MATH 537
ALGEBRAIC TOPOLOGY - I

Note Title 24.06.2021

METU Math. Department

Fall of 2020

<u>Textbook</u>: Algebraic Topology by
Allen Hatcher

(Textbook is available at his webpage.)

Introduction:

Definition of a topological space: A topology on a set X is a collection $\mathcal{T}$ of subsets of X satisfying the following axioms:

1) $\phi, X \in \mathcal{T}$.

2) If $U_1, \ldots, U_n \in \mathcal{T}$ then so is $U_1 \cap \cdots \cap U_n \in \mathcal{T}$.

3) If $U_\alpha \in \mathcal{T}$ for all $\alpha \in J$, for some index set J, then $\bigcup\limits_{\alpha \in J} U_\alpha \in \mathcal{T}$.

Elements of $\mathcal{T}$ are called open subsets of the topology $\mathcal{T}$. The pair $(X, \mathcal{T})$ is called a topological space.

A subset $C \subseteq X$ is called closed if $X \setminus C$ is open (i.e. $X \setminus C \in \mathcal{T}$).

Let $A \subseteq X$ be any subset. The closure of A, denoted $\bar{A}$ is the subset defined by

$$\bar{A} = \bigcap_{\substack{A \subseteq F \\ F \text{ closed}}} F \quad \text{, which is closed.}$$

Similarly, Interior of A, denoted $\mathrm{Int}\, A$ or $\mathring{A}$, is the subset defined by

$$\operatorname{Int} A = \bigcup_{\substack{U \subseteq A \\ U \text{ open}}} U .$$

<u>Examples</u>: 1) Finite topologies

$X = \{a,b,c\}$, $\mathcal{T} = \{\emptyset, \{a,b,c\}, \{a\}, \{b\}, \{a,b\}\}$ is a topology on X.

2) X any set. The smallest topology on X is the topology $\{\emptyset, X\}$.
This topology is also called the weakest or coarsest topology on X.

3) X any set. The largest topology on X is the topology $\mathcal{P}(X)$.

This topology is also called the strongest or the finest topology on X.

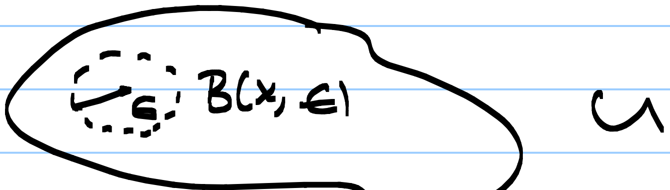
4) $X = \mathbb{R}$ $\mathcal{T}_{std} = \{U \mid U \subseteq \mathbb{R}$, if $x \in U$ then

$(x-\epsilon, x+\epsilon) \subseteq U$ for some $\epsilon > 0\}$.

$\mathbb{R}_{std}$, $\mathbb{R}^n_{std}$ $U \subseteq \mathbb{R}^n$ open if

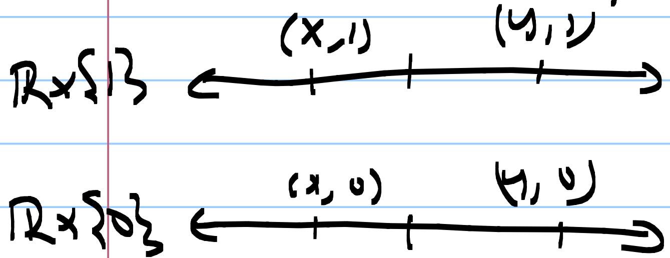
for any $x \in U$ there is some $\epsilon > 0$ so that

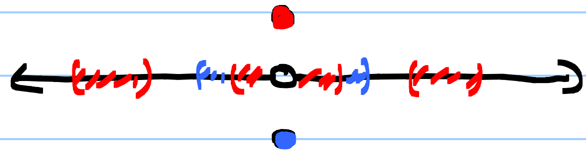
$B(x,\epsilon) = \{y \in \mathbb{R}^n \mid \|x-y\| < \epsilon\} \subseteq U$.



5) Real line with double origin.

$X = \mathbb{R}\times\{0,1\} \,/\, (x,0)\sim(x,1)$ unless $x\neq 0$





<u>Definition</u>: A topological space $(X,\mathcal{J})$ is called T_0 if for any $x,y\in X$ with $x\neq y$ then there is an open set U so that either

$(x\in U$ and $y\notin U)$ or $(y\in U$ and $x\notin U)$.

Similarly, X is called T_1 if for any $x,y\in X$ with $x\neq y$ there is an open set U with

$x\in U$ and $y\notin U$.

Finally, X is called T_2 (or Hausdorff) if for any $x,y\in X$ with $x\neq y$ there are open subsets U,V so that

$x\in U$, $y\in V$ and $U\cap V=\emptyset$.

Clearly, $T_2 \Rightarrow T_1 \Rightarrow T_0$.

<u>Example</u> The real line with double origin is T_1 but <u>not</u> T_2.

Equivalence of Topologies:

A function $f:(X,\mathcal{T}_X)\to(Y,\mathcal{T}_Y)$ is called continuous if

$f^{-1}(U)$ is open ($f^{-1}(U)\in\mathcal{T}_X$)

U is open in Y ($U\in\mathcal{T}_Y$).

A continuous bijection $f:(X,\mathcal{T}_X)\to(Y,\mathcal{T}_Y)$ is called a homeomorphism if its inverse

$f^{-1}:(Y,\mathcal{T}_Y)\to(X,\mathcal{T}_X)$ is also continuous.

In this case $(X,\mathcal{T}_X)$ and $(Y,\mathcal{T}_Y)$ are called homeomorphic. We regard homeomorphic spaces as the same in topology.

Basis and Subbasis:

Let $(X,\mathcal{T})$ be a space. A subcollection $\mathcal{B}$ of open subsets of $(X,\mathcal{T})$ is called basis if the followings are satisfied.

1) For any open subset U of X and point $x\in U$ there is some $B\in\mathcal{B}$ so that

$$x\in B\subseteq U.$$

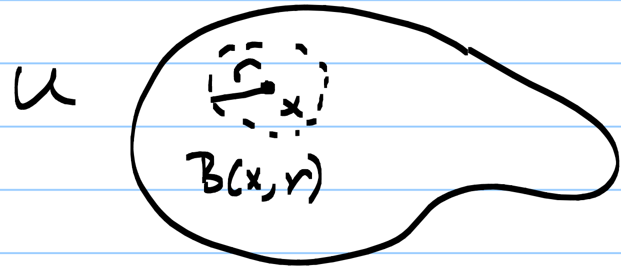
2) If $B_1,B_2\in\mathcal{B}$ and $x\in B_1\cap B_2$ then there is $B_3\in\mathcal{B}$ so that

$$x\in B_3\subseteq B_1\cap B_2.$$

Video 2

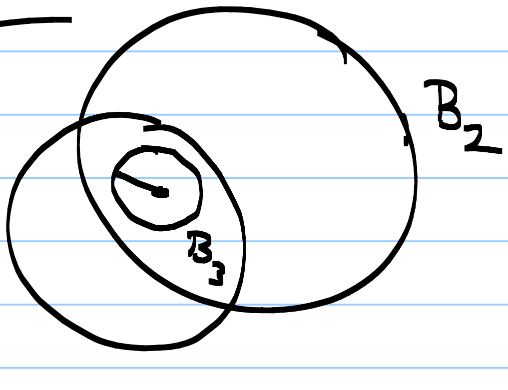
Example: $X = \mathbb{R}^2_{std}$

$\mathcal{B} = \{ B(x,r) \mid x \in \mathbb{R}^2, r > 0 \}$



1) ✓

2)

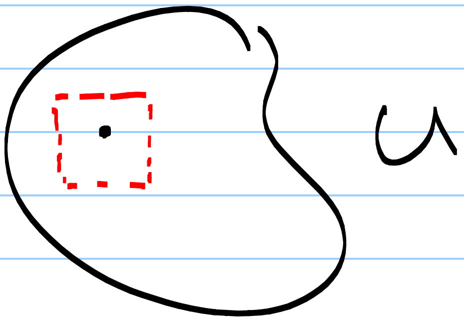


Hence, $\mathcal{B}$ is a basis for the standard topology on $\mathbb{R}^2$.

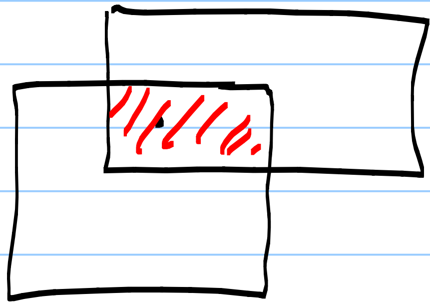
2) $\mathcal{C} = \{ (a,b) \times (c,d) \mid a<b,\ c<d \in \mathbb{R} \}$

is also a basis for the standard topology on $\mathbb{R}^2$.

1) ✓



2)



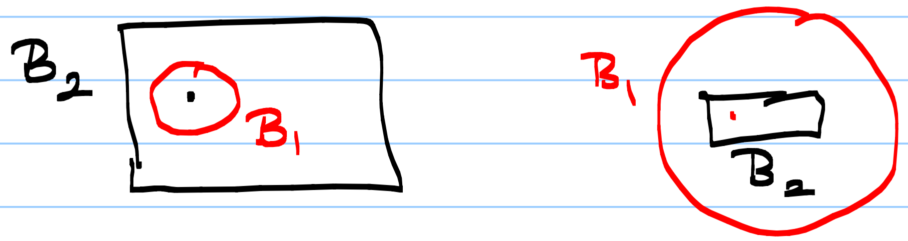
Comparison of Topologies: Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on a set X. We say that $\mathcal{T}_1$ is stronger or finer than $\mathcal{T}_2$ if

$$\mathcal{T}_2 \subseteq \mathcal{T}_1 .$$

Remark: Let $\mathcal{B}_i$ be bases for $\mathcal{T}_i$, $i=1,2$, two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on a set X.

Then $\mathcal{T}_1$ is stronger that $\mathcal{T}_2$ if and only if the following holds:

For any $B_2 \subseteq \mathcal{B}_2$ and point $x \in B_2$ then is some $B_1 \in \mathcal{B}_1$ so that $x \in B_1 \subseteq B_2$.



Definition: Let X be any set and $\mathcal{B}$ be a collection of subsets of X. The collection of subsets consisting of finite intersections of elements of $\mathcal{B}$ and arbitrary unions of finite intersections form a topology on X and it is called the topology generated by subbasis $\mathcal{B}$.

$$X = \bigcup_{B \in \mathcal{B}} B$$

New Topologies from Old Ones

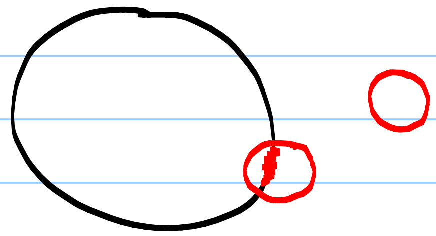
Subspace Topology $(X, \mathcal{T})$ topological space.

Any subset A of X inherits a topology from

$(X, \mathcal{T})$ called the subspace topology, defined by

$$\mathcal{T}_A = \{ U \cap A \mid U \in \mathcal{T} \}.$$

Example: $S^1 = \{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \} \subseteq \mathbb{R}^2_{std}$



Definition: Let $f: (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be a one and continuous map. Then $f(X)$, the image of X, is a subspace of Y.

If the map $f: (X, \mathcal{T}_X) \to (f(X), \mathcal{T}_Y|_{f(X)})$

is a homeomorphism (i.e., "f^{-1}" is also continuous), then f is called a topological embedding of X into Y.

Example $X = [0, \infty)$ with the basis

$$\mathcal{B} = \{ (a,b) \mid 0 < a < b \} \cup \{ [0,a) \cup (b, \infty) \mid a, b > 0 \}.$$

One can check that $\mathcal{B}$ is a basis for a topology on X.

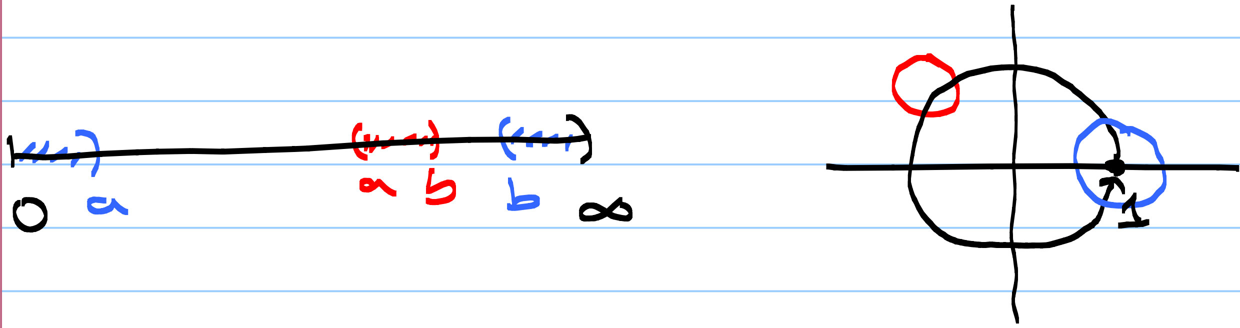
Claim: The topological space $(X, \mathcal{T})$, where $\mathcal{T}$ is generated by the basis $\mathcal{B}$ is homeomorphic to the subspace topology on S^1.

Video 3

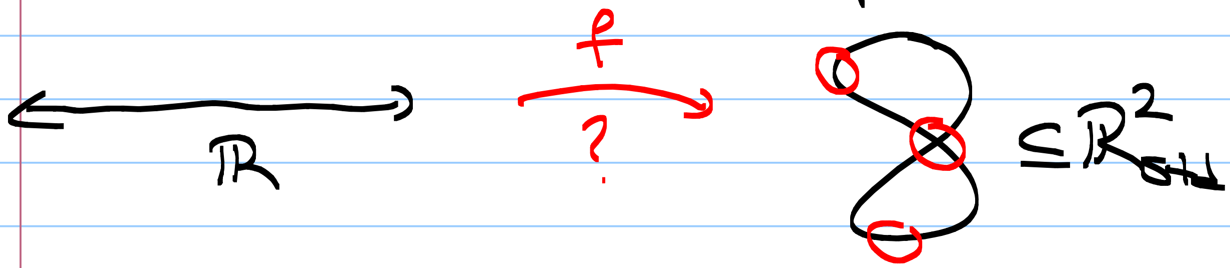
A homeomorphism is given by

$f : [0,\infty) \longrightarrow S^1$, $f(t) = e^{2\pi i t / 1+t}$

$f(t) = \left(\cos \frac{2\pi t}{1+t}, \sin \frac{2\pi t}{1+t}\right)$, $t \in [0,\infty)$.



<u>Exercise</u>: Put a topology on $\mathbb{R}$ so that it is homeomorphic to the figure eight in $\mathbb{R}^2_{std}$.



<u>Product Spaces</u>

$(X_\alpha, \mathcal{T}_\alpha)_{\alpha\in\Lambda}$ a family of topological spaces

$$\prod_{\alpha\in\Lambda} X_\alpha = \{ f : \Lambda \longrightarrow \dot{\bigcup_{\alpha\in\Lambda}} X_\alpha \mid f(\alpha) \in X_\alpha, \forall \alpha \in \Lambda \}.$$

$\prod X_\alpha \neq \emptyset$ by the Axiom of Choice

Product topology on $\prod X_\alpha$ is given by the basis $\mathcal{B}$ consisting of elements of the form $\prod_{\alpha\in\Lambda} U_\alpha$, $U_\alpha \subseteq X_\alpha$ open and

$U_\alpha = X_\alpha$ for all but finitely many $\alpha\in\Lambda$.

<u>Box Topology</u>: This is given by the basis consisting of products of the form

$\prod_{\alpha\in\Lambda} U_\alpha$, $U_\alpha \subseteq X_\alpha$ open for all α.

Box topology is much stronger than the product topology.

<u>Fact</u>: $\varphi : X \longrightarrow \prod_\alpha X_\alpha$, X, X_α top. spaces.

$P_\beta : \prod_\alpha X_\alpha \longrightarrow X_\beta$, $f \in \prod_\alpha X_\alpha$, $f : \Lambda \longrightarrow \dot{\bigcup_\alpha} X_\alpha$

$f \longmapsto f(\beta)$

Each P_β is clearly continuous.

φ is continuous if and only if each coordinate function

$P_\beta \circ \varphi : X \longrightarrow X_\beta$ is continuous.

Quotient Topology

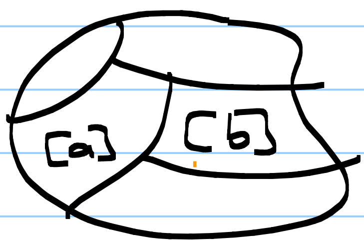
Let $f: X \to Y$ be an onto map so that X is a topological space and Y is any set. Call a subset V of Y open if and only if $f^{-1}(V)$ is open in X.

This defines a topology on Y and it is the strongest topology on Y so that f is continuous. This topology is called the quotient topology on Y determined by $f: X \to Y$.

Remark: Any onto map $f: X \to Y$ defines an equivalence relation on X as follows:

$$x_1, x_2 \in X, \quad x_1 \sim x_2 \iff f(x_1) = f(x_2).$$

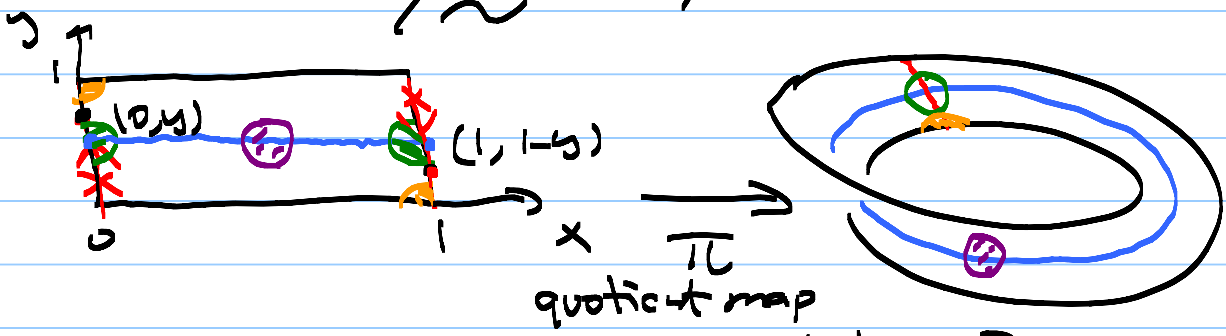
$[x] = \{x' \in X \mid x' \sim x\}$ the equivalence class of x.



The set equivalence classes can be identified with Y.

$$X/\sim \; = Y$$

Ex:



Möbius Band

$$MB = [0,1] \times [0,1] / \sim$$

$(x_1, y_1) \sim (x_2, y_2)$ if and only if $|x_1 - x_2| = 1$ and $y_1 + y_2 = 1$.

Proposition: Let $\pi : X \to Y$ be a quotient space. Let $f : Y \to Z$ be a map. Then f is continuous if and only if $f \circ \pi : X \to Z$ is continuous.

$$\begin{tikzcd} X \arrow[dr, "f\circ\pi"] \arrow[d, "\pi"'] & \\ Y \arrow[r, "f"'] & Z \end{tikzcd}$$

Corollary: Let $\pi : X \to Y$ be a quotient map and $g : X \to Z$ be a map.

$$\begin{tikzcd} x \arrow[d, mapsto] & X \arrow[r, "g"] \arrow[d, "\pi"'] & Z \\ {[x]} & Y \arrow[ur, dashed, "?f"'] & \end{tikzcd}$$

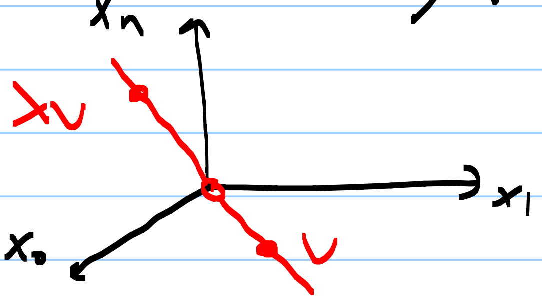
There is a map $f : Y \to Z$ such that $f \circ \pi = g$ if and only if g is constant on each $\pi^{-1}(y)$ (fiber).

