proves that N cannot embedded into $\mathbb{R}^3$ or S^3.

$$N = \Sigma_g \# \mathbb{RP}^2 \quad \text{or} \quad \Sigma_g \#_2 \mathbb{RP}^2$$



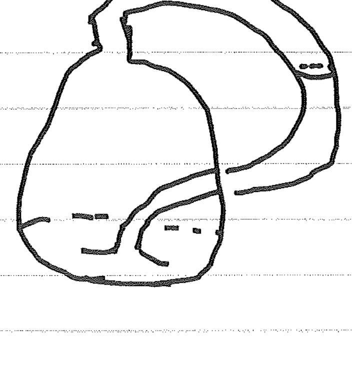
Σ_g



$\mathbb{RP}^2$

KB: $\Sigma_0 \#_2 \mathbb{RP}^2$

$\Rightarrow$ MB $\subseteq N$ and therefore we can repeat the above proof for N.



2) <u>Theorem</u> (Gauss-Bonnet Theorem)

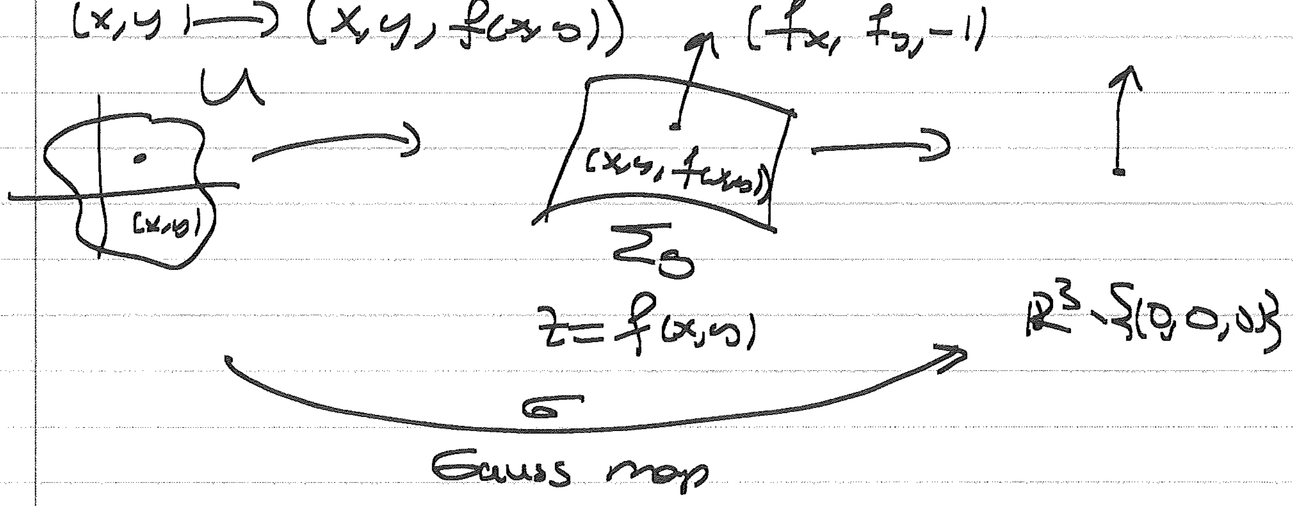
If Σ_g is a genus g orientable surface in $\mathbb{R}^3$ then

$$\int_{\Sigma_g} K(p)\,dS = 2\pi\,\chi(\Sigma_g) = 4\pi(1-g),$$

where $K : \Sigma_g \to \mathbb{R}$ is the Gaussian curvature function on Σ_g.

<u>Proof</u>: <u>Step 1</u>: $\sigma : U \to \mathbb{R}^3 \setminus \{(0,0,0)\}$

$U \subseteq \mathbb{R}^2$, $\sigma(x,y) = (f_x, f_y, -1)$ the Gauss map of the surface $\Sigma_g \subseteq \mathbb{R}^3$ parametrized by a local coordinate system



$H^2_{DR}(\mathbb{R}^3 \setminus \{(0,0,0)\}) \simeq \mathbb{R} = \langle [\omega] \rangle$, where

$$\omega = \frac{x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy}{(x^2+y^2+z^2)^{3/2}} \in \Omega^2(\mathbb{R}^3\setminus\{0\}).$$

$$\int_{S^2} \omega = 4\pi \quad \text{(Exercise!)}$$

$$dS = \sqrt{1+f_x^2+f_y^2}\, dx\, dy$$

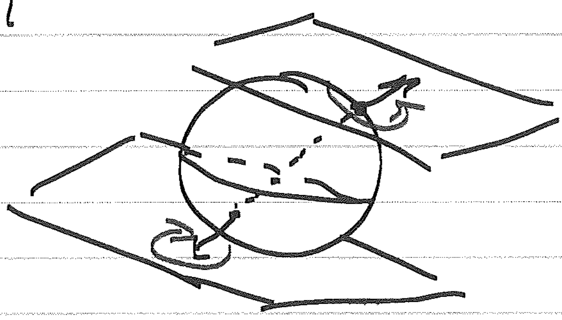
<u>Claim:</u> $\sigma^*(\omega) = K\, dS$, where

$$K(x,y) = \frac{f_{xx} f_{yy} - f_{xy}^2}{(1+f_x^2+f_y^2)^2} \quad \text{the Gaussian curvature.}$$

Proof is left as an exercise.

Step 2) $\Sigma_g \subseteq \mathbb{R}^3$, $\sigma : \Sigma_g \longrightarrow \mathbb{R}^3 \setminus \{(0,0,0)\}$ Gauss map

$\frac{\sigma}{\|\sigma\|} : \Sigma_g \longrightarrow S^2$: the space of oriented 2-planes in $\mathbb{R}^3$.



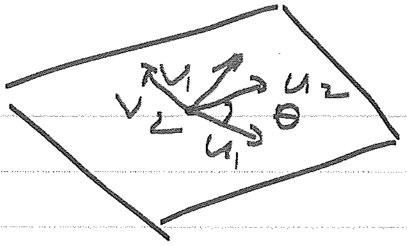
σ is homotopic to $\sigma/\|\sigma\|$, we can replace σ by $\sigma/\|\sigma\|$. So we assume that $\sigma : \Sigma_g \to S^2$.

$$\sigma : \Sigma_g \longrightarrow S^2 = Gr^+_{\mathbb{R}}(3,2) \subseteq Gr^+_{\mathbb{R}}(n,2)$$

$$\mathbb{R}^3 \subseteq \mathbb{R}^n$$
$$(x,y,z) \longmapsto (x,y,z,0,\dots,0)$$

$$Gr^+_{\mathbb{R}}(n,2) = \{(u,v) \in S^{n-1} \times S^{n-1} \mid u \perp v\} \big/ \sim$$

$$(u_1,v_1) \sim (u_2,v_2) \iff \begin{cases} u_2 = \cos\theta\, u_1 - \sin\theta\, v_1 \\ v_2 = \sin\theta\, u_1 + \cos\theta\, v_1 \end{cases}, \quad \theta \in \mathbb{R}$$



$Gr^+_{\mathbb{R}}(n,2)$ is a smooth manifold of dimension $2(n-2)$. (Exercise!)

We have a map $\Phi : Gr_{\mathbb{R}}(n,2) \to \mathbb{CP}^{n-1}$, by

$\Phi([u,v]) \doteq [u+iv]$ $\quad u+iv \in \mathbb{C}^n \setminus \{0\}$

$u, v \in \mathbb{R}^n$

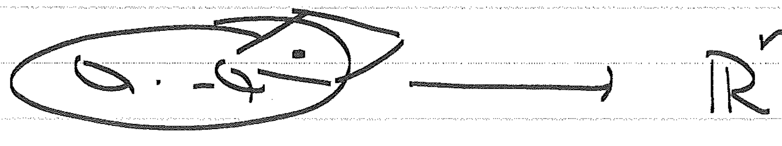
<u>Claim:</u> $\Phi(Gr_{\mathbb{R}}(n,2))$ is the quadratic hypersurface in $\mathbb{CP}^{n-1}$ given by $z_1^2 + z_2^2 + \cdots + z_n^2 = 0$.

<u>Ex</u> If $n=3$, $Gr_{\mathbb{R}}(3,2) \simeq S^2$, $\Phi(Gr_{\mathbb{R}}(3,2))$ is the quadratic curve in $\mathbb{CP}^2$ given by $z_1^2 + z_2^2 + z_3^2 = 0$.

Let $F : \Sigma_g \times [0,1] \to \mathbb{R}^n$ be a differentiable map.

$f_t \doteq F(-,t)$ $\quad f_t : \Sigma_g \to \mathbb{R}^n$ homotopy of maps.

Assume that each f_t is an immersion into $\mathbb{R}^n$.



$\sigma_t : \Sigma_g \to Gr_{\mathbb{R}}(n,2)$, $p \longmapsto Df_{t*}(T_p\Sigma_g)$

Consider the composition $\Phi \circ \sigma_t : \Sigma_g \to \mathbb{CP}^{n-1}$

$$\Sigma_g \xrightarrow{\sigma_t} Gr_{\mathbb{R}}(n,2) \xrightarrow{\Phi} \mathbb{CP}^{n-1}$$

but $a \in H^2_{DR}(\mathbb{CP}^{n-1})$ so that $\int_{\mathbb{CP}^1} a = 1/2$.

$\Rightarrow \int_{\Phi(Gr_{\mathbb{R}}(3,2))} a = 1$ because $\Phi(Gr_{\mathbb{R}}(3,2)) \to \mathbb{CP}^1$ is a double cover

$\Rightarrow K\, dS = \sigma^*(\omega) = 4\pi (\Phi \circ \sigma_t)^*(a)$ as cohomology classes.

<u>Conclusion</u>: For any two immersions of Σ_g into $\mathbb{R}^n$ the integral

$$\int_{\Sigma_g} K\, dS \quad \text{gives the same result.}$$

<u>Step 3)</u> <u>Proposition</u>: Any two immersions of Σ_g into $\mathbb{R}^n$ $(n \geq 7)$ are homotopic through immersions.

<u>Proof</u>: The vector space of all polynomials in $\mathbb{R}[x,y,z]$ of degree at most d has dimension $s = \binom{3+d}{d}$. Take any point $P = (x_0, y_0, z_0) \in \mathbb{R}^3$.

By the linear change of coordinates
$(x,y,z) \longrightarrow (x-x_0, y-y_0, z-z_0)$ we can
assume that $P = (0,0,0)$.

Let $f_1, \dots, f_{k+3} \in \mathbb{R}^S$, the vector space of polynomials
in x,y,z of degree $\leq d$.

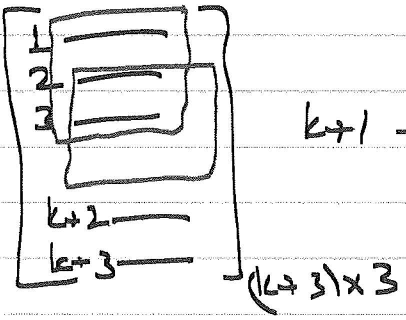
$$\phi = (f_1, \dots, f_{k+3}) : \mathbb{R}^3 \longrightarrow \mathbb{R}^{k+3}$$

The condition that $(0,0,\dots,0)$ is a critical point for ϕ is a linear condition on the first degree terms of f_i's.

$$D\phi(0,\dots,0) = \begin{bmatrix} \nabla f_1(0,0,\dots,0) \\ \vdots \\ \nabla f_{k+3}(0,0,\dots,0) \end{bmatrix}, \quad \nabla f_i(0) = \left(\frac{\partial f_i}{\partial x}(0), \frac{\partial f_i}{\partial y}(0), \frac{\partial f_i}{\partial z}(0)\right)$$

$$f_i = a_0 + a_1 x + a_2 y + a_3 z + O(2), \quad \nabla f_1(0) = (a_1, a_2, a_3)$$

$(0,0,\dots,0)$ is a critical point for Φ if and only if the matrix $D\Phi(0)$ has rank ≤ 2.



$k+1$ - independent conditions.

Hence, the subspace of all $(f_1, \dots, f_{k+3})$ in $\mathbb{R}^{S(k+3)}$ having $(0,0,\dots,0)$ as a critical point have codimension $k+1$.

Let $E = \{((x,y,z), f_1, f_2, \ldots, f_{k+3}) \in \mathbb{R}^3 \times \mathbb{R}^{s(k+3)} \mid$

$\mathrm{rk}(D(f_1, \ldots, f_{k+3})_{(x,y,z)}) \leq 2\}$.

$\pi : E \longrightarrow \mathbb{R}^3, \; ((x,y,z), f_1, \ldots, f_{k+3}) \mapsto (x,y,z)$.

All the fibers of π have the same structure and they are unions of $\binom{k+3}{2}$ linear subspaces of codim $k+1$.

Thus the set of all polynomial maps

$$\phi = (f_1, \ldots, f_{k+3}) : \mathbb{R}^3 \longrightarrow \mathbb{R}^{k+3},$$

which are not an immersion at some point, form a set in $\mathbb{R}^{s(k+3)}$ of codimension

$$(k+1) - 3 = k-2.$$

Hence, if $k \geq 4$ the set of all polynomial immersions $\phi = (f_1, \ldots, k+3) : \mathbb{R}^3 \longrightarrow \mathbb{R}^{k+3}$ is path connected because $k \geq 4 \Rightarrow k-2 \geq 4-2 = 2$.

In particular, all immersions (polynomial)

$$\mathbb{R}^3 \longrightarrow \mathbb{R}^7 \text{ is path connected.}$$

So for our surface $\Sigma_g \subseteq \mathbb{R}^3$ restriction of any immersion $\mathbb{R}^3 \to \mathbb{R}^7$ to Σ_g is also an immersion. Hence, the space of all polynomial immersions of Σ_g into $\mathbb{R}^7$ is path connected.

$\Sigma_g \overset{\phi_0}{\looparrowright} \mathbb{R}^3$, $\Sigma_g \overset{\phi_1}{\looparrowright} \mathbb{R}^3$ two immersions $\Rightarrow \exists\ \phi_t$ homotopy so that each $\phi_t : \Sigma_g \looparrowright \mathbb{R}^3$ is an immersion.

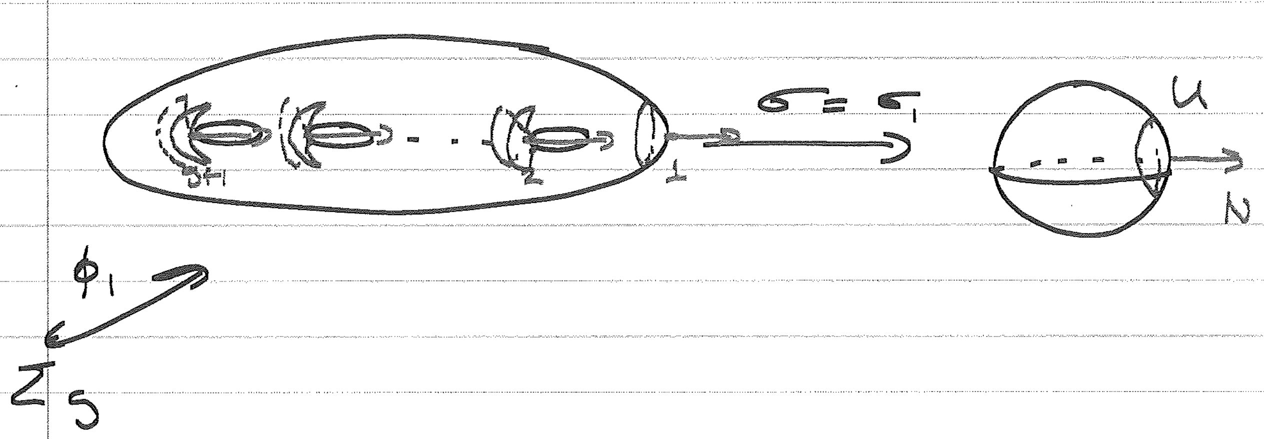
In particular, any two embeddings
$\phi_0 : \Sigma_g \hookrightarrow \mathbb{R}^3$, $\phi_1 : \Sigma_g \hookrightarrow \mathbb{R}^3$ are homotopic through immersions into $\mathbb{R}^3$.

$$\int_{\Sigma_g} \sigma_0^*(\omega) = \int_{\Sigma_g} \sigma_1^*(\omega),$$ where σ_i is the Gauss map corresponding to the embedding ϕ_i.

videos 45-46

Note Title 13.05.2020

Step 4 $\phi_0 : \Sigma_g \hookrightarrow \mathbb{R}^3$, $\phi_1 : \Sigma_g \hookrightarrow \mathbb{R}^3$ two embeddings. ϕ_0 given embedding of Σ_g. ϕ_1 is the embedding looks like



$$\int_{\Sigma_g} \kappa \, dS = \int_{\Sigma_g} \sigma^*(\omega)$$

Replace ω be a form so that it is supported in U with the same integral.

$\sigma^{-1}(U) = V_1 \cup V_2 \cup \cdots \cup V_{g+1}$ disjoint open sets.

$\sigma_i : V_i \to U$ is a diffeomorphism.

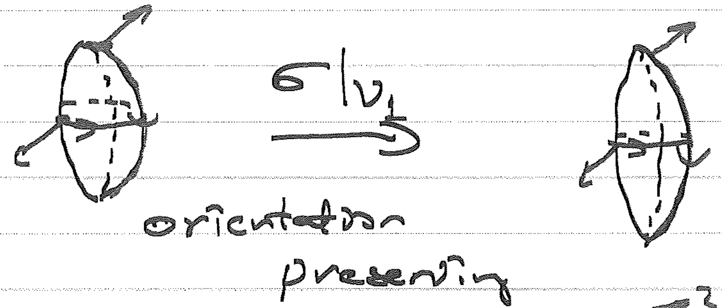
Then $\displaystyle\int_{\Sigma_g} \sigma^*(\omega) = \int_{\sigma^{-1}(U)} \sigma^*(\omega) = \sum_{i=1}^{g+1} \int_{V_i} \sigma^*(\omega)$,

where each $\displaystyle\int_{V_i} \sigma^*(\omega) = \pm \int_U \omega = \pm \int_{S^2} \omega = \pm 4\pi$

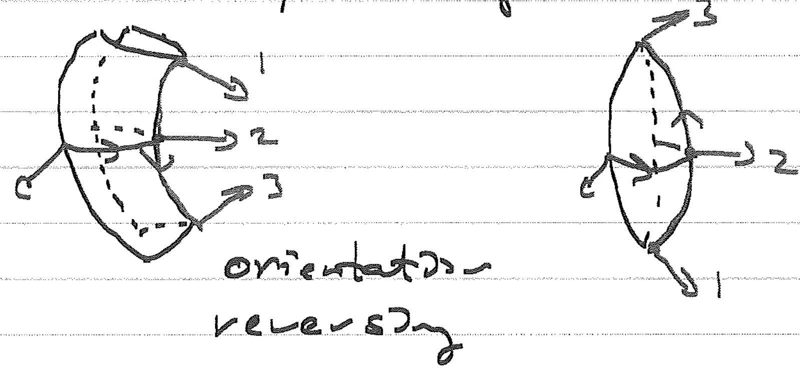
and the sign is ± 1 depending on whether

$\sigma : V_i \to U$ is orientation preserving or not.

V_1 :



V_i : $i \geq 2$



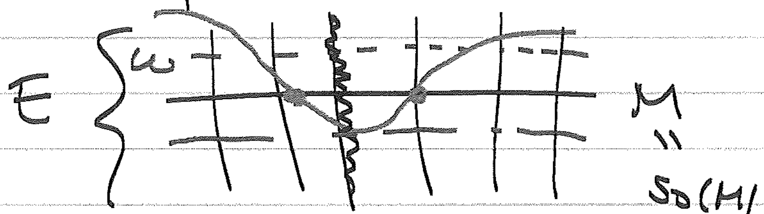
Hence, $\displaystyle\int_{\Sigma_g} \kappa(p)\, dS = \int_{\Sigma_g} \sigma^*(\omega) = 4\pi - g\, 4\pi = 2\pi\, \chi(\Sigma_g).$ ∎

CHARACTERISTIC CLASSES

Euler class $\mathbb{R}^k \to E$ oriented vector bundle
$$\begin{array}{c} E \\ \downarrow \\ M \end{array}$$

Let $s_0 : M \to E$ be the zero section.

$e(E)$: Poincaré dual of the zero section $s_0(M)$ in E.



$e(E) = [\omega]$, with $\operatorname{supp}(\omega)$ lies in a tubular neighborhood whose integral along any fiber (oriented) is equal 1.

$\omega \in \Omega^k(M)$ $\quad e(E) \in H^k_{DR}(M)$.

Some Properties of the Euler Class:

1) $E_i \to M$ $i=1,2$, oriented vector bundles.

Then $E_1 \oplus E_2 \to M$ is also an oriented vector bundle.

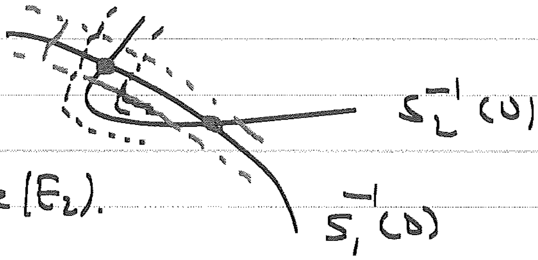
Proposition $e(E_1 \oplus E_2) = e(E_1) \cdot e(E_2)$.

Proof: $s_i : M \to E_i$ section $i = 1,2$.

$(s_1, s_2) : M \to E_1 \oplus E_2$ section.

$(s_1, s_2)^{-1}(0) = s_1^{-1}(0) \cap s_2^{-1}(0)$

Hence, $e(E_1 \oplus E_2)$ is the Poincaré dual of the intersection of the submanifolds $s_1^{-1}(0)$ and $s_2^{-1}(0)$.



Hence, $e(E_1 \oplus E_2) = e(E_1)\, e(E_2)$.

2) $f : M \to N$ smooth map, $E \to N$ oriented vector bundle. Then

<u>Proposition</u> $e(f^*(E)) = f^*(e(E))$

<u>Proof</u>

$$\begin{tikzcd} f^*(E) \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \arrow[d, bend right, leftarrow, "s"'] \\ M \arrow[r, "f"] \arrow[u, bend left, "\tilde{s}"] & N \arrow[u, bend right=60, "s"'] \end{tikzcd}$$

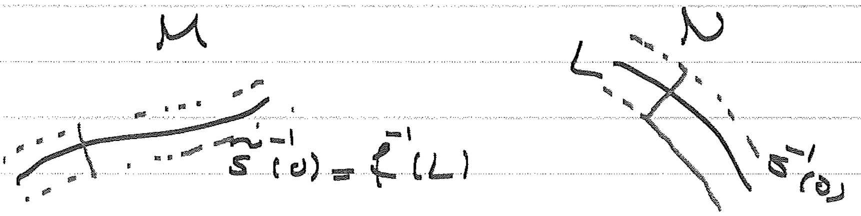
$$p \longmapsto f(p)$$

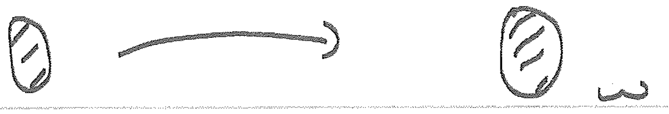
$f^*(E) = \{(p, v) \in M \times E \mid f(p) = \pi(v)\}$.

$\tilde{s}(p) \doteq (p, s(f(p)))$

Hence, $\tilde{s}^{-1}(0) = f^{-1}(s^{-1}(0))$.

Hence, choosing f and s transverse to each other $\tilde{s}^{-1}(0)$ is a submanifold in M.





$\int \omega = 1$

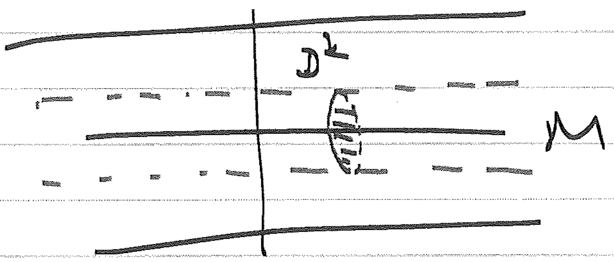
$\int f^*(\omega) = 1$

This proves the result.

3) For any oriented vector bundle E let $-E$ denote the bundle with opposite orientation. Then

$$e(-E) = -e(E).$$

<u>Proof</u>: E:



$$\int_{D^k} \omega = 1$$

$$\int_{-D^k} -\omega = +1 \qquad e(-E) = [-\omega] = -[\omega] = -e(E).$$

4) $E \to M$ oriented vector bundle and let $E^* \to M$ be the dual of $E \to M$.

$E^* = \mathrm{hom}(E, \mathbb{R})$.

Then $e(E^*) = (-1)^{\mathrm{rank}(E)/2} \, e(E)$.

<u>Proof</u>:
$$\begin{array}{c} \mathbb{C} \to L \\ \downarrow \\ M \end{array}$$
complex line bundle

$\mathbb{C} = \mathbb{R}^2 \Rightarrow$ We may regard $L \to M$ as an oriented $\mathbb{R}^2$-bundle.

$L^* = \hom(L, \mathbb{C}) \longrightarrow M$.

$\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathbb{C}^*$ transition function for L.

$\varphi_{\alpha\beta}^{-1} : U_\alpha \cap U_\beta \longrightarrow \mathbb{C}^*$, $\varphi_{\alpha\beta}^{-1}(x) = (\varphi_{\alpha\beta}(x))^{-1}$

transition function for L^*.

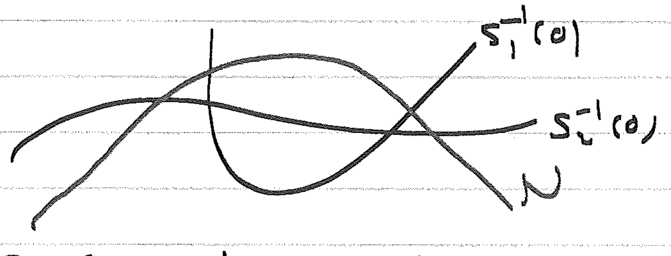
$L_1, L_2 \longrightarrow M$ cx. line bundle.

$L_1 \otimes L_2 \longrightarrow M$ another complex bundle whose transition function is the product of the transition functions of L_1 and L_2.

$s_i : M \longrightarrow L_i$ section of L_i.

$s_1 \cdot s_2 : M_1 \longrightarrow L_1 \otimes L_2$ section of $L_1 \otimes L_2$.

$(s_1 \cdot s_2)^{-1}(0) = s_1^{-1}(0) \cup s_2^{-1}(0)$.



$\mathrm{Int}(N, s_1^{-1}(0) \cup s_2^{-1}(0)) = \mathrm{Int}(N, s_1^{-1}(0)) + \mathrm{Int}(N, s_2^{-1}(0))$

$\Rightarrow PD(s_1^{-1}(0) \cup s_2^{-1}(0)) = PD(s_1^{-1}(0)) + PD(s_2^{-1}(0))$

$e(L_1 \otimes L_2) = e(L_1) + e(L_2)$.

$L \otimes L^* = \xi \longrightarrow M$ is the trivial bundle.

$0 = e(L \otimes L^*) = e(L) + e(L^*)$.

$e(L^*) = -e(L)$

$$e((L_1 \oplus L_2 \oplus \cdots \oplus L_k)^*) = e(L_1^* \oplus \cdots \oplus L_k^*)$$

$$= e(L_1^*) \cdots e(L_k^*)$$

$$= (-1)^k e_1(L_1) \cdots e(L_k)$$

$$= (-1)^k e(L_1 \oplus \cdots \oplus L_k).$$

Hence for an oriented vector bundle E of rank $2n$ we take orientation of E^* as follows,

$$e_1, \ldots, e_{2n}, \longrightarrow (-1)^n e_1^*, e_2^*, \ldots, e_{2n}^*.$$

$$e(E^*) = (-1)^n e(E), \quad \operatorname{rank}(E) = 2n.$$

<u>Special Case:</u> $T_*M \longrightarrow M$ tangent bundle.

Then T_*M as a smooth manifold is oriented, with orientation:

$\underbrace{x_1, \ldots, x_n}_{\text{on } M}, \underbrace{a_1, \ldots, a_n}$ coor. system on T_*M.

$$a_i\left(\sum \xi_i \frac{d}{dx_i}(p)\right) = \xi_i.$$

<u>Orientation of T^*M.</u>

$$\left.\begin{array}{l} x_1, \ldots, x_n, b_1, \ldots, b_n \\ b_i\left(\sum \xi_i \, dx_i\right) = \xi_i \end{array}\right\} \text{This gives an orientation on } T^*M.$$

However, this is <u>not</u> compatible with the orientation we considered above.

Instead, we take as the canonical orientation on the cotangent bundle as

$x_1, b_1, x_2, b_2, \ldots, x_n, b_n.$

<u>Remark:</u> 1) The difference of orientations given by $x_1, \ldots, x_n, b_1, \ldots, b_n$ and

$x_1, b_1, x_2, b_2, \ldots, x_n, b_n$ is $(-1)^{n(n-1)/2}$

2) T^*M has a canonical symplectic structure given by

$dx_1 \wedge db_1 + dx_2 \wedge db_2 + \ldots + dx_n \wedge db_n.$

Videos 47-48

Note Title 15.05.2020

CHERN CHARACTERISTIC CLASSES

$\pi : E \to M$, E complex vector bundle of rank r.