Moreover, if such f exists then f is continuous if and only if g is continuous.

In this case, we say that g descends to the quotient space Y.

Example: $\mathbb{RP}^n$: Real Projective Space.

$$\mathbb{RP}^n = \mathbb{R}^{n+1} \setminus \{0\} \Big/ v \sim \lambda v, \quad v = (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \setminus \{0\}, \ \lambda \in \mathbb{R} \setminus \{0\}$$



$\mathbb{RP}^n$: the set of all lines in $\mathbb{R}^{n+1}$ containing the origin.

Consider the map $\pi : \mathbb{R}^{n+1} \setminus \{0\} \longrightarrow \mathbb{R}P^n$

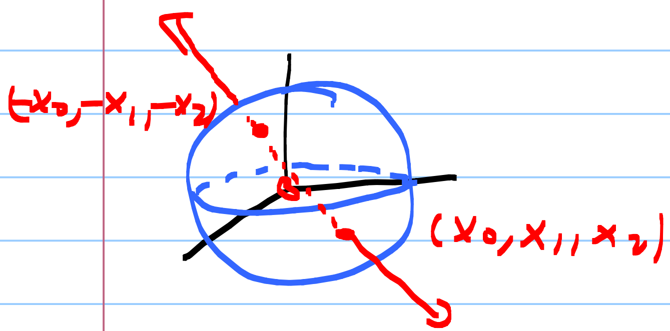
$v = (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \setminus \{0\}$, $\pi(v) \doteq [x_0 : x_1 : \dots : x_n]$

$[x_0 : \dots : x_n] = \{ \lambda (x_0, \dots, x_n) \mid \lambda \in \mathbb{R} \setminus \{0\} \}$.

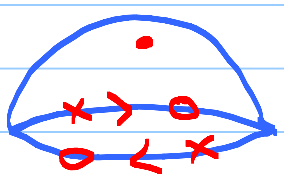
We endow $\mathbb{R}P^n$ with the quotient topology induced by the map π, where $\mathbb{R}^{n+1} \setminus \{0\}$ has the Euclidean (subspace) topology.

<u>Example</u>: $\mathbb{R}P^2$ the Real Projective Geometry

$$\mathbb{R}P^2 = \mathbb{R}^3 \setminus \{0\} \Big/ v \sim \lambda v, \quad v = (x_0, x_1, x_2) \in \mathbb{R}^3 \setminus \{0\}, \ \lambda \in \mathbb{R} \setminus \{0\}$$



$$\mathbb{R}P^2 = S^2 \Big/ (x_0, x_1, x_2) \sim (-x_0, -x_1, x_2)$$



$\mathbb{R}P^2$ does <u>not</u> embed into $\mathbb{R}^3$.
(Not easy to prove!)

<u>Proposition</u>: $\mathbb{R}P^2$ embeds into $\mathbb{R}^5$.

<u>Proof</u>:

$$\begin{tikzcd} S^2 \arrow[r, "f"] \arrow[d, "\pi"'] & \mathbb{R}^5 \\ \mathbb{R}P^2 \arrow[ur, dashed, "g"'] & \end{tikzcd}$$

<u>want</u>: g a homeomorphism onto its image.

$\pi(x_0, x_1, x_2) = [x_0 : x_1 : x_2] = \pi(-x_0, -x_1, x_2).$

$f(x_0, x_1, x_2) = (x_0^2, x_1^2, x_0x_1, x_0x_2, x_1x_2)$

$(x_0^2 + x_1^2 + x_2^2 = 1)$ f is clearly continuous on S^2

$$\begin{tikzcd} S^2 \arrow[r, "f"] \arrow[d, "\pi"'] & \mathbb{R}^5 \\ \mathbb{RP}^2 \arrow[ur, "g"', "\exists !"] & \end{tikzcd}$$

$f = g \circ \pi$

g is also continuous since f is.

Since S^2 is compact so is its image $\pi(S^2)$. Hence, $\mathbb{RP}^2$ is a compact topological space.

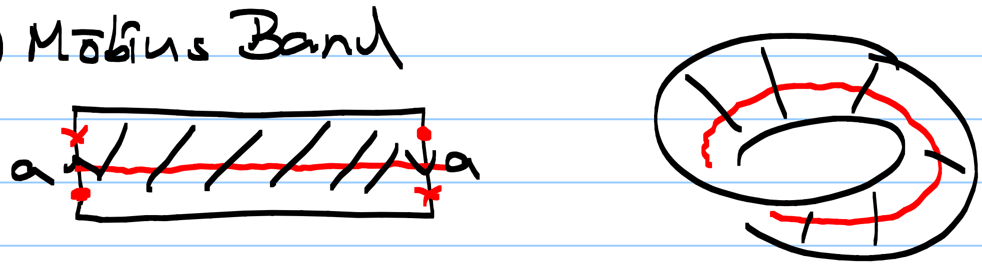
Since $\mathbb{RP}^2$ is compact and $g : \mathbb{RP}^2 \to \mathbb{R}^5$ is continuous g is a homeomorphism onto its image.

Here, we use the fact that f is $2:1$ and thus g is one to one.

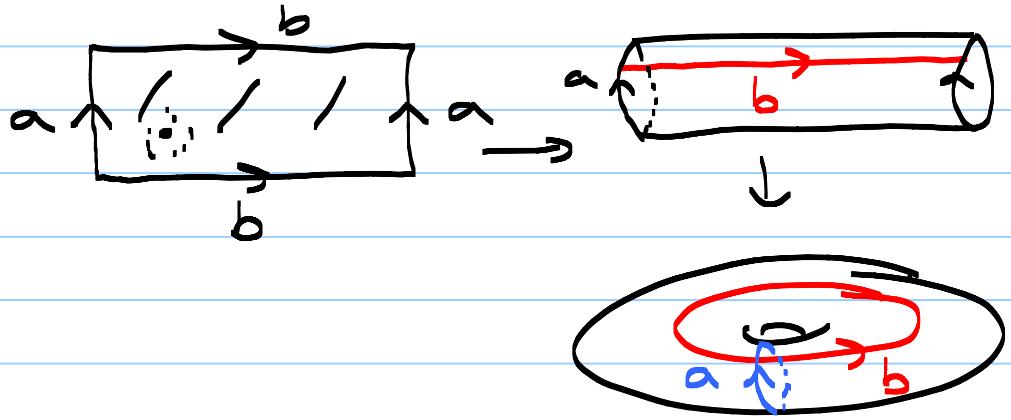
$$\left[\; f(x_0, x_1, x_2) = f(y_0, y_1, y_2) \Longleftrightarrow (x_0, x_1, x_2) = \pm (y_0, y_1, y_2) \;\right]$$

More Examples with Quotient topology

1) Möbius Band



2) Torus T^2



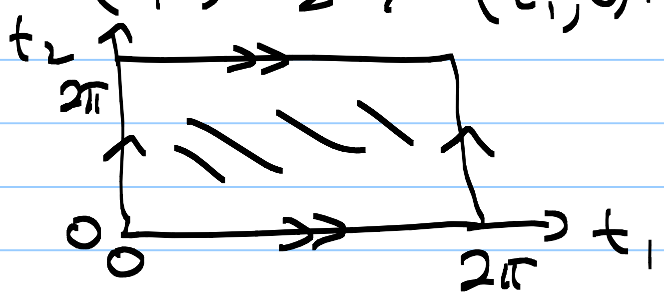
T^2 is homeomorphic to $S^1 \times S^1$ and it embeds in $\mathbb{R}^3$.

$$\begin{tikzcd}
{[0,2\pi]\times[0,2\pi]} \arrow[r, "\varphi"] \arrow[d] & S^1\times S^1 \\
{[0,2\pi]\times[0,2\pi]/\sim} \arrow[ur, "\phi\ \exists!"'] &
\end{tikzcd}$$

$$\varphi : (t_1, t_2) \longmapsto (\cos t_1, \sin t_1, \cos t_2, \sin t_2)$$

$[0,2\pi]\times[0,2\pi] \Big/ (0,t_2)\sim(2\pi,t_2),\ (t_1,0)\sim(t_1,2\pi)$, for all t_1, t_2.

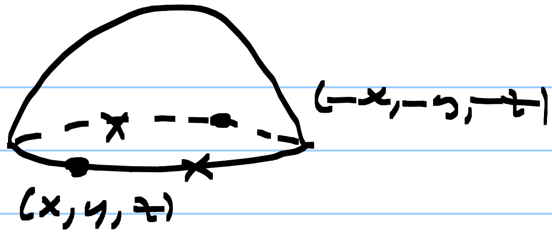
(t_1, t_2)



ϕ is a homeomorphism.

Video 5

3) $\mathbb{RP}^2$ =



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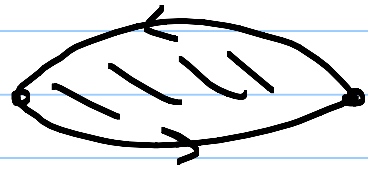
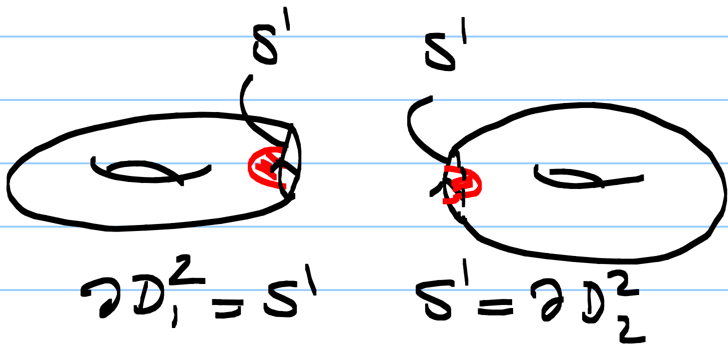
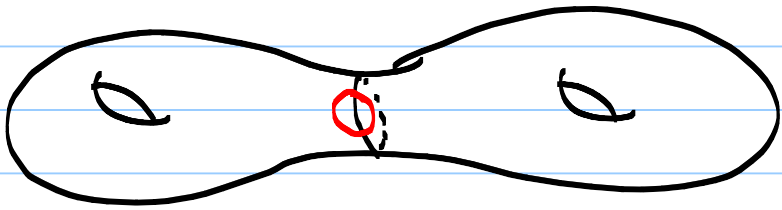


4) Connected Sum (of Surfaces):

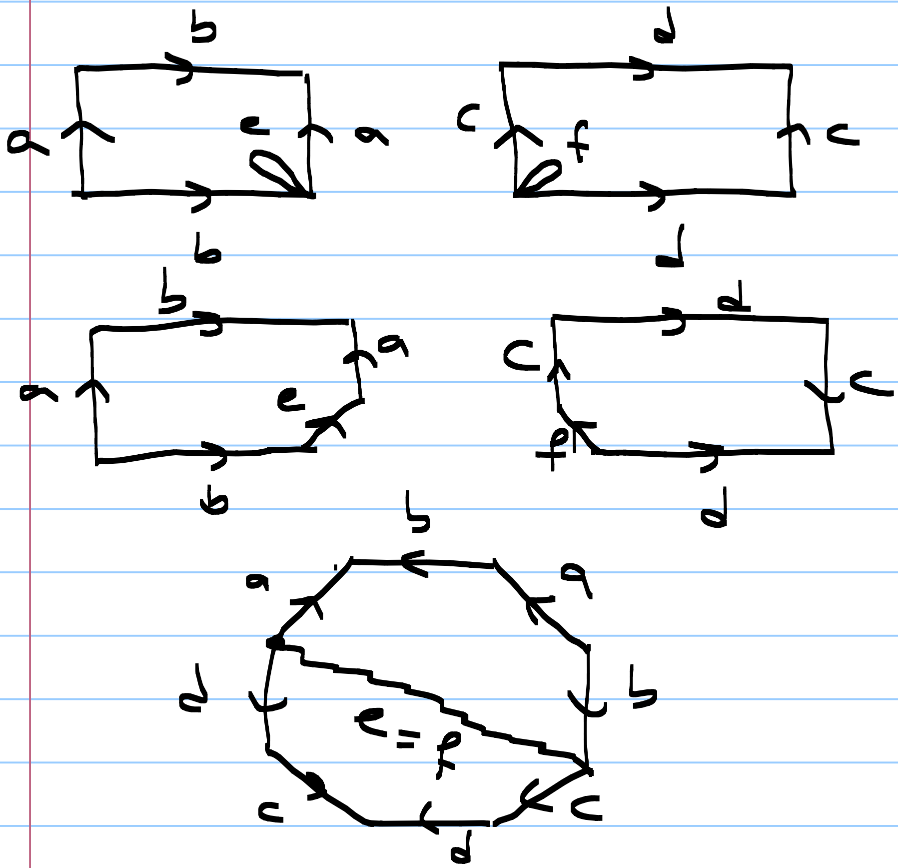
$$T^2 \# T^2 = (T^2 \setminus D_1^2) \sqcup (T^2 \setminus D_2^2) \big/ x \sim f(x)$$

$f : \partial D_1^2 \to \partial D_2^2$
a homeomorphism

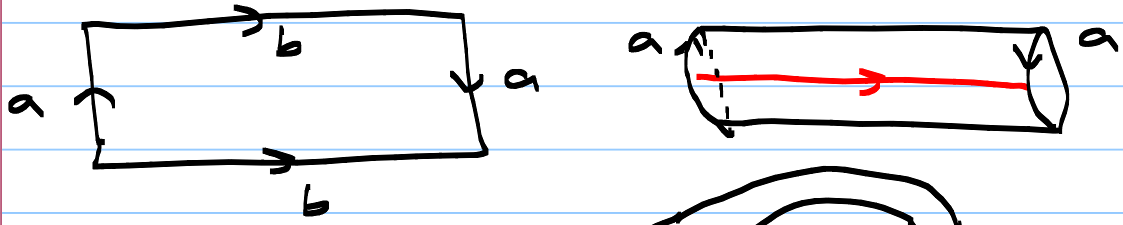


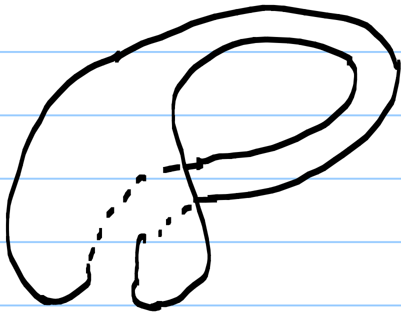


Genus two orientable surface



Exercise: Klein Bottle

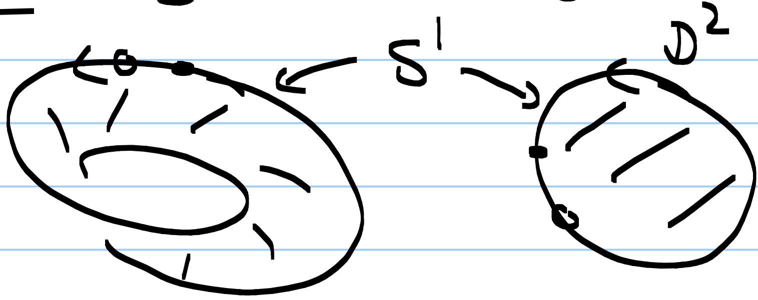




Prove that $KB = \mathbb{RP}^2 \# \mathbb{RP}^2$ and

$$KB \# \mathbb{RP}^2 = \mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2 = T^2 \# \mathbb{RP}^2.$$

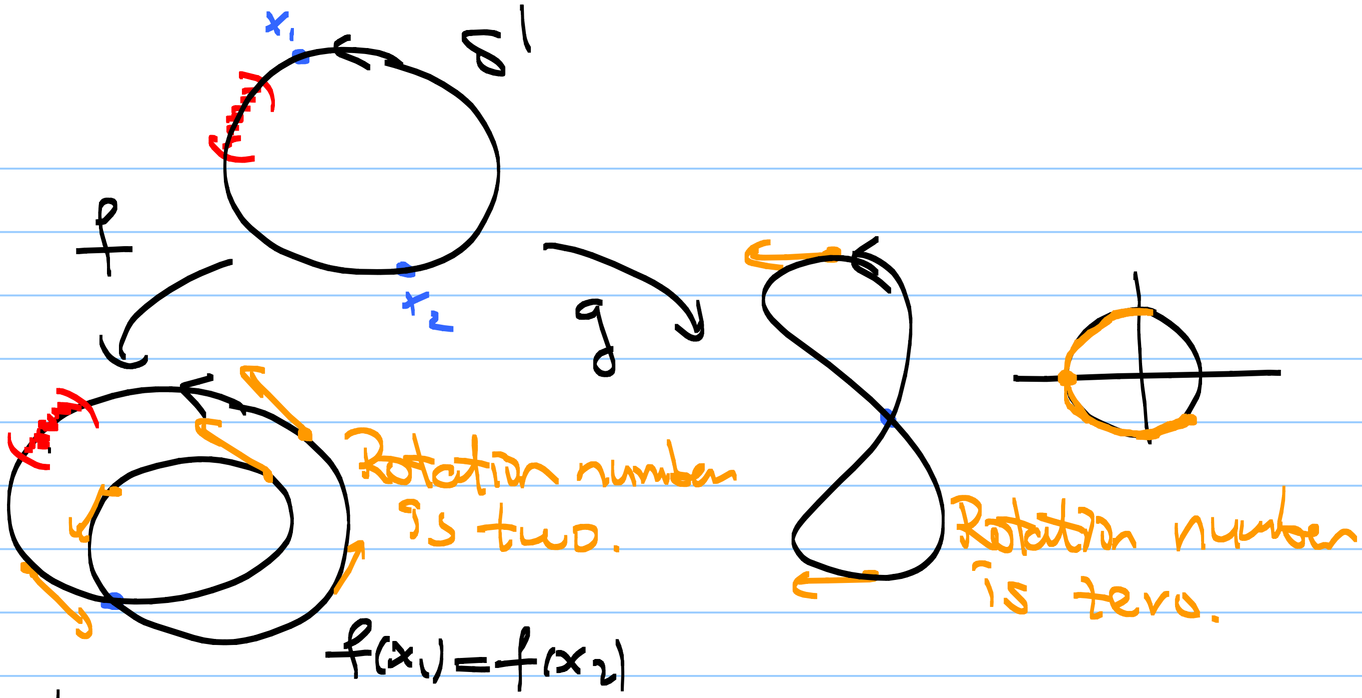
Hint: $\mathbb{RP}^2 = MB \amalg D^2$



Question: Why algebraic topology?

Answer: Comparing topological spaces is generally much more difficult than comparing algebraic objects?

Example: How can we distinguish the two immersions of the S^1 into $\mathbb{R}^2$ given below.



not an embedding.

The answer to this question is No! The above immersions are not homotopic through immersions.

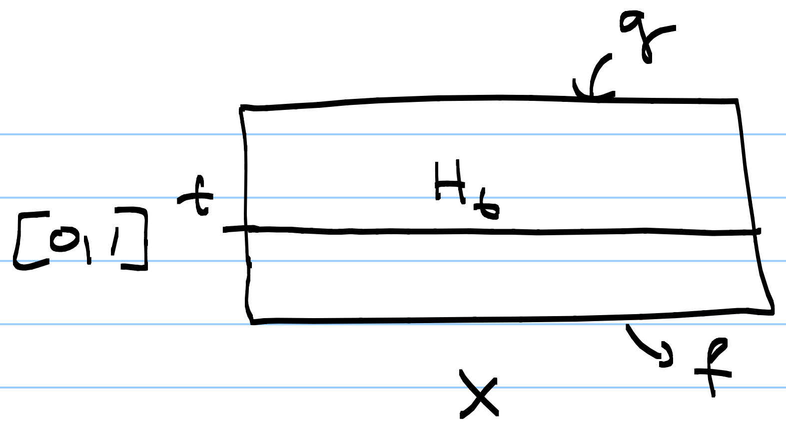
Rotation number is invariant under homotopies (through immersions) and thus the above immersions are not homotopic.

Definition of Homotopy:

Let $f: X \to Y$ and $g: X \to Y$ be two continuous maps of topological spaces. We say that f and g are homotopic maps if there is a continuous map

$$H: X \times I \to Y, \quad I=[0,1], \text{ so that}$$

$$H(x,0)=f(x) \text{ and } H(x,1)=g(x), \text{ for all } x \in X.$$



$H_t = H(x,t)$, $H_0 = f$, $H_1 = g$

(Winding and Rotation numbers, Math 709, Video 22)

Definition (Relative Homotopy)

Let (X, A) pair of topological spaces ($A \subseteq X$ subspace) and Y any topological space.

Let $f : X \longrightarrow Y$ and $g : X \longrightarrow Y$ be continuous maps. We say that f and g are homotopic relative to A if there is a homotopy

$H : X \times I \longrightarrow Y$ such that

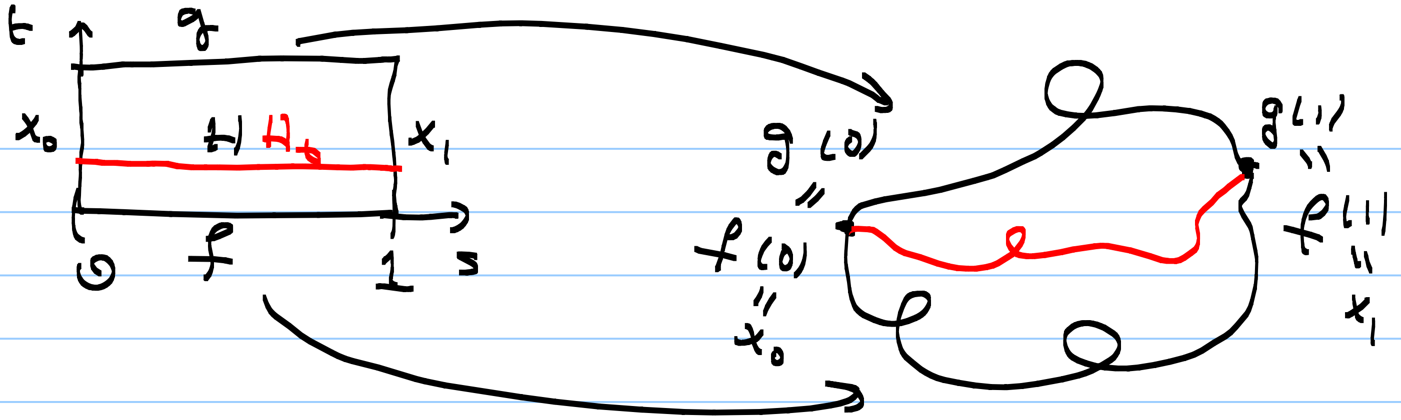
$H(x,0) = f(x)$, $H(x,1) = g(x)$, for all $x \in X$ and

$H(x,t) = f(x) = g(x)$ for all $x \in A$, and for all $t \in [0,1]$.

Example: $X = [0,1]$, $A = \partial X = \{0,1\}$.

$f, g : [0,1] \longrightarrow Y = \mathbb{R}^2$ continuous maps so that

$f(0) = g(0)$ and $f(1) = g(1)$.



f is homotopic to g rel $\{0,1\}$.

<u>Homotopy Equivalence</u> Let $f: X \to Y$ and $g: Y \to X$ be two continuous functions. If the compositions

$$f \circ g : Y \to Y \text{ and } g \circ f : X \to X \text{ are}$$

homotopic to $\mathrm{Id}_Y : Y \to Y$ and $\mathrm{Id}_X : X \to X$, respectively, then we say that the spaces X and Y are homotopy equivalent.

<u>Example:</u> $X = \mathbb{R}^n$, $Y = \{0\}$, $0 \in \mathbb{R}^n$, the origin.

$f : \mathbb{R}^n \to \{0\}$, $f(x) = 0$, $\forall x \in \mathbb{R}^n$,

$g : \{0\} \to \mathbb{R}^n$, $g(0) = 0$.

$f \circ g : \{0\} \to \{0\}$, $f \circ g = \mathrm{Id}_{\{0\}}$

$H_1 : \{0\} \times I \to \{0\}$, $H_1(\cdot, t) = f \circ g$

$g \circ f : \mathbb{R}^n \to \mathbb{R}^n$, $(g \circ f)(x) = 0$, for all $x \in \mathbb{R}^n$.

Consider the homotopy

$H_2 : \mathbb{R}^n \times I \to \mathbb{R}^n$, $H_2(x,t) = tx$, when

$H_2(x,0) = 0 \cdot x = 0 = (g \circ f)(x)$, for all $x \in \mathbb{R}^n$;

and $H_2(x,1) = 1 \cdot x = x = Id_{\mathbb{R}^n}(x)$, for all $x \in \mathbb{R}^n$.

Hence the spaces $\mathbb{R}^n$ and $\{0\}$ are homotopy equivalent.

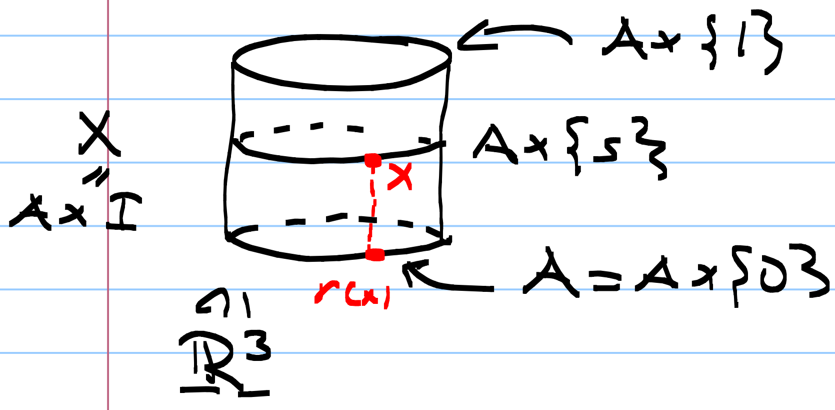
We'll see that Homotopy Equivalence is an equivalence relation on the collection of topological spaces.

<u>Definition:</u> Let (X,A) be a topological pair.

A continuous map $r : X \to A$ is called a retraction if $r(a) = a$ for all $a \in A$.

A retraction $r : X \to A$ is called deformation retraction if there is a homotopy

$H : X \times I \to X$ so that $H(x,0) = x$ for all $x \in X$ and $H(x,1) = r(x)$.



$H(x,t) = (1-t)x + t\,r(x)$.
$H(x,0) = x$, for all $x \in X$.
$H(x,1) = r(x)$, for all $x \in X$.

Video 7

Exercise: A deformation retraction is a homotopy equivalence.

Definition: A topological space is called contractible if the identity map $id_X : X \to X$ is homotopic to a constant map

$$f : X \to X, \quad f(x) = x_0, \text{ for some } x_0 \in X \text{ and for all } x \in X.$$

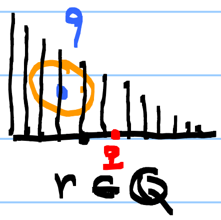
Example: The homotopy $H : \mathbb{R}^n \times I \to \mathbb{R}^n$ by

$H(x,t) = tx$ gives a contraction of $\mathbb{R}^n$ to the point $\{0\}$.

$H(x,1) = x = Id_{\mathbb{R}^n}(x)$, $\forall x \in \mathbb{R}^n$, and

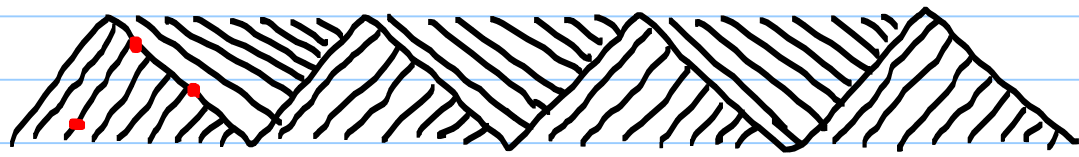
$H(x,0) = 0$, $\forall x \in \mathbb{R}^n$.

Remark: Exercise #6 of Chapter 0.

a) 

$X = \{(x,y) \mid x \in [0,1], y = 0\}$
$\cup \{(x,y) \mid x \in [0,1] \cap \mathbb{Q}, y \in [0, 1-x]\}$

X deformation retracts to any point on the x-axis but not to any other point.

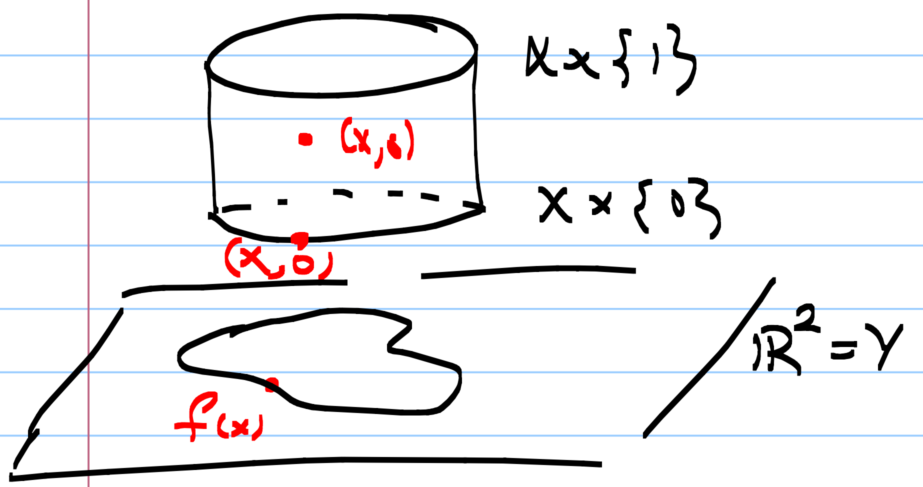
c) Y 

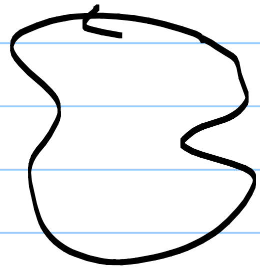
Y is contractible but does not deformation retract to any point.

<u>Mapping Cylinder</u>: Let $f: X \to Y$ be any continuous map. The topological space

$$M_f \doteq X \times I \,\dot{\cup}\, Y \Big/ (x,0) \sim f(x),\ \forall x \in X$$

$X = S^1$, $Y = \mathbb{R}^2$, $f: S^1 \to \mathbb{R}^2$





<u>Proposition</u>: M_f deformation retracts onto Y.

<u>Proof</u>: $M_f = X \times \underset{s}{I} \,\dot{\cup}\, Y \Big/ (x,0) \sim f(x),\ \forall x \in X$.

$H: M_f \times \overset{t}{I} \longrightarrow M_f$, $H((x,s),0) = (x,s)$ for all $(x,s) \in M_f$, and

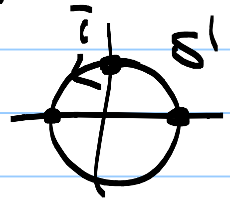
$H((x,s),1) = f(x)$, for all $(x,s) \in M_f$.

$r: M_f \longrightarrow Y$, $r(x,s) = f(x)$

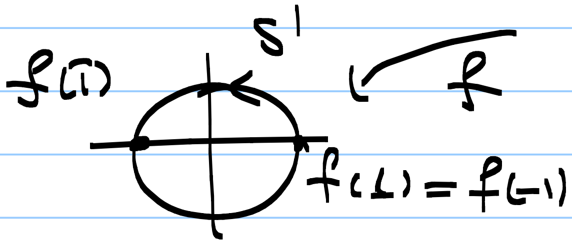
Also define H on Y as identity. Then H is continuous and gives the desired deformation retraction.

<u>Example</u>: Let $f: S^1 \to S^1$ be defined as

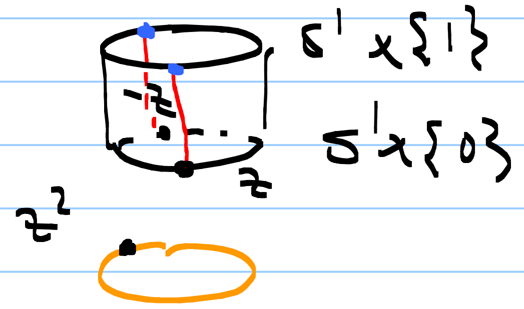
$f(z) = z^2, \quad z \in S^1 \subseteq \mathbb{C}$



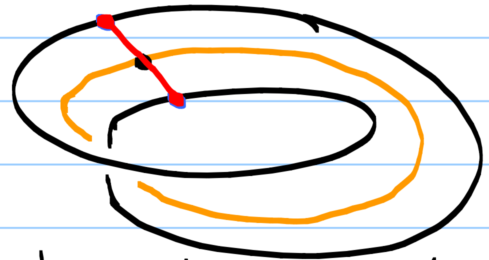
S^1 the unit circle in $\mathbb{C}$.



$X = S^1 = Y$, $M_f = S^1 \times I \cup S^1 / (z,0) \sim f(z) = z^2$



$M_f = MB$



For a proof just cut the above Möbius Band the the orange center circle.

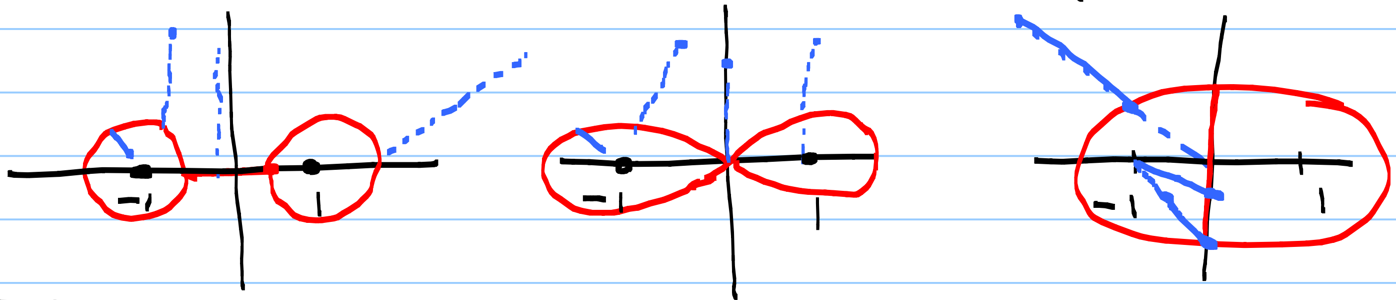
<u>Proposition</u>: Homotopy equivalence is an equivalence relation.

<u>Remark</u>: Deformation retraction is <u>not</u> an equivalence relation.

<u>Remark</u>: Deformation retraction is not an equivalence relation.

Video 8

Example: The three subspaces of $\mathbb{R}^2 \setminus \{(\pm1,0)\}$ below are deformation retraction of $\mathbb{R}^2 \setminus \{(\pm1,0)\}$ but they are not deformation retraction of each other:

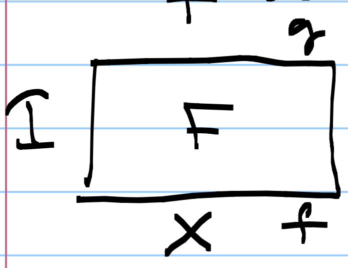


They are not deformation of each other because none of them is a subspace of any other.

Proposition: Homotopy equivalence is an equivalence relation.

Fact 1) If $f: X \to Y$ is homotopic to some $g: X \to Y$ then g is homotopic to f.

Proof Let $F: X \times I \to Y$ be a homotopy from f to g.



$F(x,0) = f(x)$, $F(x,1) = g(x)$, for all $x \in X$.

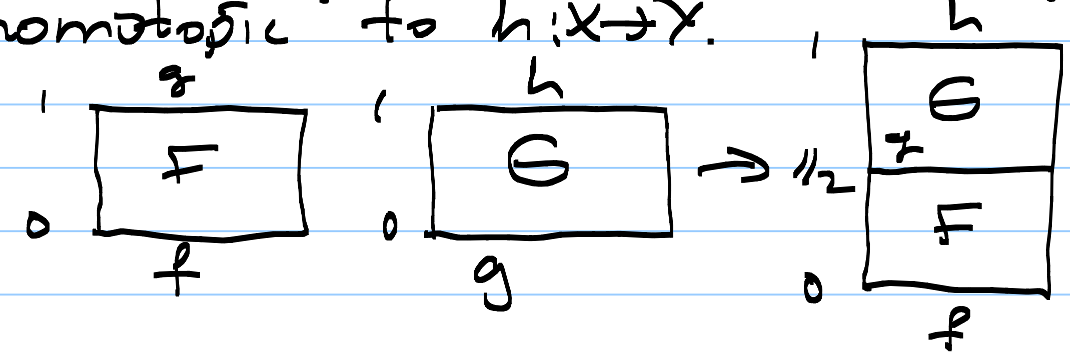
Note that $G: X \times I \to Y$, $G(x,t) = F(x,1-t)$ is a homotopy from

$$G(x,0) = F(x,1-0) = F(x,1) = g(x) \quad \text{to}$$
$$G(x,1) = F(x,1-1) = F(x,0) = f(x), \text{ for all } x \in X.$$

Fact: Clearly any function $f: X \to Y$ is homotopic to itself via the homotopy $F(x,t) = f(x)$, $x \in X$, $t \in [0,1]$.

Fact: If $f: X \to Y$ is homotopic to $g: X \to Y$ and g is homotopic to $h: X \to Y$, then f is homotopic to $h: X \to Y$.

Proof



Let $F: X \times I \to Y$ and $G: X \times I \to Y$ be homotopies from f to g and g to h, respectively. Then

$$H: X \times I \to Y \text{ by } H(x,t) = \begin{cases} F(x,2t), & 0 \le t \le 1/2 \\ G(x,2t-1), & 1/2 \le t \le 1 \end{cases}$$

is the desired homotopy (pasting lemma).

So these three facts show that being homotopic is an equivalence relation.

Proof of the proposition:

1) Reflexivity ✓

2) Symmetry: Assume that X and Y are homotopy equivalent. Then there are maps $f: X \to Y$ and $g: Y \to X$ such that

$f \circ g : Y \to Y$ and $g \circ f : X \to X$ are homotopic to id_Y and id_X, respectively. This is clearly symmetric.

3) Assume X is homotopy equivalent to Y and Y is homotopy equivalent to Z.

$$\begin{tikzcd} X \arrow[r, bend left, "f_1"] & Y \arrow[l, bend left, "g_1"] \arrow[r, bend left, "f_2"] & Z \arrow[l, bend left, "g_2"] \end{tikzcd}$$

$g_1 \circ f_1 \sim id_X$, $f_1 \circ g_1 \sim id_Y$, $g_2 \circ f_2 \sim id_Y$ and $f_2 \circ g_2 \sim id_Z$.

<u>must show:</u> $g_1 \circ g_2 \circ f_2 \circ f_1 \sim id_X$ and

$f_2 \circ f_1 \circ g_1 \circ g_2 \sim id_Z$

$$g_1 \circ \underbrace{g_2 \circ f_2}_{} \circ f_1 \sim g_1 \circ id_Y \circ f_1 = g_1 \circ f_1 \sim id_X$$

$t = 0 \to t_1 = 1/2$ $\qquad$ $t = 1/2,\ t_1 = 1$

Let $\varphi : Y \times I \to Z$ be a homotopy from $g_2 \circ f_2$ to id_Y

Define $\tilde{\varphi} : X \times I \to X$, $\tilde{\varphi}(x,t) = g_1 \circ \varphi \circ f_1$

Video 9

$$\tilde{\varphi}(x,t) = g_1(\varphi(f_1(x),t))$$

$$\tilde{\varphi}(x,0) = g_1(\varphi(f_1(x),0)) = g_1(g_2 \circ f_2)(f_1(x))$$
$$= (g_1 \circ g_2 \circ f_2 \circ f_1)(x)$$

$$\tilde{\varphi}(x,1) = g_1(\varphi(f_1(x),1)) = g_1(f_1(x)) = (g_1 \circ f_1)(x)$$

$\tilde{\varphi}(x,1)$ is not id_X but it is homotopic to id_X. Composing $\tilde{\varphi}$ with a homotopy taking $g_1 \circ f_1$ to id_X we see that

$g_1 \circ g_2 \circ f_2 \circ f_1$ is homotopic to id_X.