$r = 1$, $\pi : L \to M$ complex line bundle.

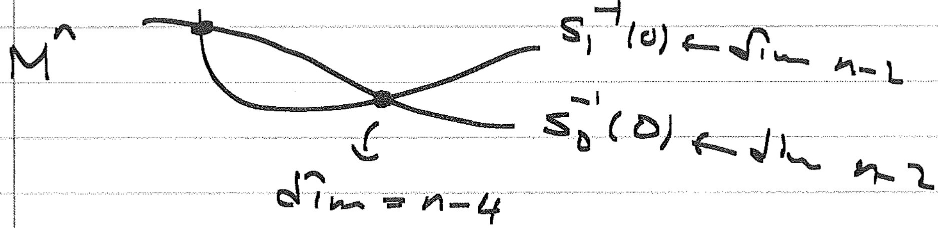
$\mathbb{C} \cong \mathbb{R}^2 \Rightarrow L \to M$ can be viewed as an oriented $\mathbb{R}^2$-bundle, denoted $L_{\mathbb{R}} \to M$.

$\left(GL^+(2,\mathbb{R}) \xrightarrow{h.e.} U(1) = S^1 \subset \mathbb{C}^* \right)$

The first Chern class of L is defined as

$c_1(L) \doteq e(L_{\mathbb{R}})$

Let $s_0, s_1, \ldots, s_k$ be sections of $L \to M$ intersecting pairwise transversally. If $2(k+1) > n$ then these sections have no common zeros.



$n - 2(k+1) < 0 \Rightarrow \bigcap_{i=0}^{k} s_i^{-1}(0) = \phi$.

Let $f : M \to \mathbb{CP}^k$ be defined by $p \mapsto [s_0(p) : \ldots : s_k(p)]$.

$i = 0, \ldots, k$, $U_i = \{p \in M \mid s_i(p) \neq 0\} \subseteq M$ open subset

Clearly, $M = U_0 \cup U_1 \cup \cdots \cup U_k$.

$$\begin{tikzcd}
L|_{U_i} \simeq U_i \times \mathbb{C} \arrow[r] \arrow[d, bend right] & \mathbb{C}^{k+1} \setminus \{0\} \arrow[d, bend left] \\
U_i \arrow[r, "f|_{U_i}"'] & \mathbb{C}P^k
\end{tikzcd}$$

$$(p, v) \longmapsto \left(\frac{s_0(p)}{s_i(p)}, \dots, 1, \dots, \frac{s_k(p)}{s_i(p)} \right)$$

Then it follows that L is isomorphic to $f^*(\xi_k)$ where ξ_k is the tautological line bundle over $\mathbb{C}P^k$.

$$\begin{array}{ccc}
\mathbb{C}^* \to \xi_k = \mathbb{C}^{k+1} \setminus \{0\} & & (z_0, \dots, z_k) \\
\downarrow & & \downarrow \\
\mathbb{C}P^k & \ni & [z_0 : \dots : z_k]
\end{array}$$

The function $f: M \to \mathbb{C}P^k$ is called a classifying map for the line bundle $L \to M$.

$[x_0 : \dots : x_k] \in \mathbb{C}P^k, [y_0 : \dots : y_\ell] \in \mathbb{C}P^\ell \to [z_0 : \dots : z_{k+\ell}] \in \mathbb{C}P^{\ell+k}$

$$\sum_{i=0}^{k} x_i t^i \qquad \sum_{i=0}^{\ell} y_i t^i \longmapsto \sum_{i=0}^{k+\ell} z_i t^i,$$

where $\displaystyle\sum_{i=0}^{k+\ell} z_i t^i = \left(\sum_{i=0}^{k} x_i t^i\right)\left(\sum_{i=0}^{\ell} y_i t^i\right)$

$z_0 = x_0 y_0$, $z_1 = x_0 y_1 + x_1 y_0$, $z_2 = x_0 y_2 + x_1 y_1 + x_2 y_0$, ...

$z_{k+\ell} = x_k y_\ell$.

This gives an embedding:

x_i $\quad$ y_j

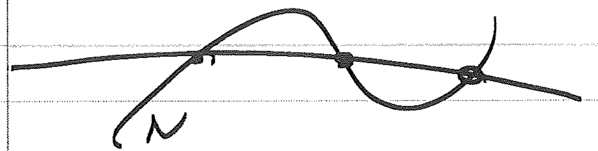
$$\phi : \mathbb{CP}^k \times \mathbb{CP}^l \hookrightarrow \mathbb{CP}^{k+l}$$

$$\phi^* : H^2_{DR}(\mathbb{CP}^{k+l}) \longrightarrow H^2_{DR}(\mathbb{CP}^k \times \mathbb{CP}^l)$$
$$\begin{array}{c} \Psi \\ a \longmapsto ? \end{array}$$

Let $H = (z_0 = 0)$ be the hyperplane in $\mathbb{CP}^{k+l}$ given $z_0 = 0$. Then the Poincaré dual of H

$$PD(H) = a \in H^2_{DR}(M) \quad \text{so that}$$

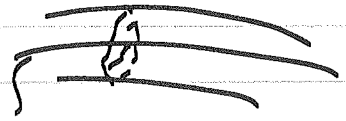
$$\int_N a = \mathrm{Int}(H \cap N), \quad \text{for any oriented submanifold } N \text{ of dimension } 2.$$



N $\quad$ H

$$\phi^{-1}(H) = \phi^{-1}(z_0 = 0) = \phi^{-1}(x_0 y_0 = 0)$$
$$= \phi^{-1}(x_0 = 0) \cup \phi^{-1}(y_0 = 0)$$
$$= \{x_0 = 0\} \times \mathbb{CP}^l \cup \mathbb{CP}^k \times \{y_0 = 0\}$$

$\phi^*(a)$ is the Poincaré dual of this union of submanifolds.

$\mathbb{CP}^k$  $\times \mathbb{CP}^l$

$x_0 = 0$

Poincaré dual of $\{x_0 = 0\} \times \mathbb{CP}^l$ is $a \otimes 1$ and Poincaré " " $\mathbb{CP}^k \times \{y_0 = 0\}$ is $1 \otimes a$.

Hence, $\phi^*(a) = a \otimes 1 + 1 \otimes a$.

Recall that $H^*_{DR}(\mathbb{CP}^k \times \mathbb{CP}^l) = H^*_{DR}(\mathbb{CP}^k) \otimes H^*_{DR}(\mathbb{CP}^l)$

Also consider the map $\bar{J} : M \longrightarrow M \times M$, $p \longmapsto (p,p)$, $p \in M$. Then we have

$\bar{J}^*(u \otimes 1) = u$, $\bar{J}^*(1 \otimes v) = v$, $\bar{J}^*(u \otimes 1 + 1 \otimes v) = u + v$.

<u>Exercise</u>: Prove this using projections

$pr_1 : M \times M \longrightarrow M$, $(p,q) \longmapsto p$ and

$pr_2 : M \times M \longrightarrow M$, $(p,q) \longmapsto q$.

— ∪ —

Let $L_1 \longrightarrow M$ and $L_2 \longrightarrow M$ be two complex line bundles over M. Also let $s_i : M \to L_i$, $i = 1,2$, be sections of L_i.

Then $s(p) = s_1(p)\, s_2(p)$ is a section of $L_1 \otimes L_2 \longrightarrow M$.

Let $f : M \to \mathbb{CP}^k$ and $g : M \to \mathbb{CP}^l$ be two classifying maps for the bundles $L_1 \longrightarrow M$ and $L_2 \longrightarrow M$, respectively:

$f = (s_0, \dots, s_k)$, $\bigcap_i s_i^{-1}(0) = \emptyset$

$g = (\tilde{s}_0, \dots, \tilde{s}_l)$, $\bigcap_i \tilde{s}_i^{-1}(0) = \emptyset$

$\phi \circ (f,g) \circ J : M \longrightarrow \mathbb{CP}^{k+l}$

$p \xmapsto{J} (p,q) \xmapsto{(f,g)} \mathbb{CP}^k \times \mathbb{CP}^l \xrightarrow{\phi} \mathbb{CP}^{k+l}$

$\phi \circ (f \circ g) \circ J\,(p) = [s_0(p)\tilde{s}_0(p) : s_0(p)\tilde{s}_1(p) + s_1(p)\tilde{s}_0(p) : \dots]$

which is a classifying map for the line bundle

$L_1 \otimes L_2 \longrightarrow M.$

Let $x = c_1(\xi_r) = e(\xi_r)$ of the 1st Chern class of the line bundle $\xi_r \longrightarrow \mathbb{CP}^r$.

$$c_1(L_1 \otimes L_2) = e\left((L_1 \otimes L_2)_{\mathbb{R}}\right)$$

$$= e\left((\phi \circ (f,g) \circ J)^*(\xi_{r\,\mathbb{R}}) \right)$$

$$= (\phi \circ (f,g) \circ J)^* e(\xi_{r\,\mathbb{R}})$$

$$= (\phi \circ (f,g) \circ J)^*(x)$$

$$= J^* \circ (f,g)^* (\phi^*(x))$$

$$= J^* \circ (f^*, g^*)(x \otimes 1 + 1 \otimes x)$$

$$= J^* \left(f^*(x) \otimes \underbrace{g^*(1)}_{=1} + \underbrace{f^*(1)}_{=1} \otimes g^*(x) \right)$$

$$= f^*(x) + g^*(x)$$

$$= f^*(c_1(\xi_k)) + g^*(c_1(\xi_\ell))$$

$$= c_1(f^*(\xi_k)) + c_1(g^*(\xi_\ell))$$

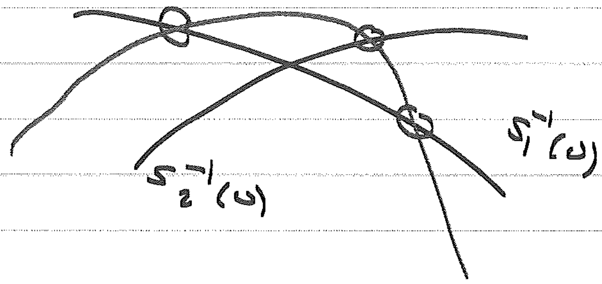
$$= c_1(L_1) + c_1(L_2).$$

$\therefore \quad c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2).$

Remark $s_i : M \longrightarrow L_i$ section

$s = s_1 \cdot s_2 : M \longrightarrow L_1 \otimes L_2$ section

$s^{-1}(0) = s_1^{-1}(0) \cup s_2^{-1}(0)$



Proposition: If k, n are positive integers with $2(k+1) > n+1$ and then there is a 1-1 correspondence between the homotopy classes of smooth maps from M to $\mathbb{CP}^k$ and the isomorphism classes of complex line bundles over M:

$$[M, \mathbb{CP}^k] \longrightarrow \mathcal{L}(M)$$

$$[f : M \to \mathbb{CP}^k] \longmapsto f^*(\xi_k)$$

$$\begin{array}{ccc} \mathbb{C} & \to & \xi_k \\ & & \downarrow \\ & & \mathbb{CP}^k \end{array}$$

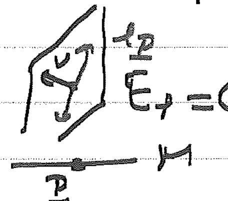
Definition of higher Chern Classes

$r > 1$, $\pi^0 : E \longrightarrow M$ complex vector bundle of rank r.

$$\begin{array}{ccc} \mathbb{C}^r & \to & E \\ & & \downarrow \pi^0 \\ & & M \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} \mathbb{C}^r \setminus \{0\} / \mathbb{C}^* = \mathbb{CP}^{r-1} & \longrightarrow & P(E) \\ & & \downarrow \pi \\ & & M \end{array}$$

$$\begin{array}{ccc} \pi^*(E) & \longrightarrow & E \\ \downarrow & & \downarrow \pi^0 \\ P(E) & \underset{\pi}{\longrightarrow} & M \end{array}$$

$$\begin{array}{ccc} \mathbb{C}^r \to \pi^*(E) = \{(\ell_p, v) \in \mathbb{P}(E)\times E \mid \pi(\ell_p) = \pi^\circ(v)\}, & \ell_p \in (E_p\setminus\{0\})/\mathbb{C}^* \\ \downarrow \qquad\qquad \big\updownarrow & p = \pi^\circ(v) \\ \mathbb{P}(E) \qquad\qquad \ell_p & \end{array}$$



The vector bundle $\pi^*(E) \xrightarrow{\pi} \mathbb{P}(E)$ has a natural line subbundle:

$$L = \{(\ell_p, v) \in \pi^*(E) \mid v \in \ell_p\} \longrightarrow \mathbb{P}(E)$$

$\ell_p \subseteq E_p$ rank 1 subspace. Let $Q_p \doteq E_p/\ell_p$ be the quotient vector space. Then we have the following sequence of vector bundles

$$0 \to L \to \pi^*(E) \to Q \to 0$$

<u>Exercise</u>: Construct $Q \to \mathbb{P}(E)$ explicitly using <u>transition functions</u>.

Now define $a \doteq c_1(L^*) \in H^2_{DR}(\mathbb{P}(E))$.

Note that $c_1(L) = -a$.

$$\begin{array}{c} \mathbb{CP}^{r-1} \to \mathbb{P}(E) \\ \downarrow \\ M \end{array} \qquad\qquad \begin{array}{ccc} L| & & L \\ \downarrow & & \downarrow \\ \mathbb{CP}^{r-1} & \to & \mathbb{P}(E) \\ & & \downarrow \\ & & M \end{array}$$

Recall that the Euler class and thus the 1st Chern class of a cx. line bundle are natural. Thus the first Chern class of the restriction of L to any fiber of $\mathbb{P}(E_p)$ is $-a$. Since $L| \to \mathbb{CP}^{r-1}$ is the tautological bundle

$a \in H^2_{DR}(\mathbb{CP}^{r-1})$ is a generator. Hence, the cohomology algebra $H^*_{DR}(\mathbb{CP}^{r-1})$ has $\mathbb{R}$-basis $\{1, a, a^2, \dots, a^{r-1}\}$.

Now $\quad \mathbb{CP}^{r-1} \longrightarrow \mathbb{P}(E) \quad \{1, a, \dots, a^{r-1}\}$

$$\begin{array}{c} \mathbb{P}(E) \\ \downarrow \\ M \end{array}$$

Now by Leray-Hirsch the set $\{1, a, \dots, a^{r-1}\}$ makes $H^*_{DR}(\mathbb{P}(E))$ a free $H^*_{DR}(M)$-module.