This finishes the proof of the proposition. ∎

Corollary 0.21 (From the Book)

Two spaces X and Y are homotopy equivalent if and only if there is a third space Z which deformation retracts onto X and Y.

Cell Complexes: We'll build a space inductively as follows:

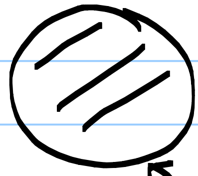
1) Start with a discrete set of points X^0, whose elements are called 0-cells.

2) Form the n-skeleton X^n from X^{n-1} by attaching n-cells e^n_α via maps

$$\varphi_\alpha : S_\alpha^{n-1} = \partial D_\alpha^n \longrightarrow X^{n-1}.$$

Hence, $X^n = X^{n-1} \coprod_\alpha D_\alpha^n \big/ x \sim \varphi_\alpha(x),\ x \in D_\alpha^n,$

where e_α^n is $\mathrm{Int}(D_\alpha^n)$.



D_α^n

$\partial D_\alpha^n = S_\alpha^{n-1}$

3) One can stop at some stage n and let $X = X^n$ or continue indefinitely, setting $X = \bigcup_n X^n$. In this case, a subset A of X will be called open (closed) if and only if $A \cap X^n$ is open (resp. closed) in X^n, for all n.

Such a space is called a CW-complex.

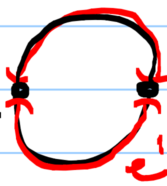
C : closure finite

W : weak topology

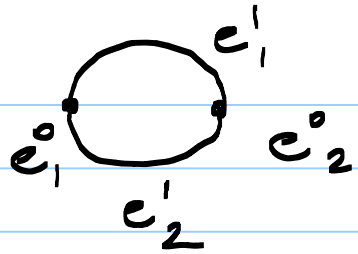
If $X = X^n$ then we say that X has dimension n.

<u>Examples</u> 1) S^0 $\cdot \quad \cdot$ $\quad (S^0 \subseteq \mathbb{R},\ x^2 = 1 \Rightarrow x = \pm 1)$

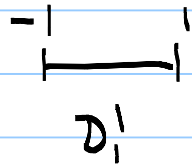
2) $S^1 \subseteq \mathbb{R}^2 \quad x^2 + y^2 = 1$



e_1^1 $\quad e_2^0$ $\quad e_1^0$ $\quad e_2^1$



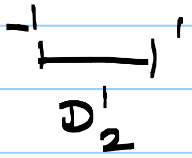
$\varphi_1 : \partial D_1^1 \longrightarrow X^0 = S^0 = \{e_1^0, e_2^0\}$



, $\partial D_1^1 = \{-1, 1\}$

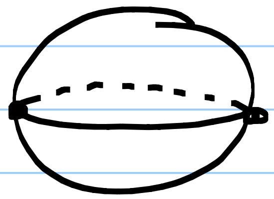
$\varphi_1(-1) = e_1^0$, $\varphi_1(1) = e_2^0$.

$\varphi_2 : \partial D_2^1 \longrightarrow X^0 = S^0 = \{e_1^0, e_2^0\}$



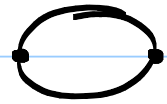
, $\partial D_2^1 = \{-1, 1\}$, $\varphi_2(-1) = e_1^0$, $\varphi_2(1) = e_2^0$.

3) $X = S^2$

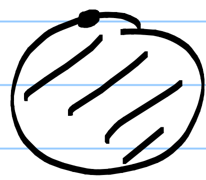


$X^0 = \{e_1^0, e_2^0\}$

$X^1 = \{e_1^0, e_2^0, e_1^1, e_2^1\}$



$X = X^2 = X^1 \sqcup D_1^2 \sqcup D_2^2 / \sim$



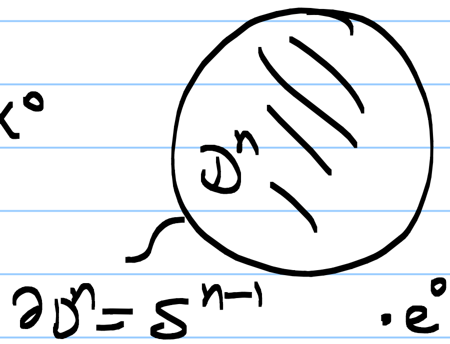
$\varphi_1 : \partial D_1^2 = S^1 \longrightarrow X^1 = S^1$

$\varphi_2 : \partial D_2^2 = S^1 \longrightarrow X^1 = S^1$

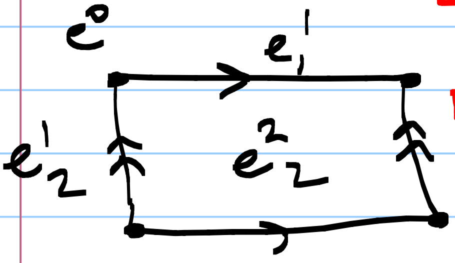
4) $S^\infty = \bigcup_{n=0}^{\infty} S^n$ (We'll see later that S^∞ is contractible, while no S^n is contractible.)

5) $S^n = X = X^0 \amalg D^n$

$X^0 = \{e^0\}$ $\varphi : \underset{S^{n-1}}{\underset{\parallel}{\partial D^n}} \longrightarrow X^{n-1} = X^0$



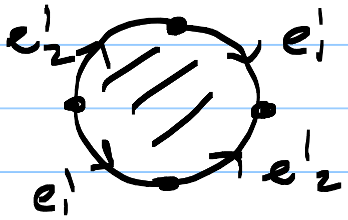
6) $T^2 = X = \underbrace{\underbrace{\underbrace{\{e^0\}}_{X^0} \cup e^1_1 \cup e^1_2}_{X^1} \cup e^2_2}_{X^2}$

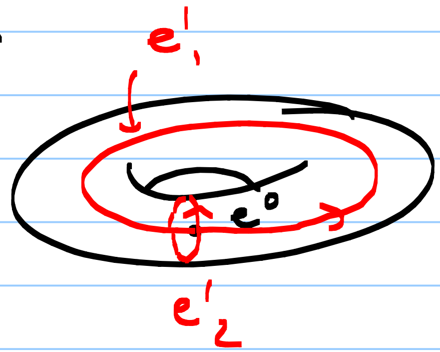


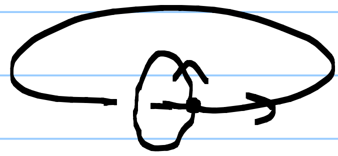
$\varphi^1_1 : \partial D^1 = \{\pm 1\} \longrightarrow X^0 = \{e^0\}$

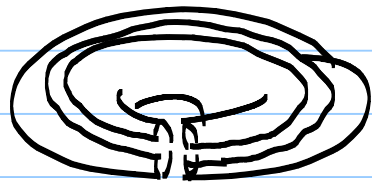
$\varphi^1_2 : \partial D^1 = \{\pm 1\} \longrightarrow X^0 = \{e^0\}$

$\varphi^2 : \partial D^2 = S^1 \longrightarrow X^1$









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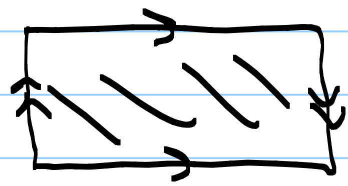
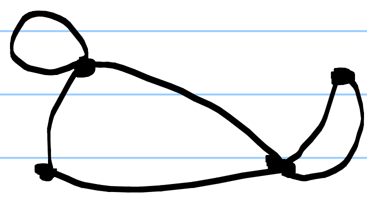


8) One dimensional CW-complexes are called graphs.



X^0: the set of vertices

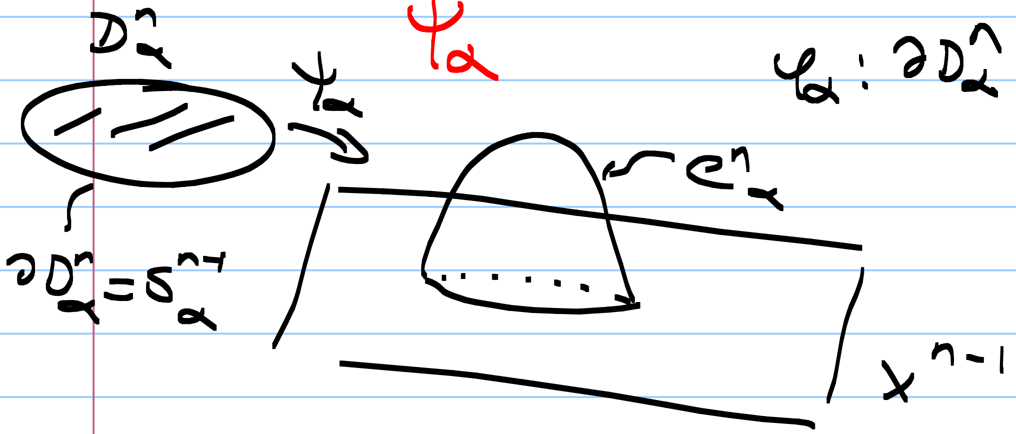
One dimensional cells are the edges.

<u>Characteristic map:</u> For any n-cell e^n_α of a CW-complex X the map

$\Phi_\alpha : D^n_\alpha \longrightarrow X^n$ is called characteristic map.

$$\begin{tikzcd} D^n_\alpha \arrow[r, hook] \arrow[rr, bend right, "\Phi_\alpha"'] & X^{n-1} \coprod_\alpha D^n_\alpha \arrow[r] & X^{n-1} \coprod_\alpha D^n_\alpha \Big/ x \sim \varphi_\alpha(x) \end{tikzcd}$$

$\varphi_\alpha : \partial D^n_\alpha = S^{n-1}_\alpha \longrightarrow X^{n-1}$

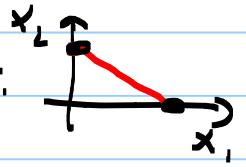


<u>Definition:</u> 1) If each characteristic map $\Phi_\alpha : D^n_\alpha \to X$ is an embedding then the CW-complex X is called regular.

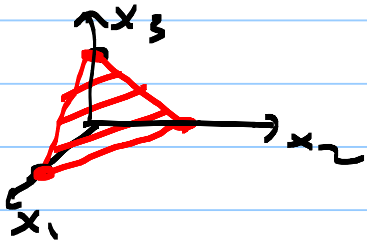
2) A simplical complex is a CW-complex, where each D^n_α is Δ^n and the gluing maps are linear homeomorphisms:

$$\Delta^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \,/\, x_i \geq 0, \ \sum_{i=1}^{n+1} x_i = 1\}.$$

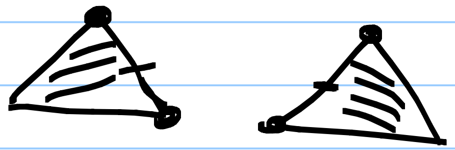
Δ^1:



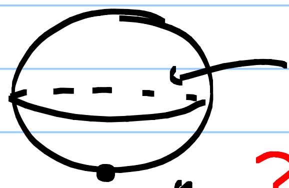
Δ^2:



Each Δ^n is homeomorphic to D^n.

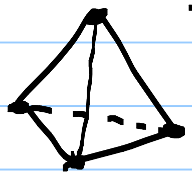


Example: $S^2 = e^2_1 \cup e^0_1$



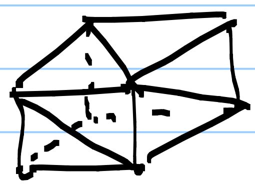
$\chi(S^2) = 1+1$
$= 2$

S^2: tetrahedron



This is a simplicial complex consisting of 4 0-simplices, 6 1-simplices and 4 2-simplices. $\chi(S^2) = 4-6+4$
$= 2$

S^2: cube



Simplicial complex having

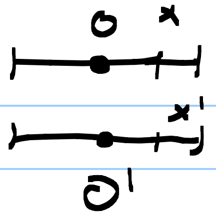
8 0-simplices
18 1-simplices
12 2-simplices

$\chi(S^2) = 8-18+12$
$= 2$

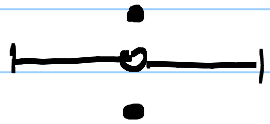
Definition: Euler characteristic of a finite cell complex X is defined to be the alternating sum of number of simplices:

$$\chi(X) = \sum_{n=0}^{\infty} (-1)^n c_n \,, \quad c_n = \# \text{ of } n\text{-simplices of } X$$

Theorem: If two finite cell complexes are homeomorphic then their Euler characteristics are the same.

Example: X : 

$/ x \sim x', \ x \neq 0$

X : 

X has 4 0-cells,
2 1-cells.

$\chi(X) = 4 - 2 = 2.$

The Fundamental Group:

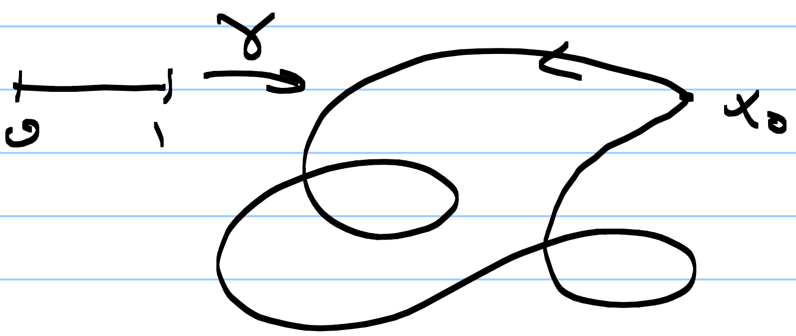
Fundamental group can be thought as a functor from the category of topological spaces with base points (continuous maps as morphisms) to the category of groups (homomorphisms as morphisms).

X topological space, $x_0 \in X$

$$(X, x_0) \longrightarrow \pi_1(X, x_0)$$

Definition: Given a based topological space (X, x_0) Let $\mathcal{L}$ be the set of all loops at x_0:

$$\mathcal{L} = \{\gamma : [0,1] \to X \mid \gamma \text{ continuous}, \ \gamma(0) = x_0 = \gamma(1)\}$$



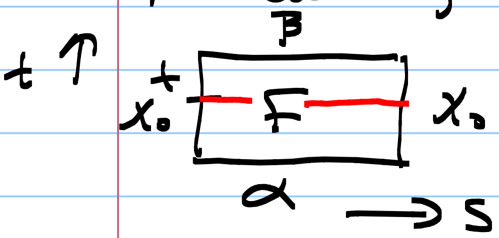
Define a homotopy relation on $\mathcal{L}$ as follows:

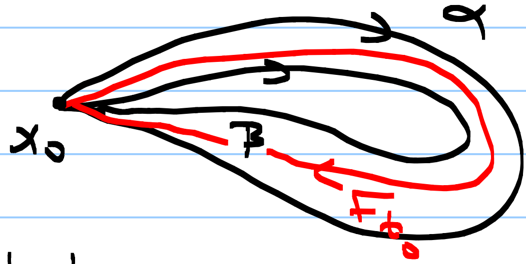
If $\alpha, \beta \in \mathcal{L}$, then $\alpha \sim \beta$ if and only if there is a homotopy

$$F : [0,1] \times [0,1] \longrightarrow X \quad \text{so that}$$

$F(s,0) = \alpha(s)$, $F(s,1) = \beta(s)$, $F(0,t) = x_0 = F(1,t)$,

for all $s, t \in [0,1]$.





$$F_{t_0} = F(s, t_0)$$

We've seen before that being homotopic is an equivalence relation.

Define $\pi_1(X, x_0)$ as the set of all equivalence classes of this relation: homotopy classes of based loops at x_0.

If α is a loop at x_0 then its homotopy class will be denoted as $[\alpha]$.

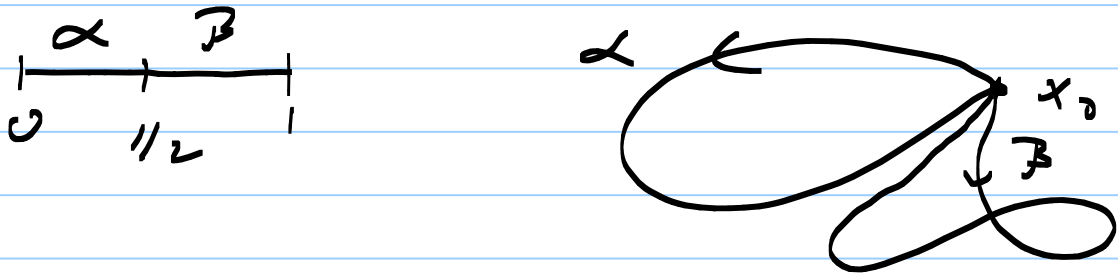
$$\pi_1(X, x_0) \doteq \mathcal{L}/\sim.$$

We define the group operation on $\pi_1(X, x_0)$ as follows:

If $[\alpha], [\beta] \in \pi_1(X, x_0)$ then let

$$[\alpha]\cdot[\beta] \doteq [\alpha\cdot\beta], \text{ where}$$

$$\alpha\cdot\beta(s) = \begin{cases} \alpha(2s), & 0 \leq s \leq 1/2 \\ \beta(2s-1), & 1/2 \leq s \leq 1. \end{cases}$$

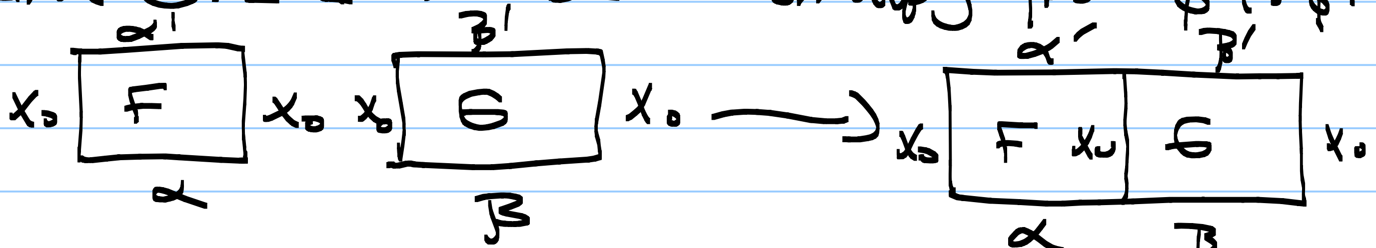


Need to show that this operation is well-defined.
Let $\alpha, \alpha', \beta, \beta'$ be loops at x_0 with

$[\alpha] = [\alpha']$ and $[\beta] = [\beta']$.

<u>must show:</u> $[\alpha\cdot\beta] = [\alpha'\cdot\beta']$.

Let $F: I\times I \to X$ be a homotopy from α to α'
and $G: I\times I \to X$ be a homotopy from β to β'.



$$H \doteq F\cdot G, \quad H(s,t) = \begin{cases} F(2s, t), & 0 \leq s \leq 1/2 \\ G(2s-1, t), & 1/2 \leq s \leq 1. \end{cases}$$

Then H is continuous by the Pasting Lemma
and $H(s,0) = \alpha\cdot\beta$ and $H(s,1) = \alpha'\cdot\beta'$.

Hence, this operation on $\pi_1(X,x_0)$ is well defined.

The identity element of $\pi_1(X,x_0)$ is defined to be the homotopy class of the constant loop at x_0:

$e:[0,1] \longrightarrow X, \quad e(s)=x_0, \forall s \in [0,1].$

$e=[e]$.

We must prove the following:

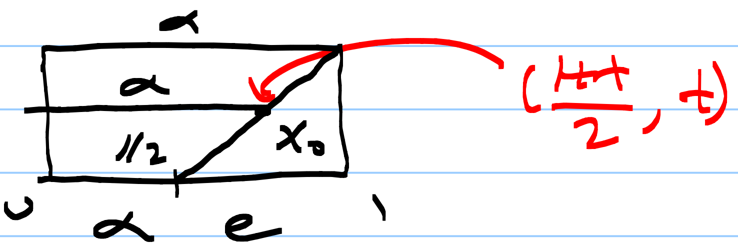
1) $e\cdot[\alpha]=[\alpha]=[\alpha]\cdot e$

2) For any $[\alpha]$ there is some $[\beta] \in \pi_1(X,x_0)$ so that $[\alpha]\cdot[\beta]=e=[\beta]\cdot[\alpha]$.