Consider the element $a^r \in H^*_{DR}(\mathbb{P}(E))$. Then

$$a^r + c_1(E)a^{r-1} + \dots + c_i(E)a^{r-i} + \dots + c_r(E) = 0$$

for some unique elements $c_1(E), \dots, c_r(E)$ in $H^*_{DR}(M)$. Now we call $c_i(E)$ as the ith Chern class of the complex vector bundle $E \to M$.

__Remark:__ 1) If rank $E = 1$ then the two definitions of $c_1(E)$ agree.

$$E = L \longrightarrow M, \quad \mathbb{P}(E) \overset{\pi}{=} M$$

$$L \simeq \pi^*(L) \longrightarrow M, \quad a = c_1(L^*) = -c_1(L) = -e(L_{\mathbb{R}})$$

$\Rightarrow a + e(L_{\mathbb{R}}) = 0.$

$\Rightarrow$ So $c_1(L) = e(L_{\mathbb{R}})$ in the new definition also.

2.) Assume that $E \to M$ is the trivial $\mathbb{C}^r$-bundle.

$$E = M \times \mathbb{C}^r \longrightarrow M$$

Then $\mathbb{P}(E) = M \times \mathbb{CP}^{r-1} \longrightarrow M$

$$H^*_{DR}(\mathbb{P}(E)) \cong H^*_{DR}(M) \otimes H^*_{DR}(\mathbb{CP}^{r-1})$$

$$a \Rightarrow a^r = 0$$

$a^r + 0 = 0 \Rightarrow c_i(E) = 0$ for all $\forall i \geq 1$.

Definition: For any complex vector bundle $E \to M$ then $c_0(E)$ is defined to be the class $1 \in H^0_{\mathbb{R}}(M)$.

$c_0(E) \doteq \underline{1}$.

Definition The total Chern class of a complex vector bundle $E \to M$ is defined to be the element

$$c(E) = c_0(E) + c_1(E) + \dots + c_r(E), \quad r = \operatorname{rank} E.$$

Proposition: Chern classes are natural. In other words, if $f: M \to N$ is a smooth map and $E \to N$ is a complex vector bundle then

$$c_i(f^*(E)) = f^*(c_i(E)), \text{ for all } i.$$

This can be proved directly using the definitions, whose details are left as an exercise.

We'll prove this using the so called Splitting Principle.

Videos 49-50

Note Title 17.05.2020

<u>Theorem</u> (Splitting Principle)
Let $\pi: E \to M$ be a complex vector bundle of rank r. Then there is a manifold $F(E)$ and a map $\phi: F(E) \to M$ so that the following hold:

1). $\phi^*(E) \to F(E)$ is a direct sum complex line bundles:
$$F(E) \cong L_1 \oplus L_2 \oplus \dots \oplus L_r,$$

2) the homomorphism $\phi^*: H^*_{DR}(M) \to H^*_{DR}(F(E))$ is injective.

Proof left as an exercise.

<u>Conclusion:</u> To prove a polynomial identity among the Chern classes of a vector bundle $E \to M$ we may assume that E is a sum of line bundles.

<u>Theorem</u> (Whitney Product Formula)

Let $E_i \to M$, $i = 1 \dots r$, be complex vector bundles. Then
$$c(E_1 \oplus E_2 \oplus \dots \oplus E_r) = c(E_1) \cdot c(E_2) \cdots c(E_r)$$

$\left(c(E) = c_0(E) + c_1(E) + \dots + c_k(E), \quad \operatorname{rank} E = k \right)$

<u>Proof.</u> Special Case: $\operatorname{rank} E_i = 1 \ \forall i$. $E_i = L_i$

$$E = E_1 \oplus E_2 \oplus \dots \oplus E_r = L_1 \oplus L_2 \oplus \dots \oplus L_r.$$

$\pi : \mathbb{P}(E) \to M$ as before and similarly, let

$\widetilde{E} = \pi^*(E) = \{(\ell_p, v) \in \mathbb{P}(E) \times E \mid \pi^\circ(v) = p,\ p \in M,\ \ell_p \in E_p \setminus K / \mathbb{C}^*\}$

$\pi^\circ : E \to M$

$\widetilde{E} \to \mathbb{P}(E)$, $\widetilde{L}_i \doteq \pi^*(L_i)$

$L \doteq \{(\ell_p, v) \in \pi^*(E) \mid v \in \ell_p\} \subseteq \widetilde{E}$ a subline bundle.

$L \subseteq \widetilde{E} = \widetilde{L}_1 \oplus \dots \oplus \widetilde{L}_r$, $s_i : L \to \widetilde{L}_i$ the restriction of the projection $\widetilde{E} \to \widetilde{L}_i$ to L.

$s_i \in \hom(L, \widetilde{L}_i) = L^* \otimes \widetilde{L}_i$, $s_i(q) : \mathbb{P}(E) \to L^* \otimes \widetilde{L}_i$ section

$V_i \doteq \{q \in \mathbb{P}(E) \mid s_i(q) \neq 0\}$.

Note that for any $q \in \mathbb{P}(E)$ at least one $s_i(q) \neq 0$.
Hence, $\mathbb{P}(E) = V_1 \cup V_2 \cup \dots \cup V_r$.

Consider $L^* \otimes \widetilde{E} = L^* \otimes (\widetilde{L}_1 \oplus \dots \oplus \widetilde{L}_r)$

$$= (L^* \otimes \widetilde{L}_1) \oplus \dots \oplus (L^* \otimes \widetilde{L}_r).$$

Now we have section

$s : \mathbb{P}(E) \to L^* \otimes \widetilde{E}$, $s(q) = (s_1(q), \dots, s_r(q))$.

By construction $s(q) \neq 0$ for all $q \in \mathbb{P}(E)$.

Hence $e((L^* \otimes \widetilde{E})_{\mathbb{R}}) = 0$.

Now, $0 = e((L^* \otimes \tilde{E})_{\mathbb{R}})$

$$= e\left((L^* \otimes \tilde{L}_1)_{\mathbb{R}} \oplus \cdots \oplus (L^* \otimes \tilde{L}_r)_{\mathbb{R}}\right)$$

$$= e((L^* \otimes \tilde{L}_1)_{\mathbb{R}}) \cdots e((L^* \otimes \tilde{L}_r)_{\mathbb{R}})$$

$$= c_1(L^* \otimes \tilde{L}_1) \cdots c_1(L^* \otimes \tilde{L}_r)$$

$$= \left(c_1(L^*) + c_1(\tilde{L}_1)\right) \cdots \left(c_1(L^*) + c_1(\tilde{L}_r)\right)$$

$a = c_1(L^*)$

$$= (a + c_1(\tilde{L}_1)) \cdots (a + c_1(\tilde{L}_r))$$

Hence, $0 = (a + c_1(\tilde{L}_1)) \cdots (a + c_1(\tilde{L}_r))$ $H^*_{DR}(P(E))$
free module on $H^*_{DR}(M)$

$$= a^r + \cdots$$

$$\Rightarrow 0 = (a + c_1(L_1)) \cdot \ldots \cdot (a + c_1(L_r))$$

Hence by the definition of higher Chern classes we see that

$$c(E) = \prod_i (1 + c_1(L_i)) = \prod_i c(L_i)$$

Here we use the naturality of Chern classes and thus we have the same identity in $H^*_{DR}(M)$.

This finishes the proof in case of each E_i is a line bundle.

For the general case $E = E_1 \oplus \cdots \oplus E_r$, it is enough to prove this for $r=2$.

Moreover, by the Splitting principle we may assume that E_1 and E_2 are sums of line bundles:

$E_1 = L_1 \oplus \cdots \oplus L_k$, $E_2 = L_1' \oplus \cdots \oplus L_\ell'$

<u>must show:</u> $c(E_1 \oplus E_2) = c(E_1)\, c(E_2)$.

$$
\begin{aligned}
c(E_1 \oplus E_2) &= c(L_1 \oplus \cdots \oplus L_k \oplus L_1' \oplus \cdots \oplus L_\ell') \\
&= \left[c(L_1) \cdots c(L_k)\right]\left[c(L_1') \cdots c(L_\ell')\right] \\
&= c(L_1 \oplus \cdots \oplus L_k)\, c(L_1' \oplus \cdots \oplus L_\ell') \\
&= c(E_1)\, c(E_2). \quad \blacksquare
\end{aligned}
$$

<u>Applications</u>: 1) $E = L_1 \oplus \cdots \oplus L_r$, $\operatorname{rank} E = r$.

$c(E) = c(L_1) \cdots c(L_r)$

$1 + c_1(E) + \cdots + c_r(E) = (1 + c_1(L_1)) \cdots (1 + c_1(L_r))$

$= 1 + \sigma_1 + \cdots + \sigma_r$, where

$\sigma_1 = c_1(L_1) + \cdots + c_1(L_r)$ 1^{st} elementary sym. poly. in $c_1(L_i)$'s.

$\sigma_2 = \sum_{i<j} c_1(L_i)\, c_1(L_j)$ 2^{nd} " " "

$\sigma_r = c_1(L_1) \cdots c_1(L_r)$. n^{th} elem. sym. poly in $c_1(L_i)$'s.

Hence, $c_k(E) = k^{\text{th}}$ elem sym. poly of $c_1(L_i)$'s.

Proposition: $E \to M$, rank $E = r$, complex vector bundle.

Then $c_r(E) = e(E_{\mathbb{R}})$.

Proof: Both c_i's and e are natural classes and thus by the splitting principle we assume that E is a sum complex line bundles:

$$E = L_1 \oplus L_2 \oplus \cdots \oplus L_r.$$

$$\begin{aligned} c_r(E) &= c_r(L_1 \oplus L_2 \oplus \cdots \oplus L_r) \\ &= c_1(L_1)\, c_1(L_2) \cdots c_1(L_r) \\ &= e(L_{1\,\mathbb{R}})\, e(L_{2\,\mathbb{R}}) \cdots e(L_{r\,\mathbb{R}}) \\ &= e((L_1 \oplus L_2 \oplus \cdots \oplus L_r)_{\mathbb{R}}) \\ &= e(E_{\mathbb{R}}). \end{aligned}$$

Proposition: Let $E \to M$ be a rank r complex vector bundle and $\overline{E} \to M$ denote the conjugate bundle. Then

$$c_i(\overline{E}) = (-1)^i c_i(E).$$

Proof: The conjugate bundle is defined as follows:

If $u_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r, \mathbb{C})$ be a transition function for the bundle $E \to M$, then the bundle whose transition functions

$$\overline{\varphi_{\alpha\beta}} : U_\alpha \cap U_\beta \longrightarrow GL(r, \mathbb{C}),\ \overline{\varphi_{\alpha\beta}}(p) = \overline{\varphi_{\alpha\beta}(p)}.$$

If $E = L$ is a line bundle then

$\overline{\varphi_{\alpha\beta}}(p) = \overline{\varphi_{\alpha\beta}(p)}$ and thus $c_1(\overline{L}) = -c_1(L)$.

$\varphi_{\alpha\beta}(p) = z = x+iy \quad \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \longleftrightarrow \theta$ radian rotation

$\overline{\varphi_{\alpha\beta}}(p) = \bar{z} = x-iy \quad \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \longleftrightarrow -\theta$ radian rotation

<u>Remark</u> $\overline{L} \cong L^*$

L	$\overline{L}$	L^*
z	$\bar{z}$	$\frac{1}{z} = \frac{\bar{z}}{\lVert z \rVert^2}$

By the Splitting Principle we may assume

$E = L_1 \oplus L_2 \oplus \cdots \oplus L_r$. Then $\overline{E} = \overline{L_1} \oplus \overline{L_2} \oplus \cdots \oplus \overline{L_r}$.

$$\begin{aligned}
c_i(\overline{E}) &= c_i(\overline{L_1} \oplus \cdots \oplus \overline{L_r}) \\
&= i^{\text{th}} \text{ elementary sym. poly. of } c_1(\overline{L_j}) \\
&= i^{\text{th}} \quad '' \quad '' \quad '' \quad '' \ -c_1(L_j) \\
&= (-1)^i \, i^{\text{th}} \text{ elem. sym. poly. of } c_1(L_j) \\
&= (-1)^i c_i(L_1 \oplus \cdots \oplus L_r) \\
&= (-1)^i c_i(E).
\end{aligned}$$

Chern Classes of $\mathbb{CP}^n$:

$H^*_{DR}(\mathbb{CP}^n) = \mathbb{R}[a]/(a^{n+1}) \qquad a \in H^2_{DR}(\mathbb{CP}^n)$

$a = PD(H), \quad H = \mathbb{CP}^{n-1};\ (z_i = 0), \quad \int_{\mathbb{CP}^1} a = 1$

$\int_{\mathbb{CP}^k} a^k = 1, \quad \mathbb{CP}^k \subseteq \mathbb{CP}^n, \ z_0 = 0, \dots, z_{n-k-1} = 0.$

<u>Claim:</u> $c(T_*\mathbb{CP}^n) = (1+a)^{n+1} = 1 + (n+1)a + \dots + (n+1)a^n + a^{n+1}$

$c_0 = 1, \ c_1 = (n+1)a, \dots, c_n = (n+1)a^n$

Proof is by Induction on n.

<u>$n=1$</u> $\mathbb{CP}^1 = S^2$, $c_0 = 1$, $c_1 = c_1(T_*\mathbb{CP}^1) = e(T_*\mathbb{CP}^1_{\mathbb{R}})$
$= 2a.$

$c(T_*\mathbb{CP}^1) = (1+2a) = (1+a)^2$

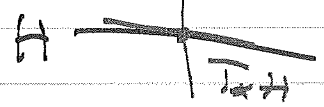
Now assume the result for n, and let's prove it for $n+1$.

$c(T_*\mathbb{CP}^n) = (1+a)^{n+1}$

<u>must show</u> $c(T_*\mathbb{CP}^{n+1}) = (1+a)^{n+2}$.

$H \subseteq \mathbb{CP}^{n+1}, \quad H : (z_0 = 0) \quad H = \mathbb{CP}^n$

$a = PD(H), \quad T_*\mathbb{CP}^{n+1}\big|_H = N \oplus T_*H$ as complex bundles.



$\mathrm{rank}_{\mathbb{C}} N = 1, \quad c_1(N) = e(N_{\mathbb{R}}) = a$

because $e(N_{\mathbb{R}})$ is the Poincaré dual of the submanifold $H \subseteq \mathbb{CP}^{n+1}$.

$i: H = \mathbb{CP}^n \hookrightarrow \mathbb{CP}^{n+1}$, $T_*\mathbb{CP}^{n+1}|_{\mathbb{CP}^n} = N \oplus T_*\mathbb{CP}^n$

$$i^*(c(T_*\mathbb{CP}^{n+1})) = c(T_*\mathbb{CP}^n)\, c(N) = (1+a)^n(1+a)$$

Since i^* is injective for $0 \leq k \leq n$, we get

$$c_k(T^*\mathbb{CP}^{n+1}) = \binom{n+2}{k} a^k, \text{ for } 0 \leq k \leq n.$$

So we need to show that $c_{n+1}(T_*\mathbb{CP}^{n+1}) = (n+2)a^{n+1}$.

$$c_{n+1}(T_*\mathbb{CP}^{n+1}) = e(T_*\mathbb{CP}^{n+1}) = \chi(\mathbb{CP}^{n+1})\, a^{n+1} = (n+2)a^{n+1}$$

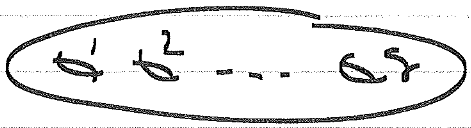
This finishes the proof. ∎

<u>Adjunction Formula</u> $M = \mathbb{CP}^2$, $C : (f=0) \subseteq \mathbb{CP}^2$

$\deg f = d$ homogeneous polynomial in z_0, z_1, z_2.

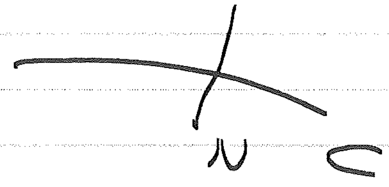
Assume that C is a smooth complex curve in $\mathbb{CP}^2$.
$C \subseteq \mathbb{CP}^2$ is an oriented surface.

$C = \Sigma_g$



g ? d

$T_*\mathbb{CP}^2|_C = N \oplus T_*C$



$$c(\mathbb{CP}^2) = (1+a)^3 = 1+3a+3a^2 \qquad c_1(\mathbb{CP}^2) = 3a, \quad c_2(\mathbb{CP}^2) = 3a^2$$

$$c_1(T_*\mathbb{CP}^2|_C) = c_1(N \oplus T_*C)$$

$$(3a) \quad = c_1(N) + c_1(T_*C)$$

$$= e(N_{\mathbb{R}}) + e(T_*C)$$

$C : f = 0$ $\qquad \displaystyle\int_C e(N_{\mathbb{R}}) = \mathrm{Int}(C,C) = d^2$

$$\int_C c_1(T_*C) = \int_C e(\underset{T_*\Sigma_g}{\underset{\parallel}{T_*C}}) = \chi(C) = 2-2g.$$

$$\int_C c_1(T_*\mathbb{CP}^2|_C) = \int_C 3a = 3\,\mathrm{Int}(C,H), \text{ where}$$
$$= 3d.$$

$H = \mathbb{CP}^1$ has Poincaré dual a.

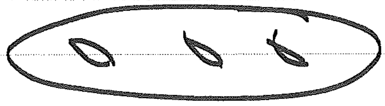
So, we get $3d = d^2 + 2 - 2g$.

$\Rightarrow 2g = d^2 - 3d + 2 = (d-1)(d-2)$

$\Rightarrow g = \frac{1}{2}(d-1)(d-2).$

<u>Ex</u> $d = 1, 2 \Rightarrow g = 0 \Rightarrow C = S^2$

$d = 3 \Rightarrow g = 1 \Rightarrow C = T^2$ Elliptic curve

$d = 4 \Rightarrow g = 3$ 

This is called the degree genus formula for curves in $\mathbb{CP}^2$.

Degree – Genus Formula for $M = \mathbb{CP}^1 \times \mathbb{CP}^1$
$$\overset{\shortparallel}{[z_0 : z_1]} \quad \overset{\shortparallel}{[w_0 : w_1]}$$

$f(z_0, z_1, w_0, w_1)$ homogeneous of bidegree d_1 and d_2 in z_i's and w_i's, respectively.

$f = z_0^3 + z_1^3 - w_0 w_1 + w_1^2$

$C : (f = 0)$ in M. $\pi_i : \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^1$ $i = 1, 2$ projection.

$T_*(\mathbb{CP}^1 \times \mathbb{CP}^1) = \pi_1^*(T_*\mathbb{CP}^1) \oplus \pi_2^*(T_*\mathbb{CP}^1)$

$c_1(T_*(\mathbb{CP}^1 \times \mathbb{CP}^1)) = \pi_1^*(2a_1) + \pi_2^*(2a_2)$

a_1 is the Poincaré dual of $\{p_0\} \times \mathbb{CP}^1$ in $\mathbb{CP}^1 \times \mathbb{CP}^1$.

a_2 " " " of $\mathbb{CP}^1 \times \{p_0\}$ in $\mathbb{CP}^1 \times \mathbb{CP}^1$.

$$\int_C a_1 = \mathrm{Int}(C, \{p_0\} \times \mathbb{CP}^1) = d_2$$

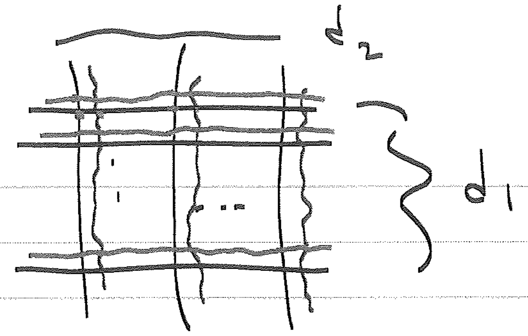
$$\int_C a_2 = \mathrm{Int}(C, \mathbb{CP}^1 \times \{p_0\}) = d_1.$$

$$\int_C c_1(T_*(\mathbb{CP}^1 \times \mathbb{CP}^1)) = 2(d_1 + d_2).$$

Also, we have $T_*(\mathbb{CP}^1 \times \mathbb{CP}^1)|_C = N \oplus T_*C$.

$c_1(N) = e(N_{\mathbb{R}})$, $\displaystyle\int_C c_1(N) = \int_C e(N_{\mathbb{R}}) = \mathrm{Int}(C, C) = \underline{\underline{2d_1 d_2}}$

$f=0$



$C = \Sigma_g$.

$$c_1(T_* \mathbb{CP}^1 \times \mathbb{CP}^1)|_C = c_1(N) + c_1(T_* C)$$

$$2(d_1+d_2) = 2d_1d_2 + 2 - 2g$$

$$\Rightarrow g = d_1d_2 - d_1 - d_2 + 1 = (d_1-1)(d_2-1).$$

The degree genus formula for smooth bidegree (d_1, d_2) curves in $\mathbb{CP}^1 \times \mathbb{CP}^1$.

Pontryagin characteristic classes

$E \to M$ real vector bundle of rank k.

$F = E \otimes_{\mathbb{R}} \mathbb{C} \to M$ complexification of E.

If $\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R})$ are the transition functions of $E \to M$, then the transition functions of $F = E \otimes_{\mathbb{R}} \mathbb{C} \to M$ are the same functions considered into $GL(k, \mathbb{C})$.

$$\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R}) \subseteq GL(k, \mathbb{C}).$$

$$E = U_\alpha \times \mathbb{R}^k / \sim, \quad (x, v) \sim (x, \varphi_{\alpha\beta}(x)(v))$$
$$\forall x \in U_\alpha \cap U_\beta, \ \forall v \in \mathbb{R}^k, \ \forall \alpha, \beta$$

$$F = U_\alpha \times \mathbb{C}^k / \sim_{\mathbb{C}}, \quad (x, v) \sim (x, \varphi_{\alpha\beta}(x)(v))$$
$$\forall x \in U_\alpha \cap U_\beta, \ \forall v \in \mathbb{C}^k, \ \forall \alpha, \beta$$

Therefore, the transition functions for E, $F = E \otimes_{\mathbb{R}} \mathbb{C}$ and $\overline{F} = E \otimes_{\mathbb{R}} \overline{\mathbb{C}}$ are the same.

In particular, F and $\overline{F}$ are isomorphic as complex vector bundles. Then

$$c_{2i+1}(F) = c_{2i+1}(\overline{F}) = (-1)^{2i+1} c_{2i+1}(F) = -c_{2i+1}(F)$$

So, $c_{2i+1}(F) = 0$, for all i.

Definition: The ith Pontryagin class $P_i(E)$ of a real vector bundle $E \to M$ is defined to be the class

$$P_i(E) \doteq (-1)^i c_{2i}(E \otimes \mathbb{C}) \in H^{4i}_{DR}(M).$$

Proposition: Let $E \to M$ be a complex vector bundle of rank r. Let $E_{\mathbb{R}}$ denote the underlying real rank $2r$ bundle. Then

$$E_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \cong E \oplus \bar{E} \quad \text{as complex vector}$$

bundles.

Proof: $\mathbb{C}^r$, $w \in \mathbb{C}^r$, $w = (u_1 + iv_1, \ldots, u_r + iv_r)$

$$\updownarrow$$

$u_i, v_i \in \mathbb{R}$ $\qquad (u_1, v_1, \ldots\ldots, u_r, v_r) \in \mathbb{R}^{2r}$

$\mathbb{C}^r \sim \mathbb{R}^{2r}$

Let $z = re^{i\theta} \in \mathbb{C}$. Then $z : \underset{\mathbb{R}^{2r}}{\overset{\shortparallel}{\mathbb{C}^r}} \longrightarrow \underset{\mathbb{R}^{2r}}{\overset{\shortparallel}{\mathbb{C}^r}}$, $w \mapsto z.w$

$$z.(u_1, v_1, \ldots, u_k, v_k, \ldots, u_r, v_r)$$
$$= (\ldots\ldots, r\cos\theta\, u_k - r\sin\theta\, v_k, r\sin\theta\, u_k + r\cos\theta\, v_k, \ldots)$$

$$z.\begin{pmatrix} u_k \\ v_k \end{pmatrix} = r\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} u_k \\ v_k \end{pmatrix}$$

$z = re^{i\theta}$
$\bar{z} = re^{-i\theta}$

Let's diagonalize this operator:

$$A_z = r\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \begin{array}{l} tr(A_z) = 2r\cos\theta \\ \det(A_z) = r^2 \end{array}$$

$$-tr(A_z) = -2r\cos\theta = -(z+\bar{z})$$

$$\det(A_z) = r^2 = z\bar{z}$$

Hence, the roots of the eigenvalue equation for A_z are z and $\bar{z}$.

In other words, eigenvalues of $w_{\mathbb{R}} \mapsto z.w_{\mathbb{R}}$ are z and $\bar{z}$.

$$z : \mathbb{R}^{2n} \otimes \mathbb{C} \longrightarrow \mathbb{R}^{2n} \otimes \mathbb{C}$$
$$w_{\mathbb{R}} \longmapsto z.w_{\mathbb{R}}$$

$$A_z = r\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad z = r\cos\theta + ir\sin\theta$$

$$A_z - z I_2 = \begin{pmatrix} r\cos\theta - z & -r\sin\theta \\ r\sin\theta & r\cos\theta - z \end{pmatrix}$$

$$= \begin{pmatrix} -ir\sin\theta & -r\sin\theta \\ r\sin\theta & -ir\sin\theta \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$+i\cancel{r\sin\theta}\, a + \cancel{r\sin\theta}\, b = 0 \Rightarrow ia + b = 0$$

$$a = 1,\ b = -i \qquad \begin{pmatrix} 1 \\ -i \end{pmatrix} \rightsquigarrow e_k - i f_k$$

$z \rightsquigarrow e_k - i f_k \quad \mathcal{B} = \{e_1 - i f_1, \ldots, e_r - i f_r\}$

$\bar{z} \rightsquigarrow e_k + i f_k \quad \overline{\mathcal{B}} = \{e_1 + i f_1, \ldots, e_r + i f_r\}$

where $\{e_1, f_1, \ldots, e_r, f_r\}$ is the standard

basis for $\mathbb{C}^r_{\mathbb{R}}$.

Then

$$\begin{array}{ccccc} \mathbb{C}^r_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} & \simeq & \langle \mathcal{B} \rangle & \oplus & \langle \overline{\mathcal{B}} \rangle \\ & & \wr\!\wr & & \wr\!\wr \\ & & \mathbb{C}^r & & \overline{\mathbb{C}}^r \\ z = i & & z = i & & \bar{z} = -i \end{array}$$

This proves the Proposition at one fiber. Since the complex structures varies from fiber to fiber smoothly this procedure gives

$E_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = E \oplus \overline{E}$ as vector bundles. ∎

$E \to M \quad P(E) \doteq \sum_{i \geq 0} P_i(E)$ the total

Pontryagin class of E. Hence, $P_0(E) = 1 \in H^0_{DR}(M)$.

Also define $\widetilde{P}(E) \doteq \sum_{i \geq 0} (-1)^i P_i(E)$.

<u>Corollary</u> If $E \to M$ is a <u>complex</u> vector bundle then $\widetilde{P}(E_{\mathbb{R}}) = c(E)\, c(\overline{E})$.

Proof: $c(E)\,c(\overline{E}) = \left(\sum c_i(E)\right)\cdot\left(\sum_j c_j(\overline{E})\right)$

$$= \left(\sum_i c_i(E)\right)\left(\sum_j (-1)^j c_j(E)\right)$$

$$= \sum_{i,j} (-1)^j c_i(E)\, c_j(E)$$

$$E_{\mathbb{R}} \otimes \mathbb{C} = E \oplus \overline{E}$$

$$\tilde{\mathcal{P}}(E_{\mathbb{R}}) = \sum_k (-1)^k \mathcal{P}_k(E_{\mathbb{R}})$$

$$= \sum_k (-1)^k (-1)^k c_{2k}(E_{\mathbb{R}} \otimes \mathbb{C})$$

$$= \sum_k c_{2k}(E \oplus \overline{E})$$

$$= \sum_k \sum_{i+j=2k} c_i(E)\, c_j(\overline{E})$$

$$= \sum_k \sum_{i+j=2k} (-1)^j c_i(E)\, c_j(E)$$

This finishes the proof. ■

Examples 1) M complex manifold of complex dimension 2 ($\dim_{\mathbb{R}} M = 4$).

Then $\tilde{\mathcal{P}}(M) := \tilde{\mathcal{P}}(T_* M) = 1 - p_1(M)$

$$= (1 + c_1(M) + c_2(M))(1 - c_1(M) + c_2(M))$$

$$\Rightarrow 1 - P_1(M) = 1 + 2c_2(M) - c_1^2(M)$$

So $-P_1(M) = 2c_2(M) - c_1^2(M)$

$$P_1(M) = c_1^2(M) - 2e(M).$$

For example, if $M = \mathbb{C}P^2$, then

$$\begin{aligned} P_1(M) &= c_1^2(M) - 2e(M) \\ &= (3a)^2 - 2\cdot(3\cdot a^2) \\ &= 9a^2 - 6a^2 = 3a^2 \end{aligned}$$

__Proposition:__ $f: M \to N$ differentiable map and $E \to N$ is a real vector bundle then

$$p(f^*(E)) = f^*(p(E)).$$

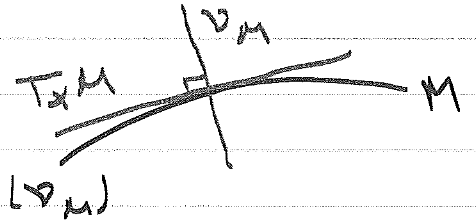
Moreover, if $E_i \to M$ are vector (real) bundles for $i = 1, 2$, then

$$p(E_1 \oplus E_2) = p(E_1)\, p(E_2).$$

Proof follows from naturality of Chern classes and related formulas for the Chern classes.