3) For any $[\alpha],[\beta],[\gamma] \in \pi_1(X,x_0)$ we must have

$$([\alpha]\cdot[\beta])\cdot[\gamma]=[\alpha]\cdot([\beta]\cdot[\gamma]).$$

Proof 1)



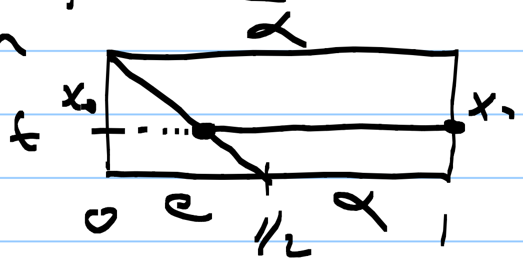
$\alpha\left(s\cdot\frac{2}{1+t}\right)$ $\quad s=0, \ \alpha(0)=x_0,$
$\quad s=\frac{1+t}{2}, \ \alpha(1)=x_0$

Let $F(s,t)=\begin{cases}\alpha\left(\frac{2s}{1+t}\right), & 0\le s\le \frac{1+t}{2},\\ x_0 & , \frac{1+t}{2}\le s\le 1.\end{cases}$

$$F(s,0)=\begin{cases}\alpha(2s), & 0\le s\le 1/2\\ \alpha_0, & 1/2\le 1\end{cases} = \alpha\cdot e$$

$$F(s,1) = \begin{cases} \alpha(s), & 0 \le s \le 1 \\ x_0, & 1 \le s \le 1. \end{cases} = \alpha.$$

For the part $[e \cdot \alpha] = \alpha$ we use the diagram

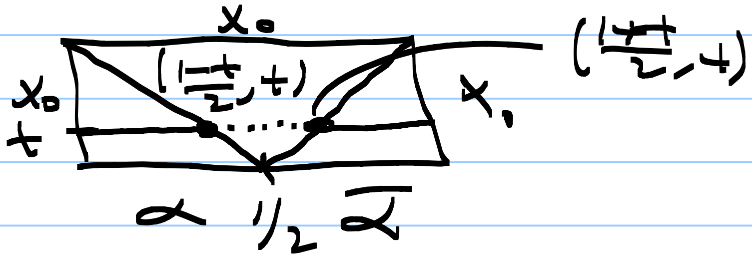


$F(s,t) = ?$

2) Given $[\alpha] \in \pi_1(X, x_0)$ let $[\alpha]^{-1}$ as $[\overline{\alpha}]$, where

$$\overline{\alpha}(s) \doteq \alpha(1-s), \quad s \in [0,1].$$

<u>must show:</u> $[\alpha \cdot \overline{\alpha}] = e$.



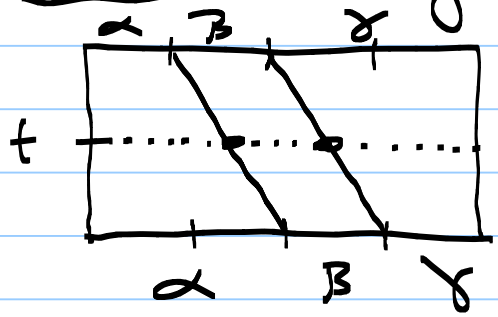
$$F(s,t) = \begin{cases} \alpha(2s) & , 0 \le s \le \frac{1-t}{2} \\ \alpha(1-t), & , \frac{1-t}{2} \le s \le \frac{1+t}{2} \\ \alpha(2-2s), & \frac{1+t}{2} \le s \le 1. \end{cases}$$

$$F(s,0) = \begin{cases} \alpha(2s), & 0 \le s \le 1/2 \\ \alpha(1) & 1/2 \le s \le 1/2 \\ \alpha(2-2s) & 1/2 \le s \le 1. \end{cases}$$

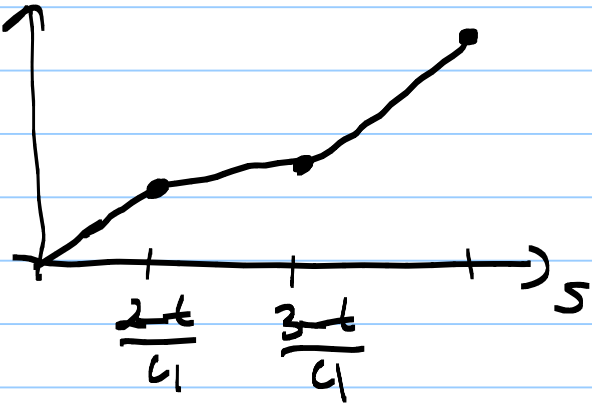
$$F(s,1) = \begin{cases} \alpha(2s) & , 0 \le s \le 0 \\ \alpha(0) & , 0 \le s \le 1 \\ \alpha(0) & , 1 \le s \le 1 \end{cases} = e$$

Exercise: $[\bar{\alpha}\cdot\alpha]=[e]$.

3) Associativity $([\alpha]\cdot[\beta])\cdot[\gamma]=[\alpha]\cdot([\beta]\cdot[\gamma])$.



$$\ell_t(s)=\begin{cases}\dfrac{4s}{2-t}, & 0\le s\le \dfrac{2-t}{4}\\[2mm] 4s+t-1, & \dfrac{2-t}{4}\le s\le \dfrac{3-t}{4}\\[2mm] \dfrac{4s+3t-1}{1+t}, & \dfrac{3-t}{4}\le s\le 1.\end{cases}$$



Let $$H(s,t)=\begin{cases}\alpha(\ell_t(s)), & 0\le s\le \dfrac{2-t}{4}\\[2mm] \beta(\ell_t(s)-1), & \dfrac{2-t}{4}\le s\le \dfrac{3-t}{4}\\[2mm] \gamma(\ell_t(s)-2), & \dfrac{3-t}{4}\le s\le 1.\end{cases}$$

Details are left as an exercise.

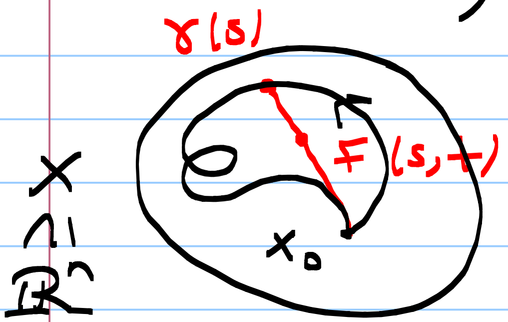
Video 11

Example For a convex subset X of $\mathbb{R}^n$ and any point $x_0 \in X$, $\pi_1(X, x_0) = (e)$, the trivial group.

Proof: Let $\gamma : [0,1] \to X$ be a loop at x_0.

Then by the line homotopy γ is homotopic to the constant loop e at x_0:

$$F : X \times I \to X, \quad F(s,t) = (1-t)\gamma(s) + t\,x_0$$



$F(s,0) = \gamma(s)$, $s \in [0,1]$
$F(s,1) = x_0$, $s \in [0,1]$.

$F(0,t) = (1-t)\gamma(0) + t x_0 = x_0$ and
$F(1,t) = (1-t)\gamma(1) + t x_0 = x_0$.

Hence, $[\gamma] = [e] = e$. ∎

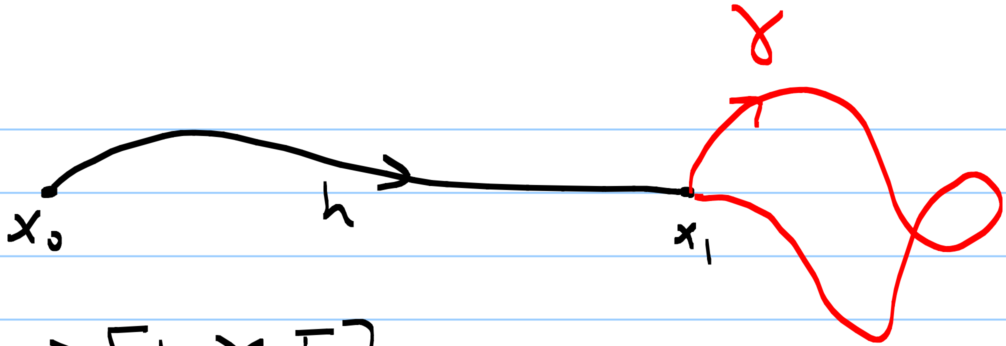
Exercise: If X is a contractible space to a point x_0, then $\pi_1(X, x_0) = (e)$.

Proposition: Let X be a topological space, $x_0, x_1 \in X$ points in X and $h : [0,1] \to X$ is a path joining x_0 to x_1: $h(0) = x_0$, $h(1) = x_1$. Then the map

$$\beta_h : \pi_1(X, x_0) \to \pi_1(X, x_1) \text{ defined by}$$

$\beta_h([\gamma]) = [\bar{h} \cdot \gamma \cdot h]$ is a group isomorphism

Proof:



$$[\gamma] \longmapsto [h\cdot\gamma\cdot\bar{h}].$$

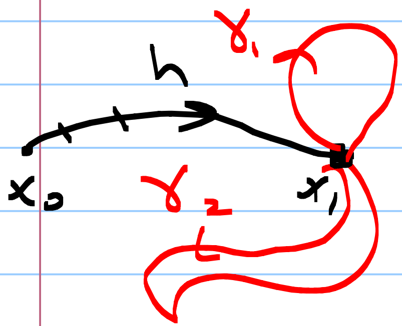
$h: [0,1] \to X$, $h(0) = x_0$, $h(1) = x_1$

$\bar{h}: [0,1] \to X$, $\bar{h}(s) = h(1-s)$, $\bar{h}(0) = h(1) = x_1$ and $\bar{h}(1) = h(0) = x_0$.

<u>β_h is a homomorphism:</u> Let $\gamma_i : [0,1] \to X$ be loops at x_1. Then $\gamma_1 \cdot \gamma_2$ is also a loop at x_1, given by
$$(\gamma_1\cdot\gamma_2)(s) = \begin{cases} \gamma_1(2s), & 0 \le s \le 1/2 \\ \gamma_2(2s-1), & 1/2 \le s \le 1. \end{cases}$$

$\beta_h([\gamma_1]) = [h\cdot\gamma_1\cdot\bar{h}]$, $\beta_h([\gamma_2]) = [h\cdot\gamma_2\cdot\bar{h}]$

$$\beta_h([\gamma_1])\cdot\beta_h([\gamma_2]) = [h\cdot\gamma_1\cdot\bar{h}]\cdot[h\cdot\gamma_2\cdot\bar{h}]$$

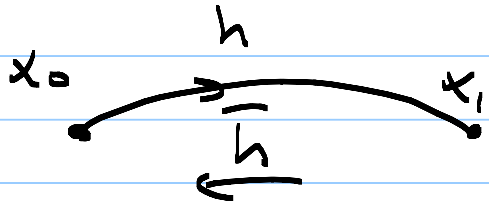


$$= [h\cdot\gamma_1\cdot\bar{h}\cdot h\cdot\gamma_2\cdot\bar{h}]$$
$$= [h\cdot\gamma_1\cdot e_{x_1}\cdot\gamma_2\cdot\bar{h}]$$
$$= [h\cdot(\gamma_1\cdot\gamma_2)\cdot\bar{h}]$$
$$= \beta_h([\gamma_1\cdot\gamma_2])$$

Hence, β_h is a group homomorphism.

<u>Claim:</u> $\beta_{\bar{h}}$ is the inverse of the homomorphism $\beta_{\bar{h}}$.

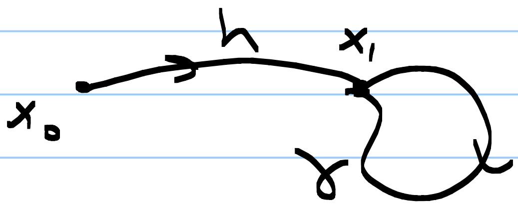
Proof:



$\beta_h : \pi_1(X, x_1) \to \pi_1(X, x_0)$, $\beta_{\bar h} : \pi_1(X, x_0) \to \pi_1(X, x_1)$.

Let $[\gamma] \in \pi_1(X, x_1)$. Then

$$(\beta_{\bar h} \circ \beta_h)([\gamma]) = \beta_{\bar h}([h \cdot \gamma \cdot \bar h])$$
$$= [\bar h \cdot (h \cdot \gamma \cdot \bar h) \cdot \bar{\bar h}] \quad (\bar{\bar h} = h)$$
$$= [(\bar h \cdot h) \cdot \gamma \cdot (\bar h \cdot h)]$$

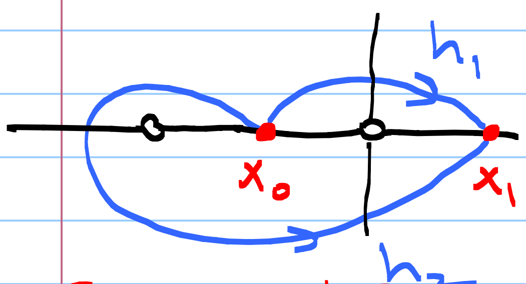


$$= [(\bar h \cdot h)] \cdot [\gamma] \cdot [\bar h \cdot h]$$
$$= e_{x_1} \cdot [\gamma] \cdot e_{x_1}$$
$$= [\gamma].$$

This finishes the proof. ∎

Remark: The isomorphism β_h from $\pi_1(X, x_1)$ to $\pi_1(X, x_1)$ is <u>not</u> canonical.

Example: $X = \mathbb{R}^2 \setminus \{(0,0), (-1,0)\}$



$\pi_1(X, x_0) \simeq F_2 \simeq \pi_1(X, x_1)$

β_{h_1} and β_{h_2} are <u>not</u> the same.

Exercise: If $\pi_1(X)$ is abelian then $\beta_{h_1} = \beta_{h_2}$ for any h_1, h_2

Video 12

Definition: A path connected space X is called simply connected if $\pi_1(X, x_0) = (e)$ for some (and thus all) $x_0 \in X$.

Proposition: A space X is simply connected if and only if for any two points x_0 and x_1 of X, there is a unique homotopy class of paths joining x_0 to x_1, where homotopies fix the end points at all times.

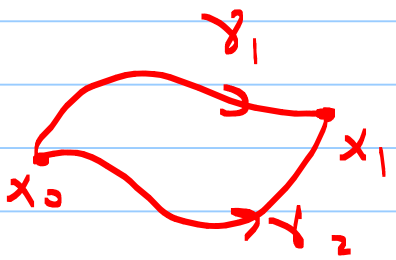
Proof: Exercise.

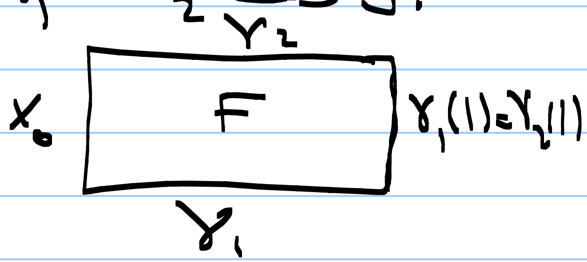
Remark: If X is simply connected space and $x_0 \in X$. Then there is a bijection between the set X and the set of homotopy classes of paths starting at x_0, where homotopies fix the end points.

$$\{\gamma : [0,1] \longrightarrow X \mid \gamma(0) = x_0\} / \sim$$

$$[\gamma] \longmapsto \gamma(1) \in X$$

$$\gamma_1 \sim \gamma_2 \iff \exists F$$





$$\varphi : \{\gamma : [0,1] \to X \mid \gamma(0) = x_0\} / \sim \; \longrightarrow X$$

$$[\gamma] \longmapsto \varphi([\gamma]) = \gamma(1).$$

The Fundamental Group of the Circle

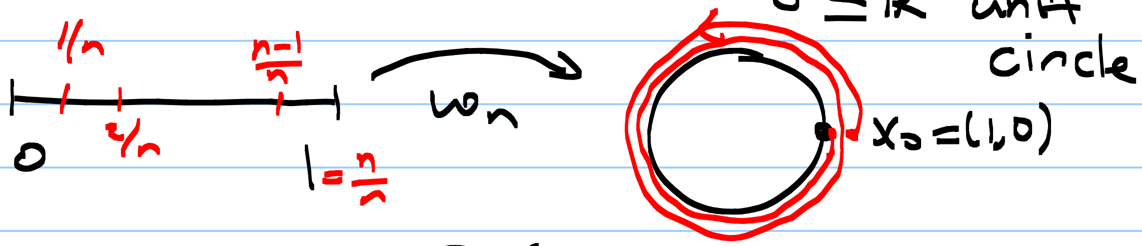
Theorem: The map $\Phi : \mathbb{Z} \to \pi_1(S^1, x_0)$, where $x_0 = (1,0)$ sending an integer n to the homotopy class of the loop

$$\omega_n : [0,1] \to S^1, \quad \omega_n(s) = (\cos 2\pi n s, \sin 2\pi n s)$$

based at $x_0 = (1,0)$ is an isomorphism.

Proof:

$S^1 \subseteq \mathbb{R}^2$ unit circle

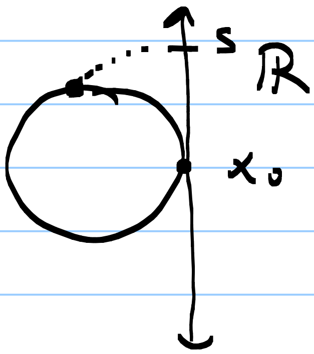


$$\begin{aligned} \mathbb{Z} &\longrightarrow \pi_1(S^1, x_0) \\ n &\longmapsto [\omega_n]. \end{aligned}$$

The proof has several stages:

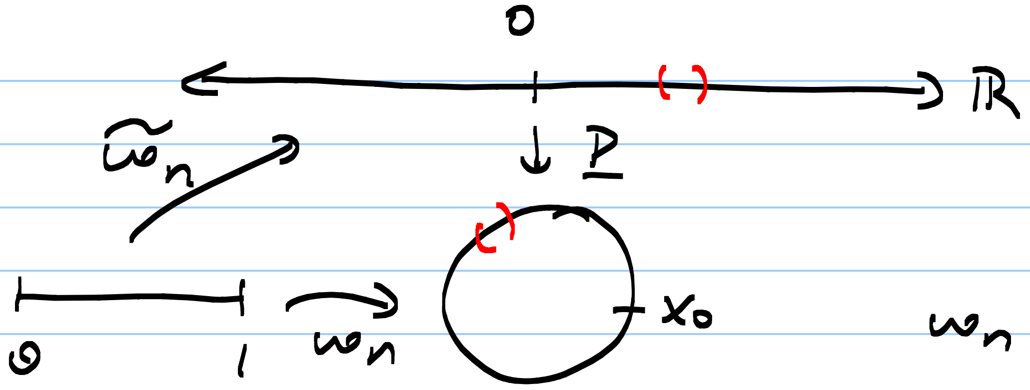
i) Consider the map $p : \mathbb{R} \to S^1$ given by

$p(s) = (\cos 2\pi s, \sin 2\pi s)$, $s \in \mathbb{R}$.



Let $\tilde{\omega}_n : [0,1] \to \mathbb{R}$ be given by $\tilde{\omega}_n(s) = ns$.

Note that $(p \circ \tilde{\omega}_n)(s) = p(\tilde{\omega}_n(s)) = p(ns) = \omega_n(s)$



$\omega_n = p \circ \tilde{\omega}_n$

Note that p is locally a homeomorphism.