__Example:__ Suppose that $M^n \subseteq \mathbb{R}^N$ a submanifold.

$$T_x\mathbb{R}^n|_M = T_xM \oplus \nu_M$$



Hence, $p(T_x\mathbb{R}^N|_M) = p(M)\, p(\nu_M)$

Since $T_*\mathbb{R}^N|_M = M \times \mathbb{R}^N$ is trivial, $p(T_*\mathbb{R}^N|_M) = 1$.

Hence, $1 = p(M)\, p(\nu_M)$.

But for example if $M = \mathbb{C}P^2 \subseteq \mathbb{R}^N$, then we have

$$1 = p(\mathbb{C}P^2)\, p(\nu_M).$$

$$1 = (1+3a^2)\, p(\nu_M) \text{ in } H^*_{DR}(\mathbb{C}P^2) = \mathbb{R}[a]/(a^3)$$

Then $p(\nu_M) = (1-3a^2)$. So $p_1(\nu_M) = -3a^2$.

ν_M is an oriented bundle of rank $N-n = N-4$.

If $N=5$, then ν_M is an oriented of rank 1 bundle. So $\nu_M = M \times \mathbb{R}$ is trivial.

$$\Rightarrow p(\nu_M) = 1, \text{ a contradiction.}$$

If $N=6$, then ν_M is an oriented rank 2 bundle. Hence, ν_M can be regarded as a complex line bundle.

$$\nu_M = L \to \mathbb{C}P^2 = M.$$

Say $c_1(L) = e(\nu_M) = \lambda a$, for some integer λ.

Hence, $p_1(L) = (-1)^1 c_2(L \otimes \mathbb{C})$

$$= - c_2(L \oplus \overline{L})$$

$$= - c_1(L)\, c_1(\overline{L})$$

$$= -(\lambda a)(-\lambda a)$$

$$= \lambda^2 a^2.$$

Therefore, we must have $\lambda^2 = -3$, a contradiction.

Hence, $M = \mathbb{CP}^2$ cannot be embedded into $\mathbb{R}^6$.

<u>Remark</u>: It is known that $\mathbb{CP}^2$ embeds in $\mathbb{R}^7$.

<u>Proposition</u>: If $E \to M$ is an oriented real vector bundle of rank $2k$, then $p_k(E) = e(E)^2$.

<u>Proof</u>: E oriented rank $2k$ vector bundle. Say $\{e_1, \ldots, e_{2k}\}$ be an oriented basis at a fiber of E. Then $E \otimes_{\mathbb{R}} \mathbb{C}$ is also a real vector bundle of rank $4k$. The orientation coming from the complex structure is given by the basis $\{e_1, ie_1, \ldots, e_{2k}, ie_{2k}\}$.

On the other hand, $E \otimes_{\mathbb{R}} \mathbb{C} = E \oplus E$ with orientation $\{e_1, e_2, \ldots, e_{2k}, ie_1, ie_2, \ldots, ie_{2k}\}$.

It follows that as real oriented bundles

$$E \otimes_{\mathbb{R}} \mathbb{C} = (-1)^{k(2k-1)} E \oplus E$$
$$= (-1)^k E \oplus E.$$

So, $p_k(E) = (-1)^k c_{2k}(E \otimes \mathbb{C})$

$= (-1)^k e(E \otimes \mathbb{C})$, since $\operatorname{rank}_{\mathbb{C}} E \otimes \mathbb{C} = 2k$

$= (-1)^k e((-1)^k E \oplus E)$

$= (-1)^k (-1)^k e(E \oplus E)$

$= e(E)\, e(E)$

$= e(E)^2$. ∎

<u>Pontryagin Numbers</u> M oriented compact manifold of dimension $4n$. Let $k_1, k_2, \ldots, k_r \geq 0$ be integers with $k_1 + 2k_2 + \cdots + rk_r = n$. Then

$p_1^{k_1}(M)\, p_2^{k_2}(M) \cdots p_r^{k_r}(M) \in H_{DR}^{4k_1 + 2 \cdot 4 k_2 + \cdots + 2rk_r = 4n}(M)$

So we may define the real number

$$p_{k_1, k_2, \ldots, k_r}(M) \doteq \int_M p_1^{k_1}(M) \cdots p_r^{k_r}(M).$$

This real number is an integer and called a Pontryagin number of M.

Clearly, each Pontryagin number is a diffeomorphism invariant of M.

<u>Proposition:</u> If $M = \partial W^{4n+1}$ is the boundary of an oriented manifold (compact) then all

Pontryagin numbers of M are zero.

Proof $p_{k_1,\dots,k_r} = \int_M p_1^{k_1}(M)\dots p_r^{k_r}(M)$, $M=\partial W$

$$= \int_W d\underbrace{(p_1^{k_1}(M)\dots p_r^{k_r}(M))}_{\in \Omega^{4n+1}(M)=0}$$

$$=0. \blacksquare$$

Example: $M=\mathbb{CP}^2$, $p_1(M)=p(\mathbb{CP}^2)=3a^2 \in H^4_{DR}(\mathbb{CP}^2)$.

$$p_1=\int_{\mathbb{CP}^2} p_1(M) = \int_{\mathbb{CP}^2} 3a^2 = 3.$$

$$k_1=0,\ k_2=2$$

$$p_2=\int_{\mathbb{CP}^2} p_2(\mathbb{CP}^2) = \int_{\mathbb{CP}^2} e(\mathbb{CP}^2) = \chi(\mathbb{CP}^2)=3.$$

In particular, $\mathbb{CP}^2$ is not the boundary of any compact orientable smooth manifold.

Theorem (René Thom)

Let M be a compact oriented $4n$-dimensional smooth manifold. If all Pontryagin numbers of M are zero, then there is a smooth compact oriented manifold W so that $\partial W = \underbrace{M\#\dots\# M}_{k}$ for some integer $k>0$.

Videos 53-54

Note Title 23.05.2020

Milnor's Exotic Spheres:

Aim: Construct smooth manifolds that are homeomorphic but not diffeomorphic to the sphere S^7.

Let $M = M^7$ be a smooth oriented closed manifold, $B = B^8$ a smooth oriented compact manifold so that $\partial B = M$ as oriented manifolds.

Lemma 1) Assume that M and B are as above and $H^3_{DR}(M) = 0 = H^4_{DR}(M)$. Then the quadratic form

$$H^4_{DR}(B) \longrightarrow \mathbb{R}, \ [\alpha] \longmapsto \int_B \alpha^2$$

is well-defined.

Proof: Step 1. Let $[\alpha] = [\alpha']$. Then we must show $\int_B \alpha^2 = \int_B \alpha'^2$.

Since $[\alpha] = [\alpha']$, $\alpha - \alpha' = d\beta$ for some $\beta \in \Omega^3(B)$. Then $\alpha = \alpha' + d\beta$ and

$$\alpha^2 = \alpha'^2 + 2\alpha' \wedge d\beta' + d\beta \wedge d\beta \text{ so that}$$

$$\int_B \alpha^2 = \int_B \alpha'^2 + \int_B 2\alpha' \wedge d\beta + \int_B d\beta \wedge d\beta.$$

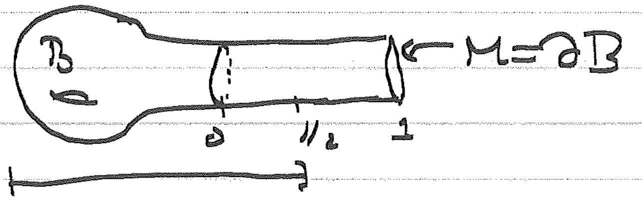
$$\Rightarrow \int_B (\alpha^2 - \alpha'^2) = \int_B d\big((2\alpha' + d\beta) \wedge \beta\big).$$

$$\int_B (\alpha^2 - \alpha'^2) = \int_{M=\partial B} (2\alpha' + d\beta) \wedge \beta.$$

Hence, to finish the proof it is enough to show that β can be chosen so that $\beta|_{M=\partial B} = 0$.

$V = M \times (0,1]$

Step 2:



$U = B - (M \times [1/2, 1])$

$U \cup V = B$ and $U \cap V = M \times (0, 1/2)$.

Now $M \times \mathbb{R}$ is diffeomorphic to $M \times (0, 1/2)$ via a proper diffeomorphism (an exercise).

Hence, $H_c^k(U \cap V) = H_c^k(M \times (0, 1/2))$

$= H_c^k(M \times \mathbb{R})$

$= H_c^{k-1}(M)$ (Poincaré lemma for comp. supp. cohomology)

Also, $H_c^k(U \cup V) = H_c^k(B) = H_{DR}^k(B)$, and

$H_c^k(U) = H_c^k(B - M)$ since again $B - M$ is diffeomorphic to U via a proper diffeomorphism.

Now consider the local cohomology sequence for compactly supported cohomology:

$U \subseteq M$ open subset,

$$0 \to \Omega_c^k(U) \to \Omega_c^k(M) \to \Omega_c^k(M)/\Omega_c^k(U) \to 0$$

is a short exact sequence. It induces a long exact sequence as follows:

$$\cdots \xrightarrow{\delta} H_c^n(U) \to H_c^n(M) \to H_c^n(M,U) \to H_c^{n+1}(U) \to \cdots$$

Take $U = M \setminus L$, $L \subseteq M$ closed manifold. Then

$$\boxed{H_c^k(M,U) = H_c^k(M, M\setminus L) \xrightarrow{\simeq} H_c^k(L).}$$

Now, $H_c^k(V, V\setminus M) \simeq H_c^k(M) = H_{DR}^k(M)$.

<u>Claim</u>: $H_c^k(V) = 0$.

<u>Proof</u>: Local cohomology sequence for the pair $(V, V\setminus M)$:

$$\cdots \to H_c^n(V\setminus M) \to H_c^n(V) \to H_c^n(V, V\setminus M) \to H_c^{n+1}(V\setminus M) \to \cdots$$

$$H_c^k(V\setminus M) = H_c^k(M \times (0,1)) = H_c^k(M \times \mathbb{R}) = H_c^{k-1}(M) = H_{DR}^{k-1}(M).$$

This gives

$$\xrightarrow{\simeq} H_{DR}^{n-1}(M) \xrightarrow{0} \underset{\substack{\| \\ 0}}{H_c^n(V)} \xrightarrow{0} H_{DR}^n(M) \xrightarrow{\simeq} H_{DR}^n(M) \to \cdots$$

$\Rightarrow$

Step 3: Mayer-Vietoris Exact Sequence for locally compactly supported cohomology for

$B = U \cup V$.

$$\cdots \to H_c^k(U\cap V) \to H_c^k(U) \oplus H_c^k(V) \to H_c^k(B) \to H_c^{k+1}(U\cap V) \to \cdots$$

Combining the above results we get

$$\to H^3_{DR}(M) \to H^4_c(B-M) \to H^4_{DR}(B) \to H^4_{DR}(M) \to \cdots$$

By assumption $H^3_{DR}(M) = 0 = H^4_{DR}(M)$ and thus

$H^4_{DR}(B) = H^4_c(B\setminus M)$. Hence, the form β in Step 1 can be choosen so that $\beta = 0$ on M.

This finishes the proof of the Lemma 1. ∎

Now we define the index of the quadratic form by $\tau(B)$.

$$H^4_{DR}(B) \to \mathbb{R}, \ [\alpha] \mapsto \int_B \alpha^2.$$

<u>Remark:</u> $\tau(-B) = -\tau(B)$

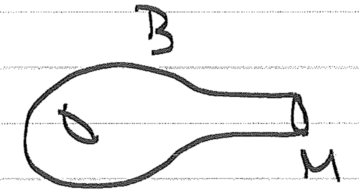
Define Pontryagin numbers of B as follows:

$q(B) \doteq Q(p_1(B)) = \int_B p_1^2(B)$, which is well-defined by Lemma 1.

We'll see later that $q(B)$ is an integer.

Finally, define $\lambda(M) \doteq 2q(B) - \tau(B) \pmod 7$.

<u>Theorem:</u> $\lambda(M)$ is independent of the choice of B and determined only by M.



<u>Corollary</u>: If $\lambda(M) \neq 0$ then M cannot be the boundary of a compact manifold B with $H^4_{DR}(B) = 0$.

Similar to $\tau(B)$, $\lambda(-M) = -\lambda(M)$, and thus we get

<u>Corollary</u>: If $\lambda(M) \neq 0$ then M does not admit an orientation reversing diffeomorphism.

<u>Proof</u>: Suppose not: Let $\varphi : M \to -M$ be a diffeomorphism. Then $\lambda(M) = \lambda(-M) = -\lambda(M)$ and thus $\lambda(M) \neq 0$, a contradiction. ∎

<u>Proof of the Theorem:</u>

<u>Step 1.</u> Let B_1 and B_2 be two compact oriented manifolds with $\partial B_i = M$, $i = 1, 2$.

Let $C = B_1 \cup_\partial -B_2$, which is an oriented manifold with $\partial C = \emptyset$. Then by Thom/Hirzebruch Signature Theorem

$$\tau(C) = \frac{1}{45} \int_C 7p_2(C) - p_1^2(C).$$

[$H^4_{DR}(C) \longrightarrow \mathbb{R}$, $[\alpha] \longmapsto \int_C \alpha^2$ (quadratic form)
$\tau(C)$ is the signature of this quadratic form]

Then we get

$$45\,\tau(C) + q(C) = \int_C 7p_2(C) - \cancel{p_1^2(C)} + \cancel{\int_C p_1^2(C)}$$
$$= 0 \pmod 7.$$

$$2q(C) + 90\,\tau(C) = 0 \pmod 7$$

$$\boxed{2q(C) - \tau(C) = 0 \pmod 7.}$$

Step 2: We'll prove that

$$\tau(C) = \tau(B_1) - \tau(B_2) \quad \text{and}$$
$$q(C) = q(B_1) - q(B_2).$$

Note that Step 2 finishes the proof of the Theorem.