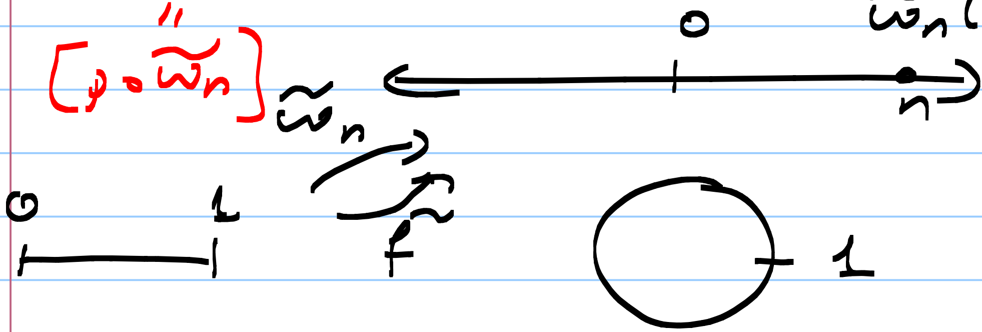
Note that $\Phi(n) = [\omega_n]$ can be defined as the homotopy class of the loop $p \circ \tilde{f}$ for any path $\tilde{f}$ in $\mathbb{R}$ from 0 to n, because any such $\tilde{f}$ is homotopic to $\tilde{\omega}_n$, keeping the end points fixed:

$t \longmapsto (1-t)\tilde{f} + t\,\tilde{\omega}_n$ and thus $p \circ \tilde{f}$ is homotopic to ω_n.

$$[p \circ \tilde{f}] = [\omega_n] = \Phi(n)$$

$\tilde{\omega}_n(0) = n = \tilde{f}(0)$

"$[p \circ \tilde{\omega}_n]$"

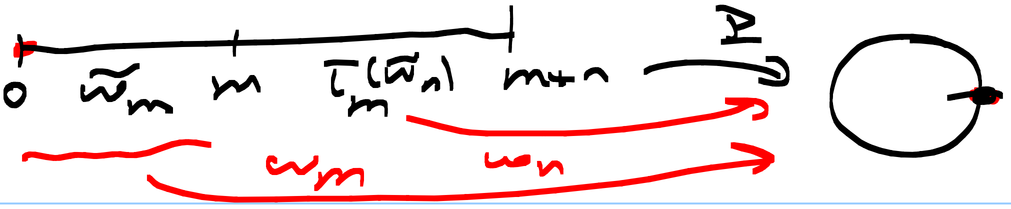


ii) <u>Claim</u>: Φ is a group homomorphism.

<u>Proof</u>: Let $\tau_m : \mathbb{R} \to \mathbb{R}$ be the translation map $\tau_m(x) = x + m$, $(m \in \mathbb{Z})$.

Then $\tilde{\omega}_m \cdot (\tau_m(\tilde{\omega}_n))$ is a path in $\mathbb{R}$ from 0 to $m+n$, so that $\Phi(m+n)$ is the homotopy

Video 13



class of $p(\tilde{\omega}_m \cdot (T_m(\tilde{\omega}_n)))$.

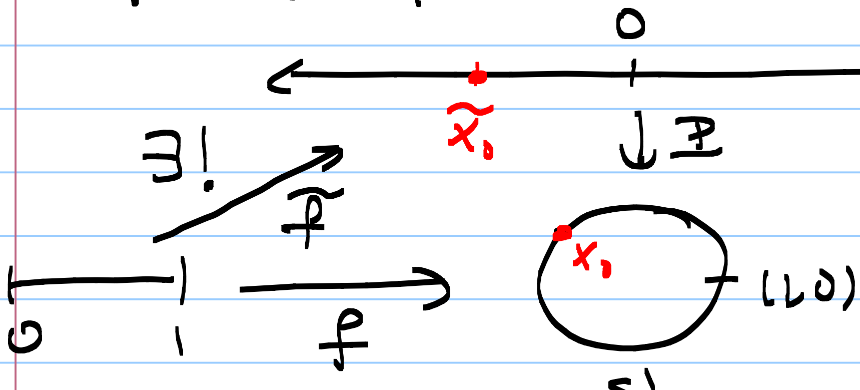
The image of the path $\tilde{\omega}_m \cdot (T_m(\tilde{\omega}_n))$ under p is just $\omega_m \cdot \omega_n$, so that

$$\Phi(m+n) = [\omega_m \cdot \omega_n] = [\omega_m] \cdot [\omega_n] = \Phi(m) \cdot \Phi(n).$$

Hence, Φ is a group homomorphism.

Next we'll show that Φ is a isomorphism.
To do so we'll use two facts:

a) For each path $f: I \to S^1$ starting a point x_0 and each $\tilde{x}_0 \in \mathbb{R}$ with $p(\tilde{x}_0) = x_0$ there is a unique lift $\tilde{f}: I \to \mathbb{R}$ starting at $\tilde{x}_0$.



$f(0) = x_0$, $\tilde{f}(0) = \tilde{x}_0$ and $p(\tilde{f}(s)) = f(s)$

b) For each homotopy $f_t: I \to S^1$ of paths

$(F: I \times I \to S^1,\ F(s,t) = f_t(s))$ starting at x_0

then there is a unique homotopy $\tilde{f}_t: I \to \mathbb{R}$

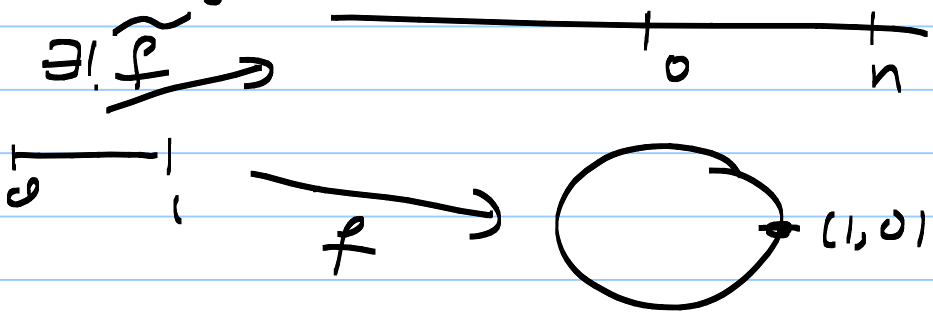
starting at $\tilde{x}_0$ and $p(\tilde{f}_t(s)) = f(s)$, for all s.

$\left(\tilde{F}: I\times I \longrightarrow \mathbb{R},\ \tilde{F}(s,t)=\tilde{f}_t(0)\right)$

iii) (a) and (b) prove the theorem.

<u>Φ is surjective</u>: Let $f: I \longrightarrow S^1$ be a loop at the base point $(1,0)$ x_0, representing an element of $\pi_1(S^1,(1,0))$.

Now by Fact (a) there is a (unique) lift $\tilde{f}$ starting at $\tilde{x}_0 = 0 \in \mathbb{R}$ ($p(0)=(1,0)$)



$f(0)=f(1)=(1,0)$. Since $p(\tilde{f}(1)) = f(1) = (1,0)$ we see that $\tilde{f}(1) \in \mathbb{Z}$ because

$p^{-1}(0) = \mathbb{Z} \subseteq \mathbb{R}$. Say $n=\tilde{f}(1)$.

By the extended definition of Φ we have

$\Phi(n) = [p\circ\tilde{f}] = [f]$ so that we are done.

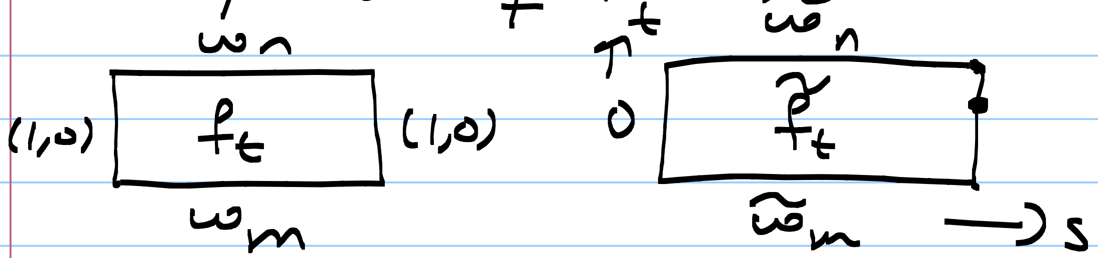
<u>Φ is injective</u>: Suppose that $\Phi(n)=\Phi(m)$, for some $m,n\in\mathbb{Z}$. So $[\omega_n]=[\omega_m]$ and thus ω_n and ω_m are homotopic.

<u>must show</u>: $m=n$.

Let f_t be a homotopy from $f_0=\omega_m$ to $f_1=\omega_n$. By (b) there is a unique lift $\tilde{f}_t$ of paths starting at 0.

The uniqueness part of (a) implies that

$\tilde{f}_0=\widetilde{\omega}_m$ and $\tilde{f}_1=\widetilde{\omega}_n$. Since $\tilde{f}_t$ is a homotopy of paths the endpoints $\tilde{f}_t(1)$ is independent of t.



This is because $\tilde{F}(1,t)=\tilde{f}_t(1)\in\mathbb{Z}$ for all $t\in[0,1]$, and thus it must be a fixed integer.

$\tilde{f}_t(1)=?$ For $t=0$ the endpoints are m

and for $t=1$, the endpoints are n, and thus $m=n$.

Now we must prove (a) and (b). Indeed, we'll prove another fact (c), which will imply both (a) and (b).

(c) Given a map $F:Y\times I\to S^1$ and another map $\tilde{F}:Y\times\{0\}\to\mathbb{R}$ lifting $F|_{Y\times\{0\}}$, then there is a unique map $\tilde{F}:Y\times I\to\mathbb{R}$ lifting F ($p\circ\tilde{F}=F$ as maps on $Y\times I$)

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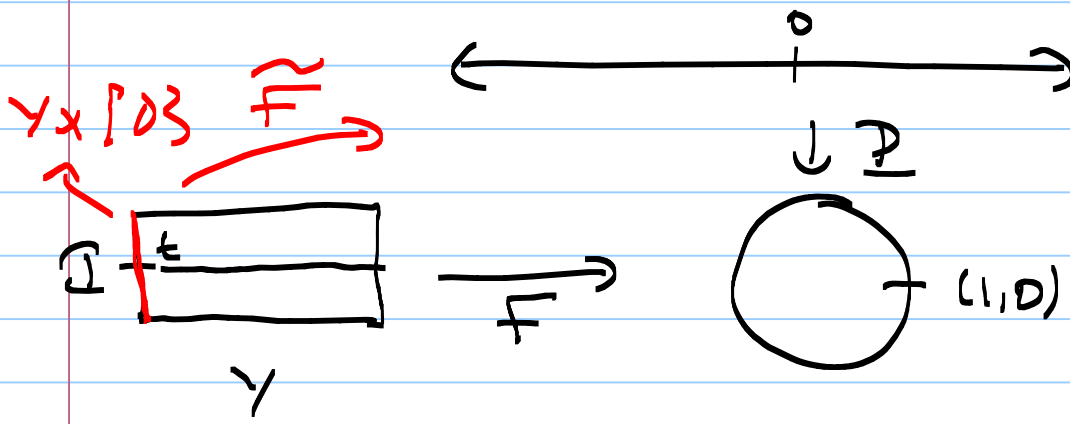
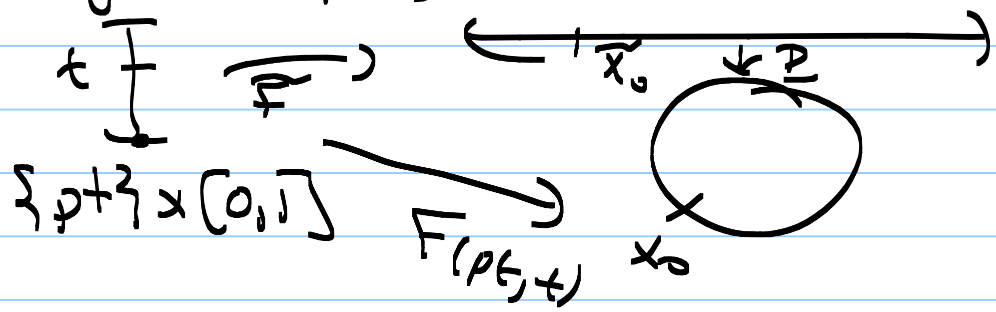
so that it restricts to the given $\tilde{F}$ on $Y \times \{0\}$.



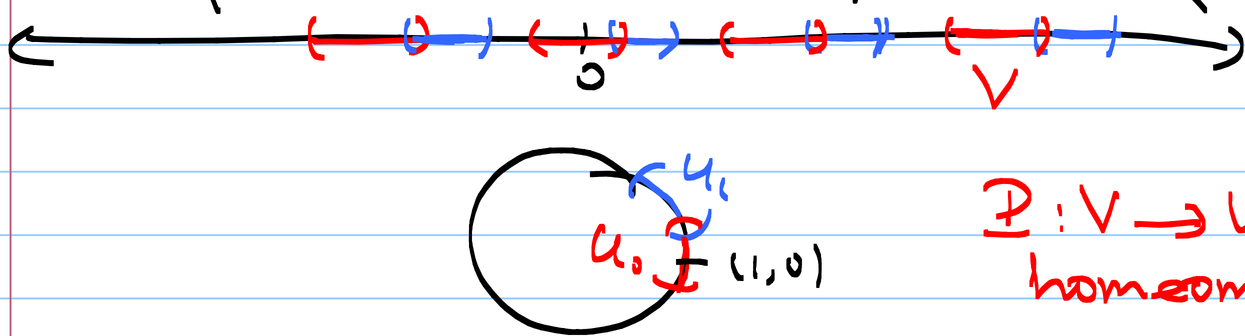
Taking $Y = \{pt\}$ (c) reduces to (a).



Similarly, taking $Y = [0,1]$ (c) becomes (b).

To finish the proof we need to prove fact (c):

To prove (c) we'll often use the following property of the map $p: \mathbb{R} \to S^1$: There is an open cover $\{U_\alpha\}$ of S^1 so that $p^{-1}(U_\alpha)$ is a disjoint union of open subsets of $\mathbb{R}$ each of which is homeomorphic to U_α by p.



$p: V \to U_0$
homeomorphism

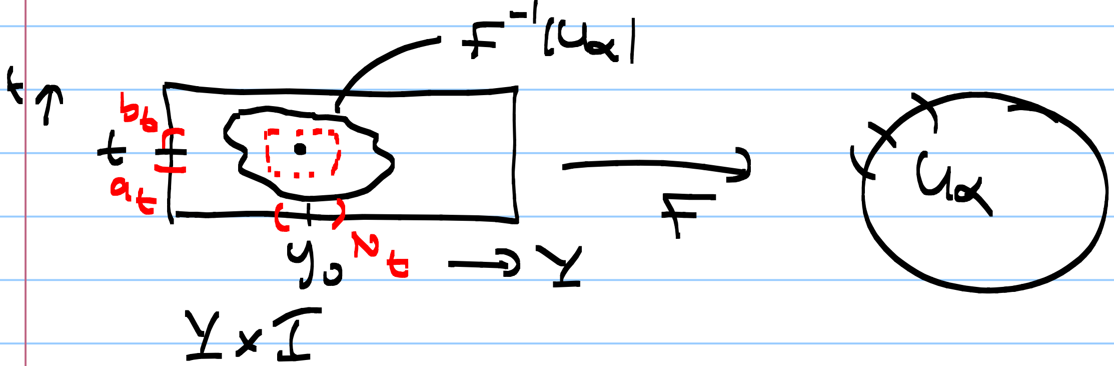
Definition: Let $p: X \to Y$ be an onto map. If Y has an open cover $\{U_\alpha\}$ so that for each α, the inverse image $p^{-1}(U_\alpha)$ is a disjoint union of open subsets each of which is homeomorphic to U_α via p, then we'll call the map $p: X \to Y$ a covering space/map.

Hence, the above $p: \mathbb{R} \to S^1$ is an example of a covering space (or map).

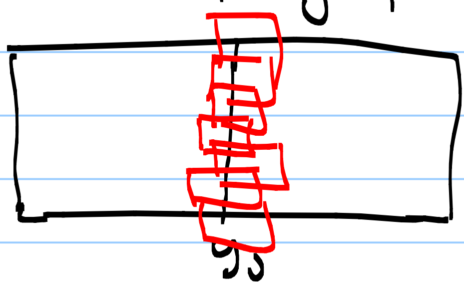
First let's construct a lift $\tilde{F}: N \times I \to \mathbb{R}$ for N some neighborhood in Y of a point $y_0 \in Y$.

Since F is continuous, every point $(y_0, t) \in Y \times I$ has a product neighborhood $N_t \times (a_t, b_t)$ such that

$$F(N_t \times (a_t, b_t)) \subseteq U_\alpha \text{ for some } \alpha.$$



Since I is compact $\{y_0\} \times I$ is compact and thus $\{y_0\} \times I$ is covered by finitely many such products, say



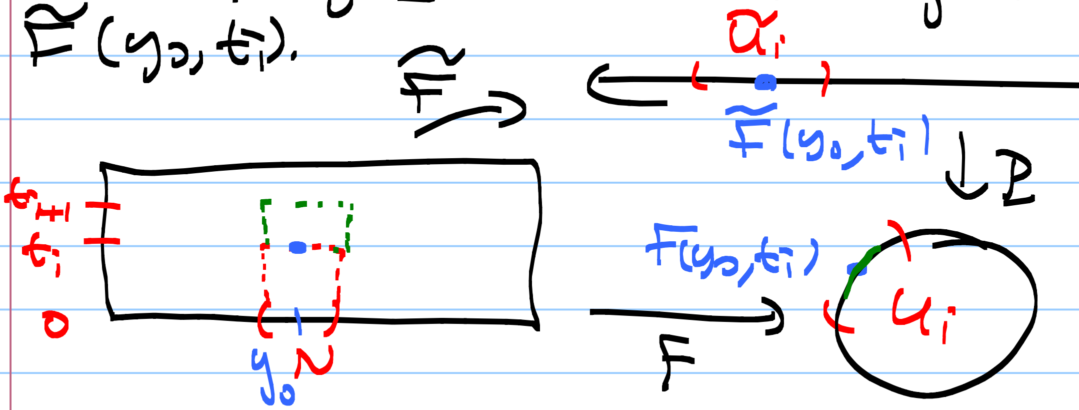
$$N_{\alpha_i} \times (a_i, b_i)$$

$i = 1, \ldots, k.$

Let $N = N_{\alpha_1} \cap N_{\alpha_2} \cap \cdots \cap N_{\alpha_k}$. Also choose $0 = t_0 < t_1 < t_2 < \cdots < t_m = 1$ so that

$$F(N \times [t_i, t_{i+1}]) \subseteq U_i \doteq U_{\alpha_i}, \quad i = 1, \ldots, m.$$

Assume that $\tilde{F}$ has been constructed on $N \times [0, t_i]$. Since $F(N \times [t_i, t_{i+1}]) \subseteq U_i$ there some $\tilde{U}_i \subseteq \mathbb{R}$ projecting homeomorphically onto U_i by P and containing the point $\tilde{F}(y_0, t_i)$.



$P: \tilde{U}_i \to U_i$ homeomorphism

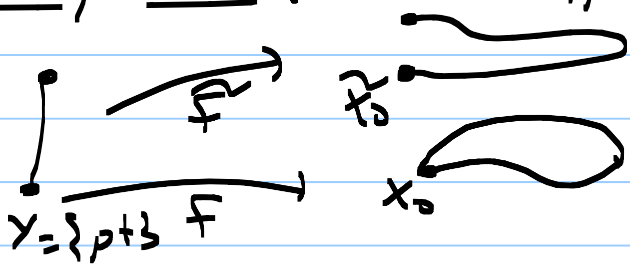
Now define $\tilde{F}$ on $N \times [t_i, t_{i+1}]$ to be the composition of F with the inverse

$$P^{-1}: U_i \to \tilde{U}_i. \text{ So, } \tilde{F} = P^{-1} \circ F.$$

Repeating this finitely many times we get the required map $\tilde{F}: N \times I \to \mathbb{R}$.

<u>Uniqueness of $\tilde{F}$ for $Y = \{pt\}$</u>

If



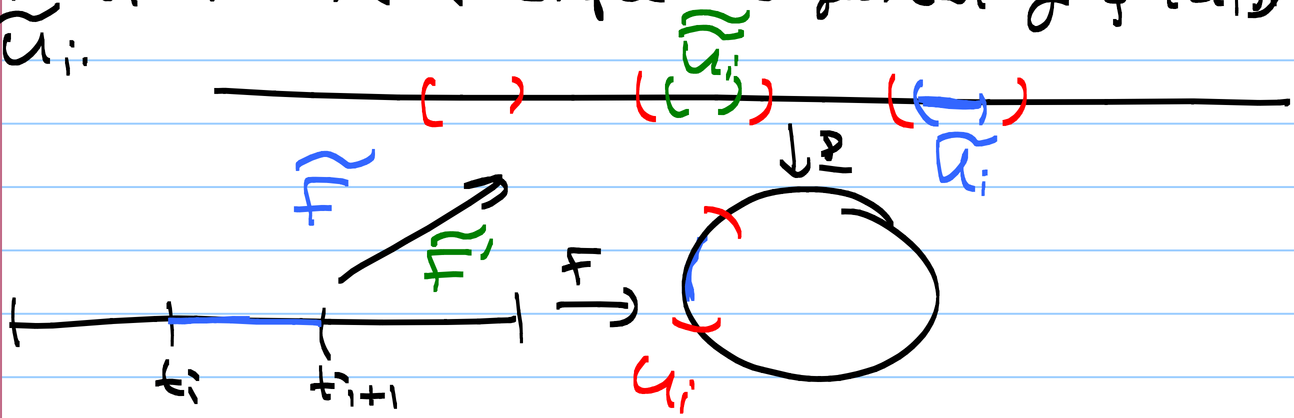
Since $Y = \{pt\}$ is a single point we may drop it from the notation.

Suppose $\tilde{F}$ and $\tilde{F}'$ are two lifts of $F : I \longrightarrow S^1$ so that $\tilde{F}(0) = \tilde{F}'(0)$. As above choose a partition

$$0 = t_0 < t_1 < t_2 < \cdots < t_m = 1 \text{ of } I$$

so that $F([t_i, t_{i+1}]) \subseteq U_i$ for some U_i.

Assume that $\tilde{F}$ and $\tilde{F}'$ agree on $[0, t_i]$. Since $[t_i, t_{i+1}]$ is connected $\tilde{F}([t_i, t_{i+1}])$ must lie in a single component of $p^{-1}(U_i)$, say $\tilde{U}_i$.



Similarly, since $\tilde{F}'$ is also a lift $\tilde{F}'([t_i, t_{i+1}])$ must lie in a single component say $\tilde{\tilde{U}}_i$.

However, $\tilde{F}(t_i) = \tilde{F}'(t_i)$ and thus $\tilde{U}_i = \tilde{\tilde{U}}_i$.

Since p is injective on $\tilde{U}_i$ and

$p \circ \tilde{F} = p \circ \tilde{F}'$ on we must have

$\tilde{F} = \tilde{F}'$ on $[t_i, t_{i+1}]$. Inductively, we see that $\tilde{F} = \tilde{F}'$ on $[0, t_m] = [0, 1]$.

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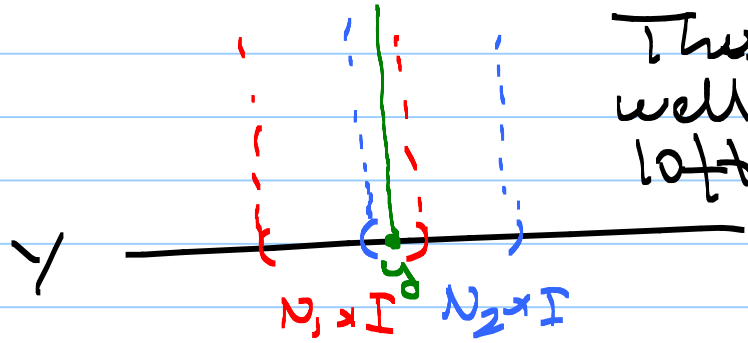
Finishing the Proof: Since $\tilde{F}$'s constructed above on the sets $N \times I$ are unique when restricted to segment $\{y\} \times I$, they must agree whenever two such $N \times I$'s overlap.



Thus we get a well-defined unique lift $\tilde{F}$ on $Y \times I$.

The unique lift $\tilde{F} : Y \times I \to \mathbb{R}$ is continuous since it is continuous on each $N \times I$.

This finishes the proof. ■

So we have proved that the map

$$\Phi : \mathbb{Z} \longrightarrow \pi_1(S^1, (1,0)), \quad m \longmapsto [\omega_m], \; m \in \mathbb{Z},$$

is a group isomorphism.

Some Applications:

Theorem: (Fundamental Theorem of Algebra)

Every nonconstant polynomial with coefficient in $\mathbb{C}$ has a root in $\mathbb{C}$.

Proof: Let $P(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$

be a polynomial with $a_i \in \mathbb{C}$, $i=1,\dots,n$.
Assume on the contrary that $P(z)$ has no roots in $\mathbb{C}$. Then for any real number $r \geq 0$, the formula

$$f_r(s) = \frac{P(re^{2\pi i s})/P(r)}{|P(re^{2\pi i s})/P(r)|}, \quad f_r : [0,1] \longrightarrow S^1$$

a continuous function for each r. Indeed,

$F : [0,1] \times \mathbb{R}^{\geq 0} \longrightarrow S^1$, $F(s,r) = f_r(s)$ is a

continuous function. Indeed, each f_r is loop on S^1 based at $(1,0) = 1 \in \mathbb{C}$,

$f_r(0) = 1$ and $f_r(1) = 1$, for all $r \geq 0$.

Hence, F defines a homotopy from f_0 to and f_r, $r \geq 0$.

$f_0 : [0,1] \longrightarrow S^1$, $f_0(s) = 1$, $\forall s \in [0,1]$, is the constant loop.

Hence, $[f_r] = [f_0] = e \in \pi_1(S^1, 1)$

Indeed, $e = 0$ in $\mathbb{Z} \cong \pi_1(S^1, 1)$.

Let $r = |a_1| + |a_2| + \dots + |a_n| + 1$. Now, for any $z \in \mathbb{C}$, with $|z| = r$, then

$$|z^n| = r^n = r \cdot r^{n-1} > (|a_1| + \dots + |a_n|)\,|z|^{n-1}$$

In particular,

$$|a_1 z^{n-1} + \dots + a_{n-1} z + a_n| \leq |a_1 z^{n-1}| + \dots + |a_{n-1} z| + |a_n|$$

$(r \geq 1)$

$$= |a_1| r^{n-1} + \dots + |a_{n-1}| r + |a_n|$$

$$\leq |a_1| r^{n-1} + \dots + |a_{n-1}| r^{n-1} + |a_n| r^{n-1}$$

$$= (|a_1| + \dots + |a_n|) r^{n-1}$$

$$< |z^n|.$$

Hence, the polynomial

$$P_t(z) = z^n + t(a_1 z^{n-1} + \dots + a_{n-1} z + a_n)$$

has no roots on the circle $|z| = r$, for any $0 \leq t \leq 1$.

Now the formula

$$\frac{P_t(re^{2\pi i s})/P_t(r)}{|P_t(re^{2\pi i s})/P_t(r)|}, \quad t \in [0,1], \text{ defines a}$$

homotopy.

$$t = 0 \Rightarrow \frac{(re^{2\pi i s})^n / r^n}{|e^{2\pi i s}|} = e^{2\pi i n s}, \quad s \in [0,1]$$

which is ω_n.

Hence, $n = [\omega_n] = [f_r] = 0$ in $\mathbb{Z} \simeq \pi_1(S^1, 1)$.

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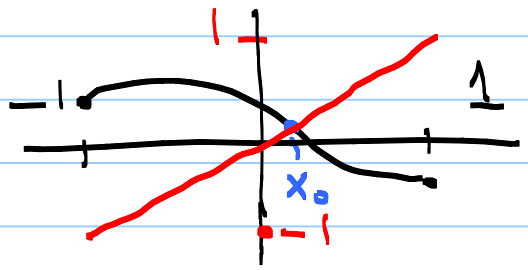
Thus $P(z)$ is a polynomial of degree $n=0$, i.e., it is a constant polynomial, a contradiction to the assumption.
Therefore, $P(z)$ must have a zero in $\mathbb{C}$. ∎

Theorem: Every continuous map $f: D^2 \to D^2$ has a fixed point.

Remark: Indeed the same holds for any $f: D^n \to D^n$.

$n=1$ is known from Intermediate Value Theorem.

$f:[-1,1] \to [-1,1]$

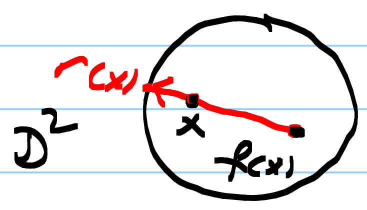


$f(x_0)=x_0$

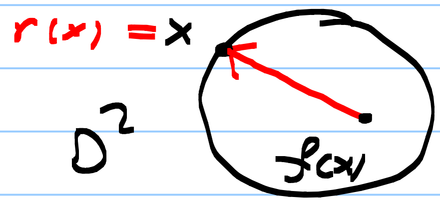
For $n \geq 3$ we'll Homology or Higher Homotopy theory.

Proof for $n=2$: (Brouwer proved this for D^n in 1910)

Assume on the contrary that the given function
$$f: D^2 \to D^2$$
has no fixed points. Hence, $f(x) \neq x$, for all $x \in D^2$. Now define a retraction
$r: D^2 \to S^1$ as follows: $r(x)$ is the intersection of $S^1 = \partial D^2$ with the ray starting at $f(x)$ and passing through x.



The continuity of $r(x)$ is left as an exercise.
Moreover, if $x \in S^1 = \partial D^2$, then $r(x) = x$.



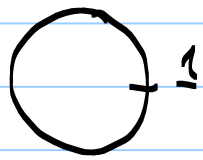
Hence, $r : D^2 \longrightarrow \partial D^2 = S^1$ is a retraction.

Now we need a fact:

__Claim:__ There is no retraction from D^2 to the boundary $\partial D^2 = S^1$.

__Proof:__ Suppose that there is a retraction $r : D^2 \longrightarrow S^1$.

$$\begin{array}{ccccc} S^1 & \overset{i}{\hookrightarrow} & D^2 & \overset{r}{\longrightarrow} & S^1 \\ x & \longmapsto & x & \longmapsto & r(x) \end{array}, \quad x \in S^1.$$



This gives homomorphism on π_1:

$$(r \circ i)(1) = r(i(1)) = r(1) = 1.$$

$$(S^1, \{1\}) \xrightarrow{\;i\;} (D^2, \{1\}) \xrightarrow{\;r\;} (S^1, \{1\})$$

__Fact:__ If $f : (X, x_0) \longrightarrow (Y, y_0)$ is a continuous map of based topological spaces then the map

$$f_\# : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0), \quad [\gamma] \longmapsto [f \circ \gamma],$$

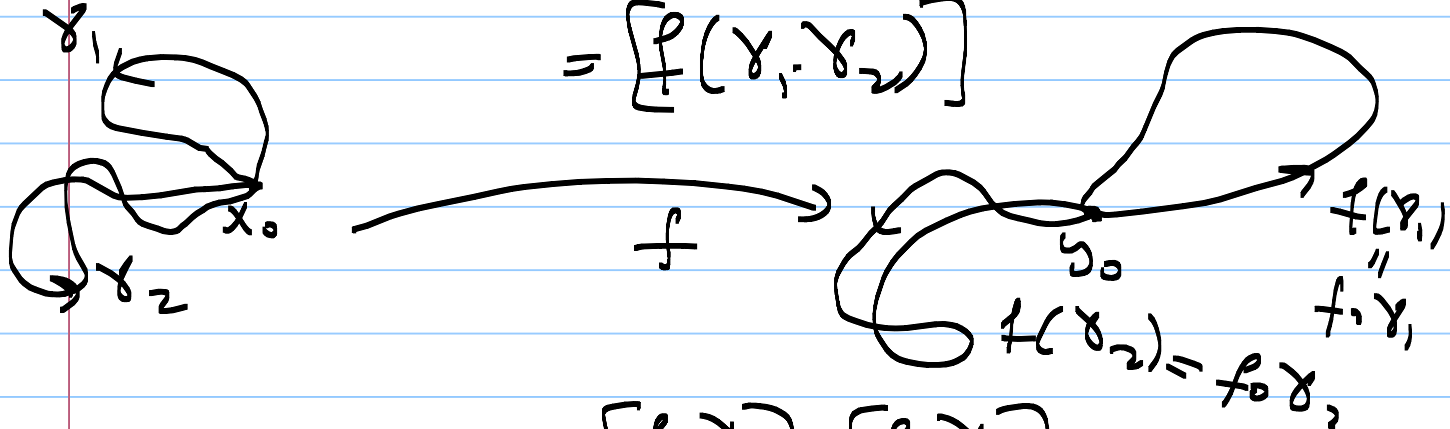
is a group homomorphism.

Proof of the fact:

i) It is well defined. If γ_t is a homotopy from γ_0 to γ_1 of based loops at x_0 then $f\circ\gamma_t$ is a homotopy of based loops at $y_0 = f(x_0)$ from $f\circ\gamma_0$ to $f\circ\gamma_1$.

ii) If $[\gamma_1]$ and $[\gamma_2]$ are two classes in $\pi_1(X,x_0)$ then

$$f_\#([\gamma_1]\cdot[\gamma_2]) = f_\#([\gamma_1\cdot\gamma_2])$$
$$= [f(\gamma_1\cdot\gamma_2)]$$



$$= [f\circ\gamma_1]\cdot[f\circ\gamma_2]$$
$$= f_\#([\gamma_1])\cdot f_\#([\gamma_2]).$$

Hence, $f_\#$ is a group homomorphism. ∎

Remark: If $f:(X,x_0)\to(X,x_0)$ is the identity function then the homomorphism

$$f_\#:\pi_1(X,x_0)\longrightarrow\pi_1(X,x_0) \text{ is the}$$

identity homomorphism.

Back to the proof of the Application:

$$(S^1, 1) \xhookrightarrow{i} (D^2, 1) \xhookrightarrow{r} (S^1, 1)$$

$$r \circ i = \mathrm{id}_{(S^1,1)}.$$

$$\begin{tikzcd}
\pi_1(S^1,1) \arrow[r, "i_\#"] \arrow[rr, bend right, ""] & \pi_1(D^2,1) \arrow[r, "r_\#"] & \pi_1(S^1,1) \\
\mathbb{Z} \arrow[u, phantom, "\cong"] \arrow[r] & (e) \arrow[u, phantom, "="] \arrow[r] & \mathbb{Z} \arrow[u, phantom, "\cong"]
\end{tikzcd}$$

$$(r \circ i)_\# = (\mathrm{id}_{(S^1,1)})_\# = \mathrm{id}_{\pi_1(S^1,1)}$$

This is a contradiction since the identity isomorphism of $\mathbb{Z}$ passes through the trivial group.

This finishes the proof. ∎

Theorem (Borsuk–Ulam Theorem)

For every continuous map $f: S^2 \to \mathbb{R}^2$, where S^2 is the unit sphere in $\mathbb{R}^3$ there exists a pair of antipodal points x and $-x$ on S^2 so that $f(x) = f(-x)$.

$$f(x) = (\underset{\text{air pressure}}{P(x)}, \underset{\text{temp.}}{T(x)})$$

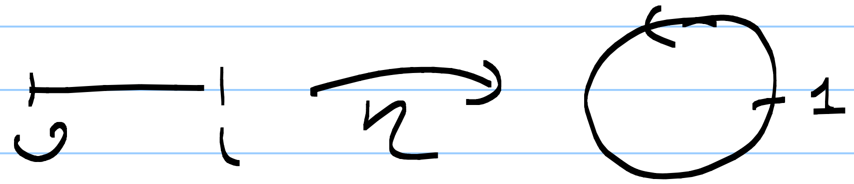
$x = (a, b, c)$, $-x = (-a, -b, -c)$

Proof: Suppose that $f(x) \neq f(-x)$, for all $x \in S^2$. We must arrive at a contradiction!

Define a map $g: S^2 \longrightarrow S^1$ as follows:

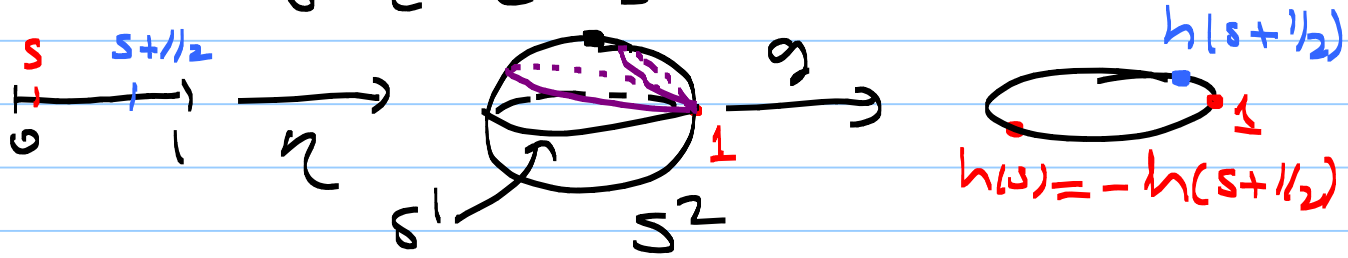
$$g(x) = \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|}, \quad x \in S^2.$$

Let $\eta : [0,1] \longrightarrow S^1 \subseteq S^2$ by $\eta(s) = (\cos 2\pi s, \sin 2\pi s, 0)$



$[\eta] = [\omega_1] = 1 \in \pi_1(S^1, 1) \simeq \mathbb{Z}$

Let $h = g \circ \eta : [0,1] \longrightarrow S^1$



Note that $g(-x) = -g(x)$, for all $x \in S^2$.

Hence, $h(s+1/2) = g(\eta(s+1/2))$

$$\begin{aligned}
&= g(\cos 2\pi(s+1/2), \sin 2\pi(s+1/2)) \\
&= g(\cos(2\pi s + \pi), \sin(2\pi s + \pi)) \\
&= g(-\cos 2\pi s, -\sin 2\pi s) \\
&= -g(\cos 2\pi s, \sin 2\pi s) \\
&= -g(\eta(s))
\end{aligned}$$

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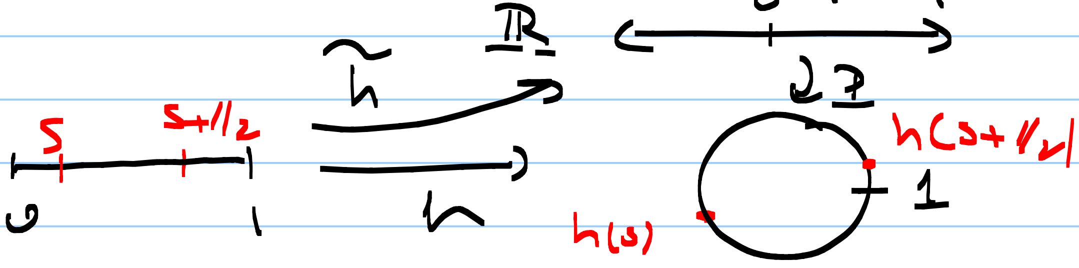
$$= -h(s), \quad s\in[0,1].$$

Let $\tilde h : [0,1]\to\mathbb{R}$ be the lift of h with $\tilde h(0)=0$.



Since $h(s+1/2) = -h(s)$ we get

$$\tilde h(s+1/2) = \tilde h(s) + \frac{q}{2} \text{ for some odd integer } q\in\mathbb{Z},$$

because $p(\tilde h(s+1/2)) = h(s+1/2) = -h(s) = -p(\tilde h(s))$

Note that since $q = 2(\tilde h(s+1/2) - \tilde h(s))$ and $\tilde h$ is continuous q must be independent of s.
In particular,

$$\tilde h(1) = \tilde h(1/2) + q/2 = \tilde h(0) + q \Rightarrow [h] = q \in \mathbb{Z} \cong \pi_1(S^1,1).$$

Since q is an odd integer $[h]\neq 0\in\mathbb{Z}$ and thus h is <u>not</u> null homotopic.
However, $h = g\circ\eta$ and η is clearly null homotopic. Thus h is null homotopic, and thus we arrived at a contradiction.
This finishes the proof. ■

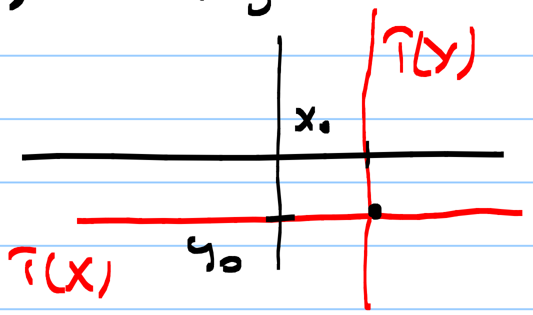
<u>Proposition:</u> Let (X,x_0) and (Y,y_0) be path connected based spaces. Then $(X\times Y,(x_0,y_0))$ is a path connected based space and

$\pi_1(X\times Y,(x_0,y_0))$ is isomorphic to $\pi_1(X,x_0)\times\pi_1(Y,y_0)$.