Using similar ideas used in the proof of Lemma 1 we obtain a commutative diagram where each arrow is an isomorphism:

$$\begin{tikzcd} H^4_c(C) \arrow[d] & H^4_c(B_1 - M) \oplus H^4_c(B_2 - M) \arrow[l] \arrow[d] \\ H^4_{DR}(M) \arrow[r] & H^4_{DR}(B_1) \oplus H^4_{DR}(B_2) \end{tikzcd}$$

$$\alpha \longmapsto \beta_1 + \beta_2$$

So, for any $\alpha \in H^4_{DR}(M)$ we can write $\alpha = \beta_1 + \beta_2$ for some $\beta_i \in H^4_{DR}(B_i)$, $i = 1, 2$, so that each β_i restricts to zero on $\partial B_i = M$.

$$\alpha = \beta_1 + \beta_2 \quad , \quad \beta_i \in H^4_{DR}(B_i), \quad \beta_i\big|_{\partial B_i = M} = 0.$$

$$\alpha^2 = \beta_1^2 + \beta_2^2 + \underbrace{2\beta_1\beta_2}_{=0}$$

$$\Rightarrow \tau(C) = \int_C \alpha^2 = \int_{B_1 - B_2} \beta_1^2 + \beta_2^2 = \int_{B_1} \beta_1^2 - \int_{B_2} \beta_2^2$$

$$= \tau(B_1) - \tau(B_2).$$

So $q(C) = q(B_1) - q(B_2)$ just note that

Pontryagin classes are natural and then

we get $C = B_1 \cup -B_2 \Rightarrow p_1(C) = p_1(B_1) \pm p_1(B_2)$

$\Rightarrow p_1^2(C) = p_1^2(B_1) + p_1^2(B_2) \Rightarrow q(C) = q(B_1) - q(B_2)$. ∎

Tangent Bundle of S^4.

$S^4 = \mathbb{H} \dot\cup \mathbb{H} / p \sim 1/p, \quad p \in \mathbb{H}^* = \mathbb{H} \setminus \{0\}$.

$T_* S^4 = T_*\mathbb{H} \dot\cup T_*\mathbb{H} \Big/ (p,v) \sim (1/p, D\varphi_p(v))$, where

$\varphi : \mathbb{H}^* \to \mathbb{H}^*, \ \varphi(p) = 1/p$.

Let's compute $D\varphi_p(v)$, for $p \in \mathbb{H}^*$, $v \in \mathbb{H} = T_*\mathbb{H}^*$.

$$D\varphi_p(v) = \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{\varphi(p+hv) - \varphi(p)}{h}$$

$$= \lim_{h\to 0} \frac{1/(p+hv) - 1/p}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{\bar p + h\bar v}{\|p+hv\|^2} - \dfrac{\bar p}{\|p\|^2}}{h}, \quad \text{since } q\bar q = \|q\|^2.$$

$$= \lim_{h\to 0} \frac{(\bar p + h\bar v)\|p\|^2 - \|p+hv\|^2 \bar p}{\|p+hv\|^2 h \|p\|^2}$$

$$= \lim_{h\to 0} \frac{(\bar p + h\bar v)\|p\|^2 - (\bar p + h\bar v)(p+hv)\bar p}{\|p+hv\|^2 h \|p\|^2}$$

$$= \lim_{h\to 0} \frac{(\bar p + h\bar v)\left[\|p\|^2 - (p+hv)\bar p\right]}{\|p+hv\|^2 h \|p\|^2}$$

$$= \lim_{h\to 0} \frac{(\bar p + h\bar v)\left[\cancel{\|p\|^2} - \cancel{\|p\|^2} - hv\bar p\right]}{\|p+hv\|^2 h \|p\|^2}$$

$$= \lim_{h \to 0} \frac{(\bar{p} + h\bar{v})(-\cancel{h} v \bar{p})}{\|p+hv\|^2 \cancel{h} \|p\|^2}$$

$$= \lim_{h \to 0} -\left(\frac{\bar{p} + h\bar{v}}{\|p+hv\|^2}\right) v \left(\frac{\bar{p}}{\|p\|^2}\right)$$

(denominators are nonzero real numbers, so they commute)

$$= -\frac{\bar{p}}{\|p\|^2} v \frac{\bar{p}}{\|p\|^2}$$

$$= -\frac{1}{p} v \frac{1}{p}.$$

$\mathbb{R}^4$ – bundles over S^4:

$S^2 = \mathbb{CP}^1 = \mathbb{C} \dot\cup \mathbb{C} / z \sim \phi(z) = \frac{1}{z}$, $z \neq 0$.

$\Rightarrow S^4 = \mathbb{HP}^1 = \mathbb{H} \dot\cup \mathbb{H} / p \sim \frac{1}{p} = \frac{\bar{p}}{\|p\|^2} = \phi(p)$

$p \in \mathbb{H} = \mathbb{R}^4$, $p = (x, y, z, w) = x + iy + jz + kw$.

$\bar{p} = x - iy - jz - kw$.

$T_* S^4 = T_* \mathbb{H} \dot\cup T_* \mathbb{H} / (p, v) \sim \left(\frac{1}{p}, \phi'(p)(v)\right)$

$\phi'(p) : T_p \mathbb{H} \longrightarrow T_{\phi(p)} \mathbb{H}$

$$\phi'(p)(v) = \lim_{h \to 0} \frac{\phi(p + hv) - \phi(p)}{h}$$

$$= -\frac{1}{p} v \frac{1}{p} \quad \left(\neq -\frac{1}{p^2} v\right)$$

In $\mathbb{H}^* = \mathbb{R}^4 \setminus \{0\}$ there is a path joining -1 to 1. Using this path we can find a homotopy joining the map

$(p, v) \longmapsto -\frac{1}{p} v \frac{1}{p}$ to the map

$(p, v) \longmapsto \frac{1}{p} v \frac{1}{p}$, so that these two maps give isomorphic bundles.

Now for any pair of integers (h, j) define the bundle

$$\xi_{h,j} \longrightarrow S^4 \quad \text{as}$$

$$\xi_{h,j} = \mathbb{H} \times \mathbb{H} \,\dot\cup\, \mathbb{H} \times \mathbb{H} \Big/ {\scriptstyle (p,v) \sim \left(\frac{1}{p},\, p^h v\, p^j\right)}$$

$$(p, v) \in \mathbb{H}^* \times \mathbb{H}.$$

Note that $\xi_{-1,-1} = T_* S^4$.

<u>Lemma:</u> The characteristic classes p_1 and e of $\xi_{h,j} \longrightarrow S^4$ are given by

$p_1(\xi_{h,j}) = 2(h-j)\nu$ and $e(\xi_{h,j}) = -(h+j)\nu$,

where $\nu \in H^4_{DR}(S^4)$ with $\int_{S^4} \nu = 1$.

Videos 55-56

Note Title 25.05.2020

$\mathbb{H} \to \xi_{h,j} \to S^4$, $h, j \in \mathbb{Z}$
$\overset{\shortparallel}{\mathbb{R}^4}$

$$\xi_{h,j} = \mathbb{H} \times \mathbb{H} \,\dot\cup\, \mathbb{H} \times \mathbb{H} \Big/ {}_{(p,v) \sim (\frac{1}{p},\, p^h v p^j),\ p \neq 0}.$$

$T_* S^4 = \xi_{-1,-1}$.

<u>Lemma:</u> $p_1(\xi_{h,j}) = 2(h-j)\nu$ and $e(\xi_{h,j}) = -(h+j)\nu$,

where $\nu \in H^4_{DR}(S^4)$ so that $\int_{S^4} \nu = 1$.

<u>Proof</u>: A) $e(\xi_{h,j}) = -(h+j)\nu$

<u>Case 1</u> $h+j \leq 0$

Note that the functions $s_i : \mathbb{H} \to T_* \mathbb{H}$

$$s_i(p) = (p, 1 + \bar{p}^{-h-j}),\ i = 1,2,\ p \in \mathbb{H},$$

satisfy the identity $p^j s_1(p) p^h = s_2(1/p)$, $\forall p \neq 0$

and therefore they define a section of $\xi_{h,j}$

$$s : S^4 = \mathbb{H}_1 \,\dot\cup\, \mathbb{H}_2 / {}_{\substack{p \sim 1/p \\ p \neq 0}} \longrightarrow \xi_{h,j}$$

$$s(p) = \begin{cases} s_1(p) & p \in \mathbb{H}_1 \\ s_2(p) & p \in \mathbb{H}_2 \end{cases} \qquad (p,v) \sim (1/p,\, p^h v p^j)$$

$$\bar{p}^{J} s_1(p)\, p^h = \bar{p}^{J}\left(1 + \bar{p}^{-h-J}\right) p^h$$
$$= p^{\bar{J}+h} + 1$$
$$= \left(\frac{1}{p}\right)^{-h-J} + 1$$
$$= s_2(1/p).$$

$\int_{S^4} e(\xi_{h,J}) =$ Number of zeros of a section transverse to the zero section.

<u>Fact</u> (Eilenberg–Niven, Bull. Amer. Math. Soc. 1944)

The degree of the polynomial map

$\mathbb{H} \to \mathbb{H}$, $p \mapsto p^k$, $k \in \mathbb{Z}^+$, is k.

Hence $\deg(1 + \bar{p}^{h-J} : \mathbb{H} \to \mathbb{H}) = -(h+J)$.

So, $\int_{S^4} e(\xi_{h,J}) = -(h+J)$ and thus

$$e(\xi_{h,J}) = -(h+J)\nu, \qquad \int_{S^4} \nu = 1.$$

<u>Case 2</u> $h+J > 0$.

Note that the homotopy, $t \in [0,1]$, given by

$$(t,(p,v)) \longmapsto \left(\frac{1}{p}, \frac{p^h}{t+(1-t)\|p\|^h}\, v\, \frac{p^J}{t+(1-t)\|p\|^J}\right)$$

joins the maps

$t=0$, $(p,v) \mapsto \left(\frac{1}{p}, \left(\frac{p}{\|p\|}\right)^h v \left(\frac{p}{\|p\|}\right)^j\right)$ and

$t=1$, $(p,v) \mapsto \left(\frac{1}{p}, p^h v p^j\right)$.

$S^4 = D^4 \dot{\cup} D^4 / p \sim \frac{1}{p}, \ p \in \partial D^4 = S^3$.

Now let's reverse the orientation of the fibers of $\xi_{h,j}$.

$v \mapsto u = \bar{v}$ and thus the gluing function of the bundle becomes

$$(p,v) \longmapsto (1/p, \ p^h v p^j)$$

$$(p, \bar{v}) \longmapsto (1/p, \ \overline{p^h v p^j}) \qquad \overline{ab} = \bar{b}\bar{a}$$

$$\overline{p^h v p^j} = \bar{p}^j \bar{v} \bar{p}^h = p^{-j} \bar{v} \, p^{-h}$$

$\bar{p} = \frac{1}{p} = p^{-1}$

Hence, we get $(p,u) \mapsto \left(\frac{1}{p}, \bar{p}^j u \bar{p}^h\right)$, so that the effect of changing the orientation of the fiber of $\xi_{h,j}$ results in the bundle $\xi_{-j,-h}$.

$\therefore \ -\xi_{h,j} = \xi_{-j,-h}$.

Thus, $e(\xi_{h,j}) = -e(-\xi_{h,j}) = -e(\xi_{j-h})$

$$= -((h+j)\nu)$$

$$= -(h+j)\nu.$$

This finishes the proof of part (A).

B) $p_1(\xi_{h,j}) = 2(h-j)\nu$.

Now let's change the orientation of both the base and the fiber of $\xi_{h,j} \to S^4$.

$$(p,v) \longmapsto (\bar{p}, \bar{v}) = (q, \tilde{v}).$$

$\xi_{h,j} : (p,v) \longmapsto (1/p, p^h v p^j)$

This becomes in the $(q,\tilde{v})$ coordinates

$$(q,\tilde{v}) \longmapsto (1/q, q^j \tilde{v} q^h)$$

$$\begin{array}{ccc} (p,v) & \longmapsto & (1/p, p^h v p^j) \\ \downarrow & & \downarrow \\ (\bar{p},\bar{v}) & \longmapsto & \left(\frac{1}{\bar{p}}, \bar{p}^j \bar{v} \bar{p}^h\right) \\ \| & & \\ (q,\tilde{v}) & \longmapsto & \left(\frac{1}{q}, q^j \tilde{v} q^h\right) \end{array}$$

$\overline{\frac{1}{p}}$

So the bundle $\xi_{h,j}$ became $\xi_{j,h}$.

What about the effect of this change of orientation on p_1?

As oriented real vector spaces $\mathbb{H}\otimes\mathbb{C}$ and $-\mathbb{H}\otimes\mathbb{C}$ are isomorphic.

$\mathbb{H}$ $\{e_1, e_2, e_3, e_4\}$ $\qquad$ $-\mathbb{H}$ $\{-e_1, e_2, e_3, e_4\}$

$\mathbb{H}\otimes\mathbb{C}$: $\{e_1, ie_1, e_2, ie_2, e_3, ie_3, e_4, ie_4\}$

$-\mathbb{H}\otimes\mathbb{C}$: $\{-e_1, -ie_1, e_2, ie_2, e_3, ie_3, e_4, ie_4\}$

On the other hand, changing the orientation on S^4 replaces ν by $-\nu$.

$$\begin{tikzcd} \mathbb{H} \arrow[r] & \xi_{h,j} \arrow[d] & & \mathbb{H} \arrow[r] & \xi_{j,h} \arrow[d] \\ & S^4 \arrow[rr, squiggly] & & & S^4 \end{tikzcd}$$

$p_1(\xi_{h,j}) = c\nu$ $\qquad$ $p_1(\xi_{j,h}) = c(-\nu)$

So, $p_1(\xi_{j,h}) = -p_1(\xi_{h,j})$.

<u>Example</u> So $p_1(T_* S^4) = p_1(\xi_{-1,-1}) = -p_1(\xi_{-1,-1})$

$\Rightarrow p_1(T_* S^4) = 0$.

Indeed, $p_1(\xi_{h,h}) = 0$, for all $h \in \mathbb{Z}$.

<u>Lemma:</u> $p_1(\xi_{h,j}) = p_1(\xi_{h-1,j-1})$.

<u>Proof:</u> $\psi_1 : \mathbb{H} = \mathbb{R}^4 \longrightarrow \mathbb{R}^4 = \mathbb{H}$, $v \longmapsto vp$ and
$\psi_2 : \mathbb{H} = \mathbb{R}^4 \longrightarrow \mathbb{R}^4 = \mathbb{H}$, $v \mapsto \overline{vp} = \bar{p}\bar{v}$ $\quad (p \neq 0)$

when considered as $\mathbb{R}$-linear maps from $\mathbb{R}^4$ to $\mathbb{R}^4$ are <u>not</u> homotopic in $GL(4,\mathbb{R})$ because the $\psi_2 = \overline{\psi_1}$ and conjugation changes orientation so that the determinants in a fixed basis have different signs.

$B = \{e_1, e_2, e_3, e_4\}$ $\quad p = a + ib + jc + kd = ae_1 + be_2 + ce_3 + de_4$

$v = \sum_{i=1}^{4} v_i e_i$, $\quad \overline{e_1 \cdot p} = \overline{1 \cdot p} = a \mp \bar{i} b \mp \bar{j} c \mp \bar{k} d$

$\overline{e_2 \cdot p} = \overline{i \cdot p} = -b + ia + \bar{j} d + \bar{k} c$

$\Rightarrow v_1 \quad \overline{e_3 \cdot p} = \overline{j \cdot p} = -c \mp kd \mp a\bar{j} \mp bk$

$$[\psi_1]_B = \begin{bmatrix} a & -b & -c & -d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{bmatrix}, \quad [\psi_2]_B = \begin{bmatrix} a & -b & -c & -d \\ -b & -a & -d & c \\ -c & -d & -a & -b \\ -d & -c & -b & -a \end{bmatrix}$$

$e_4 \cdot p = -d + ci + bj + ak$

However, the linear maps

$\psi_1 \otimes id_{\mathbb{C}} : \mathbb{C}^4 = \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}^4$ are

homotopic as maps into $GL(4,\mathbb{C})$.

$(\psi_1 \otimes id_{\mathbb{C}})_{\mathbb{R}} : \mathbb{R}^8 \longrightarrow \mathbb{R}^8$

$a = a + i\cdot 0 \quad \begin{bmatrix} a\cos\theta & -a\sin\theta \\ a\sin\theta & a\cos\theta \end{bmatrix} \quad \theta \in [0, \pi]$

$\theta = \pi \longrightarrow \begin{bmatrix} -a & 0 \\ 0 & -a \end{bmatrix} \longrightarrow -a.$

Hence, replacing vp by $\overline{vp} = \bar{p}\bar{v}$ does not change the isomorphism type of the bundle.

$$\xi_{h,j} \quad (p,v) \longmapsto (1/p,\ p^h v p^{\bar{j}}) \sim (1/p,\ p^{h-1}\bar{v}\,\bar{p}^{j-1})$$

$$p^h (\overline{vp})\, \bar{p}^{j-1} = p^h \bar{p}\,\bar{v}\,\bar{p}^{j-1} = \|p\|^2 p^{h-1} \bar{v} p^{j-1}$$

$$(p,\bar{v}) \longrightarrow (p,v) \longrightarrow (1/p,\ p^{h-1}\bar{v}\,\bar{p}^{j-1})$$
$$\parallel$$
$$(p,u) \longmapsto (1/p,\ p^{h-1} u \bar{p}^{j-1}) \to \xi_{h-1,j-1}.$$

Changing the orientation on the fiber does not change the orientation of the complexified bundle and thus the bundles $\xi_{h,j}$ and $\xi_{h-1,j-1}$ are isomorphic once they are complexified.

Hence, $p_1(\xi_{h,j}) = p_1(\xi_{h-1,j-1})$ ∎

Using this result j times consequtively we obtain

$$p_1(\xi_{h,j}) = p_1(\xi_{h-j,0}).$$

Also note that the map $g_h : S^4 = \mathbb{H} \cup \mathbb{H}_\infty \to \mathbb{H}\cup\mathbb{H}_\infty = S^4$ given by $g_h(p) = p^h$ satisfies

$g_h^*(\xi_{1,0}) \simeq \xi_{h,0}$ (Exercise!) and thus

$$p_1(\xi_{h,0}) = p_1(g_h^* \xi_{1,0}) = g_h^*(p_1(\xi_{1,0})) = \deg(g_h)\, p_1(\xi_{1,0})$$

$\Rightarrow p_1(\xi_{h,0}) = h\, p_1(\xi_{1,0})$ since $\deg(\theta_h) = h$.

Hence, we just need to compute $p_1(\xi_{1,0})$.

<u>Claim</u> The bundle $\xi_{1,0} \longrightarrow S^4$ admits a complex structure.

<u>Proof</u>: $p = a + ib + jc + kd = A + jB$, $A = a + ib$, $B = c - di$

$v = e + if + jg + kh = C + jD$, $C = e + if$, $D = g - hi$

$$p \cdot v \longleftrightarrow \begin{bmatrix} A & -\bar{B} \\ B & \bar{A} \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix}$$

$$v \cdot z \doteq \begin{bmatrix} C \\ D \end{bmatrix} z = \begin{bmatrix} Cz \\ Dz \end{bmatrix}, \quad z \in \mathbb{C}.$$

This finishes the proof. ∎

So, $p_1(\xi_{1,0}) = c_1^2(\xi_{1,0}) - 2c_2(\xi_{1,0})$ since $\xi_{1,0}$ is a complex vector bundle of rank two.

$\Rightarrow p_1(\xi_{1,0}) = 0 - 2e(\xi_{1,0}) = -2(-1-0)\nu = 2\nu$

where $\nu \in H^4_{DR}(S^4)$ with $\int_{S^4} \nu = 1$.

<u>Corollary</u> For the bundle $\xi_{h,j} \longrightarrow S^4$ oriented by the natural orientation of $\mathbb{H}$, we have

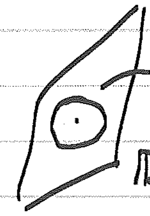
$e(\xi_{h,j}) = -(h+j)\nu$ and $p_1(\xi_{h,j}) = 2(h-j)\nu$.

Milnor's Exotic Spheres

For a given odd integer k choose $h, j \in \mathbb{Z}$ so that $h+j=-1$, $h-j=k$.

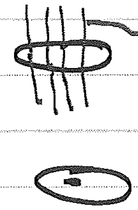
Let $B_k = B_k^8$ be the total space of the disk bundle

$$\xi_{h,j} \longrightarrow S^4$$



Let $M_k^7 = \partial B_k^8$ be the total space of the corresponding unit sphere bundle.

$$\begin{array}{ccc} D^4 \longrightarrow B_k & \qquad & S^3 \longrightarrow M_k \\ \downarrow & & \downarrow \\ S^4 & & S^4 \end{array}$$



<u>Theorem</u> $\lambda(M_k) = k^2 - 1 \equiv 0 \pmod 7$.

$$\begin{array}{c} \xi_{h,j} \\ \downarrow \end{array}$$

<u>Proof</u>: Consider the projection map $\pi : B_k \to S^4$.

Then $T_* B_k = \pi^*(T_* S^4) \oplus \pi^*(\xi_{h,j})$

Let $\alpha = \pi^*(\nu) \in H^4_{DR}(B_k)$.

By the Whitney Product Formula

$$p_1(B_k) = p_1(\pi^*(T_* S^4)) + p_1(\pi^*(\xi_{h,j}))$$

$$= \pi^*(\overbrace{p_1(T_* S^4)}^{=0}) + \pi^*(p_1(\xi_{h,j}))$$

$$= \pi^*(2(h-j)\nu) = 2(h-j)\alpha = 2k\alpha.$$

On the other hand, $h+j=-1$ implies that $e(\xi_{h,j}) = 1\cdot\nu = \nu$.

<u>Claim:</u> $\int_B \alpha^2 = 1$.

<u>Proof:</u> $\pi^*: H^4_{DR}(S^4) \to H^4_{DR}(B_k)$ is an isomorphism (B_k is homotopy equivalent to S^4) and thus $H^4_{DR}(B_k) = \langle\alpha\rangle$, $\alpha = \pi^*(\nu)$. Let S^4 denote the zero section of the bundle $D^4 \to B_k \to S^4$. Also let β be the Poincaré dual of S^4 in B_k. Hence, $\beta = a\alpha$ for some $a \in \mathbb{R}$. Since the number of $\pi: B_k \to S^4$ is one the self intersection of S^4 in B_k is one:

$$1 = \mathrm{Int}_{B_k}(S^4, S^4) = \int_{S^4} \beta = \int_{B_k} \beta^2.$$

On the other hand, $\int_{S^4} \alpha = 1$ so that $\alpha = \beta$ and

$$\int_{B_k} \alpha^2 = \int_{B_k} \beta^2 = 1.$$

∎

Now it follows that $\tau(B_k) = 1$ and

$$q(B_k) = \int_{B_k} p_1^2(B_k) = \int_{B_k} (2k\alpha)^2 = 4k^2 \int_{B_k} \alpha^2 = 4k^2.$$

Hence, $\lambda(M_k) = 2q(B_k) - \tau(B_k) = 8k^2 - 1 = k^2 - 1\ (7).$

Video 57

Note Title 27.05.2020

Theorem: For any odd integer $k \in \mathbb{Z}$ the manifold M_k is homeomorphic to S^7. However, M_k is not diffeomorphic to S^7 provided that
$$\lambda(M_k) = k^2 - 1 \neq 0 \pmod 7.$$

Proof: If S^7 is diffeomorphic to M_k then $\lambda(M_k) = \lambda(S^7) = 0$ because $S^7 = \partial D^8$ and $H^4_{DR}(D^8) = 0$, which is a contradiction.

So we just need to show that M_k is homeomorphic to S^7.

M_k is the total space of the unit sphere bundle of $\xi_{h,j}$, $(h+j=-1,\ h-j=k)$.

$$\xi_{h,j} = \mathbb{H} \times \mathbb{H} \,\dot\cup\, \mathbb{H} \times \mathbb{H} / (p,v) \sim \left(\tfrac{1}{p}, p^h v p^j\right),\ p \neq 0$$

Thus
$$M_k = \mathbb{H} \times S^3 \,\dot\cup\, \mathbb{H} \times S^3 / (p,v) \sim \left(\tfrac{1}{p}, \underbrace{\tfrac{1}{\|p^{h+j}\|}}_{= \frac{1}{\|p\|^{h+j}} = \|p\|} p^h v p^j\right),\ p \neq 0$$

$$(p,v) \sim \left(\tfrac{1}{p}, \|p\| p^h v p^j\right) = (q,u)$$

Define a function $F: M_k \to \mathbb{R}$ as follows:

$F(p,v) = \dfrac{\mathrm{Re}(v)}{\sqrt{1+\|p\|^2}}$ and on the other coordinate

chart $F(q,u) = \dfrac{\mathrm{Re}(u)}{\sqrt{1+\|q\|^2}}$, where

$q = \frac{1}{p}$, $w = q\frac{1}{u}$. We must check that F is well-defined:

$$w = q\frac{1}{u} = \frac{1}{p}\,\frac{1}{\|p\|\,p^h v p^j}$$

$$w = \frac{1}{p\|p\|}\,(p^j)^{-1}\,v^{-1}\,(p^h)^{-1}$$

$$= \frac{1}{p\|p\|}\,\frac{\bar{p}^j}{\|p\|^{2j}}\,\bar{v}\,\frac{\bar{p}^h}{\|p\|^{2h}}$$

$$= \frac{\bar{p}}{\|p\|^2\,\|p\|}\,\frac{\bar{p}^j\,\bar{v}\,\bar{p}^h}{\|p\|^{2(-1)}}$$

$$= \frac{1}{\|p\|}\,\bar{p}^{j+1}\,\bar{v}\,\bar{p}^h$$

$\mathrm{Re}(ab) = \mathrm{Re}(ba)$

$$F(q,u) = \frac{\mathrm{Re}(w)}{\sqrt{1+\|q\|^2}}$$

$$= \frac{\frac{1}{\|p\|}\,\mathrm{Re}(\bar{p}^{j+1}\,\bar{v}\,\bar{p}^h)}{\sqrt{1+1/\|p\|^2}}$$

$$= \frac{\mathrm{Re}(\bar{p}^{j+1}\,\bar{v}\,\bar{p}^h)}{\sqrt{1+\|p\|^2}}$$

$$= \frac{\mathrm{Re}(\bar{p}^{(j+1+h)}\,\bar{v})}{1+\|p\|^2}$$

$$= \frac{\mathrm{Re}(\bar{p}^0\,\bar{v})}{1+\|p\|^2}$$

$$= \frac{\mathrm{Re}(v)}{1+\|p\|^2}$$

$F : M_k \longrightarrow \mathbb{R}$ smooth function.

__Exercise:__ F has exactly two critical points which are both nondegenerate.

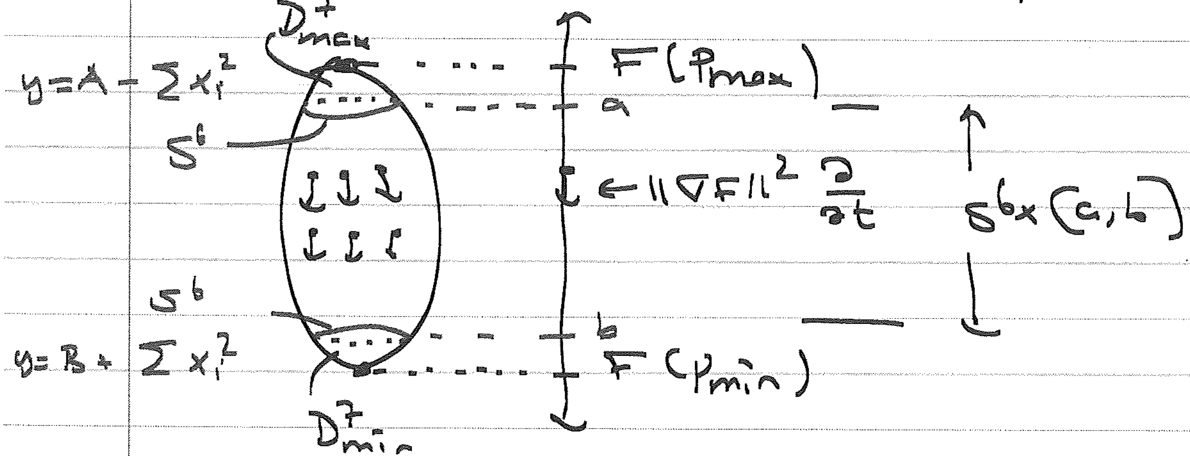
Say P_{min} and P_{max} are the critical points of F.

$\mathrm{Hess}(F)(P_{min})$ is positive definite and $\mathrm{Hess}(F)(P_{max})$ is negative definite.

<u>Morse Lemma</u>: Around p_{min} (resp. p_{max}) there is a coordinate chart on which F is given by

$$F(x_1, x_2, x_3, x_4) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2$$

$$(\text{resp. } -x_1^2 - x_2^2 - x_3^2 - x_4^2 - x_5^2 - x_6^2 - x_7^2)$$



The gradient vector field $-\nabla F$ is never zero on $M_k \setminus \{p_{min}, p_{max}\}$.

This vector field has flow, i.e., a family of diffeomorphisms $\varphi_t : M_k \to M_k$, $\varphi_0 = id$ with

$$\dot{\varphi}_t(p) = -\nabla F(p)$$

Hence M_k is homeomorphic to $S^6 \times [a, b] \cup D^7_{max} \cup D^7_{min}$ which is S^7 (topologically).

(Reeb's Sphere Theorem, 1946).