Proof: $i:(X,x_0)\longrightarrow(X\times Y,(x_0,y_0))$, $x\longmapsto(x,y_0)$, $x\in X$

$j:(Y,y_0)\longrightarrow(X\times Y,(x_0,y_0))$, $y\longmapsto(x_0,y)$, $y\in Y$.

Clearly i and j are continuous maps.



We have also projection maps:

$P_X:(X\times Y,(x_0,y_0))\longrightarrow(X,x_0)$, $(x,y)\longmapsto x$, $(x,y)\in X\times Y$,

and $P_Y:(X\times Y,(x_0,y_0))\longrightarrow(Y,y_0)$, $(x,y)\longmapsto y$, $(x,y)\in X\times Y$.

$P_X\circ i:(X,x_0)\longrightarrow(X,x_0)$, $x\longmapsto(x,y_0)\longmapsto x$, $x\in X$.

so that $P_X\circ i=\mathrm{id}_{(X,x_0)}$

and similarly $P_Y\circ j=\mathrm{id}_{(Y,y_0)}$.

$$\begin{tikzcd} \pi_1(X) \arrow[r, "i_\#"] & \pi_1(X\times Y) \arrow[l, bend left, "P_{X\#}"] \end{tikzcd}, \quad \begin{tikzcd} \pi_1(Y) \arrow[r, "j_\#"] & \pi_1(X\times Y) \arrow[l, bend left, "P_{Y\#}"] \end{tikzcd}.$$

Consider the map

$\varphi:\pi_1(X)\times\pi_1(Y)\longrightarrow\pi_1(X\times Y)$ given by

$\varphi([f],[g]) = [h]$, where $h: I \to X \times Y$,
$h(s) = (f(s), g(s))$.

<u>must show:</u> i) φ is well defined
ii) φ is a homomorphism
iii) φ has an inverse

$$\varphi^{-1} : \pi_1(X \times Y) \longrightarrow \pi_1(X) \times \pi_1(Y)$$

$$\varphi^{-1}([h]) = ([f],[g])$$

Rest is exercise!

<u>Corollary</u> $\pi_1(S^1 \times S^1) \simeq \mathbb{Z} \times \mathbb{Z}$ and in general

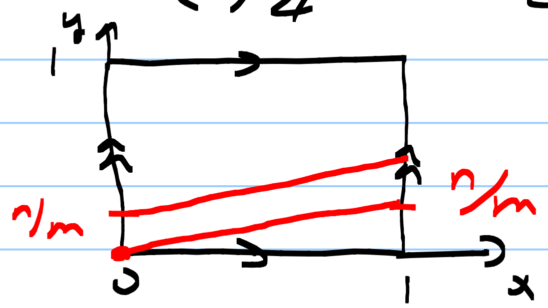
$$\pi_1(\underbrace{S^1 \times S^1 \times \dots \times S^1}_{n\text{-copies}}) \simeq \underbrace{\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}}_{n\text{-copies}}.$$

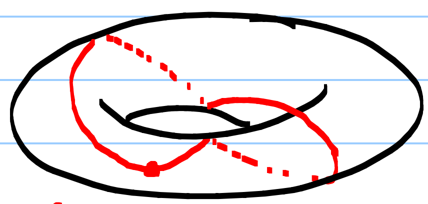
<u>Example:</u> $(m,n) \in \mathbb{Z} \times \mathbb{Z}$

$S^1 = \mathbb{R}/\mathbb{Z}$ $\quad x \sim x+n, \ \forall n \in \mathbb{Z}$

$S^1 \times S^1 = (\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z})$

$(m,n) \longmapsto (e^{2\pi i s m}, e^{2\pi i s n})$





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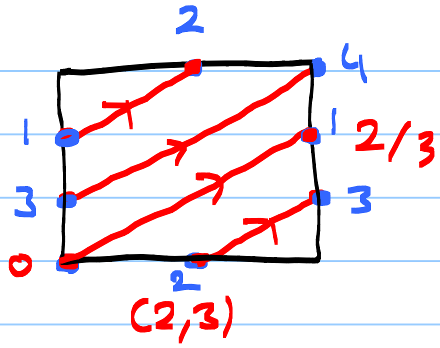
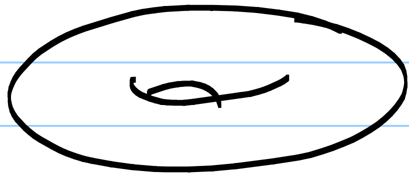
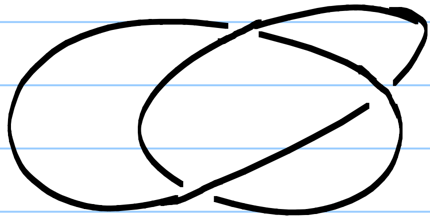






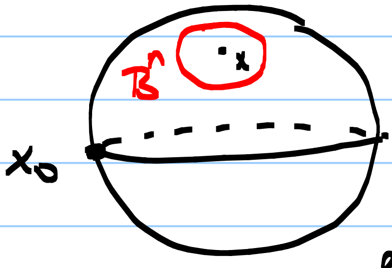
Trefoil Knot.

Exercise: The homotopy class of the loop corresponding to $(m,n) \in \mathbb{Z} \times \mathbb{Z} \simeq \pi_1(T^2)$ is represented by an embedded circle if and only if m and n are relatively prime.

Theorem: $\pi_1(S^n, x_0) = (e)$ if $n \geq 2$.

Proof: Let $f : [0,1] \to S^n$ be any loop based at x_0.

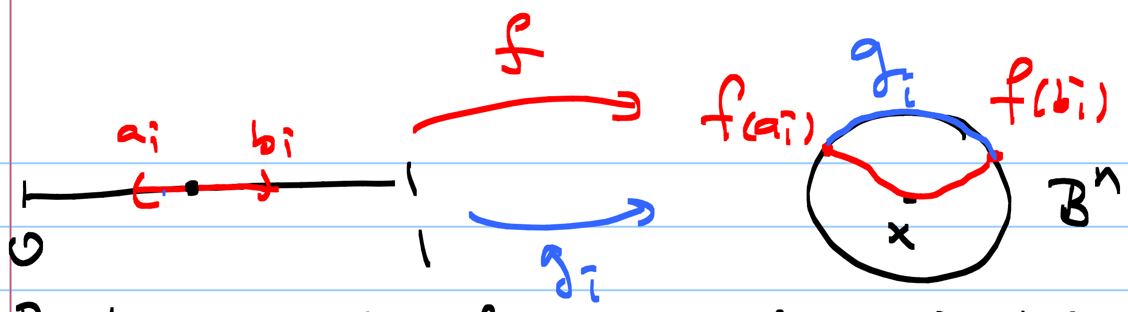
Let $x \in S^n$, $x \neq x_0$ and choose a small open ball B^n around. The inverse image of B^n, $f^{-1}(B^n)$ is a disjoint union of open intervals (a_i, b_i) in I. Since $f^{-1}(x)$ is a compact subset of $[0,1]$ and it is covered by the open intervals (a_i, b_i) $f^{-1}(x)$ is covered by finitely many (a_i, b_i). Now for each these finitely many intervals choose some



$$g_i : [a_i, b_i] \to S^n \text{ so that } g_i \text{ is homotopic}$$

to $f_i|_{[a_i,b_i]}$, $f(a_i) = g(a_i)$, $f(b_i) = g(b_i)$ and

$g_i([a_i, b_i]) \subseteq \partial B^n$.



Replacing each $f|_{[a_i,b_i]}$, for finitely many i, by suitable g_i, we obtain the loop represented by g so that $[f]=[g]$ and $g(I)\subseteq S^n\setminus\{x\}$.

$g:[0,1]\longrightarrow S^n\setminus\{x\}\simeq \mathbb{R}^n$ is null homotopic and thus $[f]=[g]=e$ in $\pi_1(S^n,x_0)$.

This finished the proof. $\blacksquare$

<u>Examples</u>: 1) $x\in\mathbb{R}^n$, then $\mathbb{R}^n\setminus\{x\}\simeq \mathbb{R}^{>0}\times S^{n-1}$ homeomorphic.

$$\varphi:\mathbb{R}^n\setminus\{x\}\longrightarrow \mathbb{R}^{>0}\times S^{n-1},\quad y\longmapsto \left(\|y-x\|, \frac{y-x}{\|x-y\|}\right),$$

$$\mathbb{R}^n\setminus\{x\}\simeq \mathbb{R}^{>0}\times S^{n-1}\simeq \mathbb{R}\times S^{n-1} \text{ homeomorphism.}$$
$$(t,p)\longmapsto(\ln t, p)$$

$$\pi_1(\mathbb{R}^n\setminus\{x\})\simeq \pi_1(\mathbb{R}\times S^{n-1})=\pi_1(\mathbb{R})\times\pi_1(S^{n-1})$$
$$\simeq (e)\times\pi_1(S^{n-1})$$
$$\simeq \pi_1(S^{n-1}) = \begin{cases}\mathbb{Z} & \text{if } n=2\\ (e) & \text{if } n>2\end{cases}$$

<u>Corollary</u>: Hence $\mathbb{R}^2$ is <u>not</u> homeomorphic to any $\mathbb{R}^n$ if $n\neq 2$.

Proof: $n=1$ $\mathbb{R}^2 \ ? \ \mathbb{R}$

If $f: \mathbb{R}^2 \to \mathbb{R}$ is a homeomorphism then

$f: \mathbb{R}^2 \setminus \{p\} \to \mathbb{R} \setminus \{f(p)\}$ would be still a

homeomorphism. This is a contradiction because $\mathbb{R}^2 \setminus \{p\}$ is connected and $\mathbb{R} \setminus \{f(p)\}$ is not.

$n \geq 2$: $f: \mathbb{R}^2 \to \mathbb{R}^n$. This time we would get a homeomorphism from $\mathbb{R}^2 \setminus \{p\}$ to

$\mathbb{R}^n \setminus \{f(p)\}$, where $\mathbb{R}^n \setminus \{f(p)\}$ is simply connected and $\mathbb{R}^2 \setminus \{p\}$ is not simply connected.

Proposition: Let $\varphi: X \to Y$ be a homotopy equivalence.

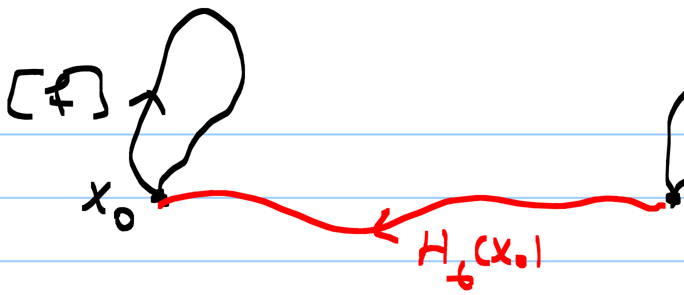
Then $\varphi_\#: \pi_1(X, x_0) \to \pi_1(Y, \varphi(x_0))$ is an isomorphism, for any $x_0 \in X$.

Proof: By the assumption there is a continuous map $\psi: Y \to X$ so that

$\psi \circ \varphi: X \to X$ is homotopic to id_X and

$\varphi \circ \psi: Y \to Y$ is homotopic to id_Y.

Let $x_1 = (\psi \circ \varphi)(x_0) = \psi(\varphi(x_0))$



$$(\psi\circ\varphi)_\#([f])$$

$$\psi\circ\varphi \sim \mathrm{Id}_X$$

$$H : X\times I \longrightarrow X$$

$$H(x,t) = H_t(x)$$

$$H_0(x) = (\psi\circ\varphi)(x)$$

$$H_1(x) = \mathrm{Id}_X(x) = x$$

The map $t \longmapsto H_t(x_0)$ is a path from x_0 to x_1

$$\pi_1(X,x_0) \xrightarrow[(\psi\circ\varphi)_\#]{} \pi_1(X,x_1) \xrightarrow[\beta_{H_t(x_0)}]{\simeq} \pi_1(X,x_0)$$

$$\varphi_\# : \pi_1(X,x_0) \longrightarrow \pi_1(Y,\varphi(x_0))$$

$$\psi : \pi_1(Y,\varphi(x_0)) \longrightarrow \pi_1(X,x_1)$$

Rest is exercise.

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Van Kampen's Theorem:

Free Products of Groups: Let $\{G_\alpha\}_{\alpha\in\Lambda}$ be family of groups. Then the free product of this family is defined to be group by means of the following universal property:

Notation: $\ast_\alpha G_\alpha$: Free product of G_α's.

P: If $\varphi_\alpha : G_\alpha \to H$ is a homomorphism for each $\alpha\in\Lambda$, then there is a unique homomorphism $\Phi : \ast_\alpha G_\alpha \to H$ so that the diagram below is commutative:

$$\begin{tikzcd} \ast_\alpha G_\alpha \arrow[r, "\Phi"] & H \\ G_\beta \arrow[u, "i_\beta"] \arrow[ur, "\varphi_\beta"'] & \end{tikzcd}$$

$i_\beta : G_\beta \to \ast_\alpha G_\alpha$ is a monomorphism.

Theorem: There is a unique (up to isomorphism) group satisfying the property P.

Idea of the proof: Elements of $\ast_\alpha G_\alpha$ are finite words of the form

$$g_1 g_2 g_3 \cdots g_n, \text{ where } g_i \in G_{\alpha_i}, i=1,\dots,n.$$

Group operation: If $g_1 g_2 \cdots g_n$ and $g_1' g_2' \cdots g_m'$ are two words then

$$(g_1 g_2 \cdots g_n)\cdot(g_1' g_2' \cdots g_m') = g_1 g_2 \cdots g_n g_1' g_2' \cdots g_m'$$

Inverse: $(g_1 g_2 \cdots g_n)^{-1} = g_n^{-1} \cdots g_2^{-1} g_1^{-1}$ so that

$$(g_1 g_2 \cdots g_n)(g_n^{-1} \cdots g_2^{-1} g_1^{-1}) = g_1 g_2 \cdots (\underbrace{g_n g_n^{-1}}_{}) \cdots g_2^{-1} g_1^{-1}$$

$$\overset{e \in G_{\alpha_n}}{\shortparallel}$$

$$= g_1 g_2 \cdots (g_{n-1} g_{n-1}^{-1}) \cdots g_2^{-1} g_1^{-1}$$
$$\vdots$$
$$= g_1 g_1^{-1}$$
$$= e$$

If we choose each G_α to be the cyclic group $(\mathbb{Z}, +)$ then the group $\underset{\alpha}{*} G_\alpha$ is called the free group on $|\Lambda|$ letters.

In particular, $|\Lambda| = n$ then we obtain

$$\underbrace{\mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}}_{n\text{-copies}}$$ the free group on n-letters.

<u>Notation:</u> $F_n \doteq \mathbb{Z} * \cdots * \mathbb{Z}$

<u>Examples</u> 1) $F_2 = \mathbb{Z} * \mathbb{Z} = \langle a \rangle * \langle b \rangle$

$$= \langle a, b \mid - \rangle$$

Elements of F_2 are words on the alphabeth a, a^{-1}, b, b^{-1}. For example,

$$ab,\ a^2 b,\ abab,\ aba^{-1},\ ab^2 a^{-1} b a b^{-1} a^2 b^3, \ldots$$

$$abb^{-1}a^3 = aea^3 = aa^3 = a^4$$

<u>Definition:</u> Suppose that G is a group on the alphabeth $\{g_\lambda\}$ and let $R_\gamma \in G$, for each $\gamma \in R$, the set of relations.

Let N denote the smallest normal subgroup of G containing each R_γ, $\gamma \in R$. Then the quotient group G/N is called the group with generators $\{g_\lambda\}$ and relations $\{R_\alpha\}$.

$$G/N = \langle g_\lambda \mid R_\gamma \rangle$$

<u>Example:</u> $\mathbb{Z}_2 = \langle a \mid a^2 \rangle$, $\mathbb{Z}_n = \langle a \mid a^n \rangle$

$\mathbb{Z} = \langle a \mid - \rangle$.

$F_2 = \langle a, b \mid - \rangle$.

$\mathbb{Z}_2 * \mathbb{Z}_3 = \langle a, b \mid a^2, b^3 \rangle \doteq F_2/N$

$N =$ normal closure of the set $\{a^2, b^3\}$.

$\mathbb{Z}_2 * \mathbb{Z}_3 \simeq PSL(2, \mathbb{Z})$

$a = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$.

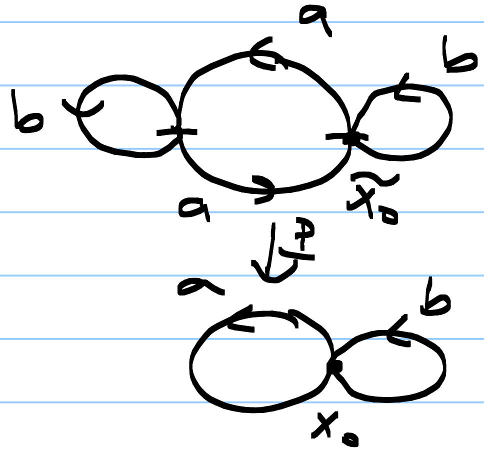
<u>Example:</u> $\mathbb{Z} \times \mathbb{Z}$ free abelian group of rank 2.

$\mathbb{Z} \times \mathbb{Z} = \langle a, b \mid aba^{-1}b^{-1} \rangle = F_2/F_2'$, where

$F_2' = [F_2, F_2]$ is the commutator subgroup.

In general $F_n/F_n' \simeq \mathbb{Z}^n = \mathbb{Z} \times \cdots \times \mathbb{Z}$, the free abelian group of rank n.

Example: F_2 contains F_3 as a normal subgroup of index 2



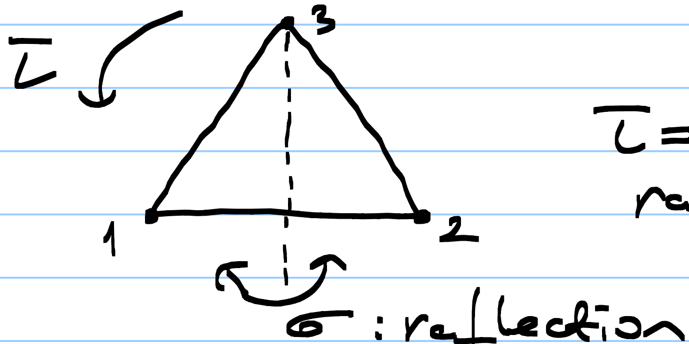
$F_2 = \langle a, b \mid - \rangle$

$H = \langle a^2, b, aba^{-1} \rangle$ the subgroup generated by a^2, b and aba^{-1}.

Claim: H is a normal subgroup of F_2 of index 2 isomorphic to F_3.

This will be proved using the theory of covering spaces.

Example: $S_3 = \langle \sigma, \tau \mid \sigma^2, \tau^3, \sigma\tau\sigma\tau \rangle$



τ = Counter clockwise $2\pi/3$ radian rotation

$\sigma = (1\,2)$, $\tau = (1\,2\,3)$

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Theorem (Seifert, Van Kampen)

Let X be a topological space and U, V path connected open subsets of X so that $U \cap V$ is a nonempty path connected subset of X with $X = U \cup V$. Then the homomorphism

$$\Phi : \pi_1(U) * \pi_1(V) \longrightarrow \pi_1(X),$$ where

$\Phi(g) = i_{U\#}(g)$ $\forall$ $g \in \pi_1(U)$ and

$\Phi(g) = i_{V\#}(g)$ $\forall$ $g \in \pi_1(V)$, is surjective.

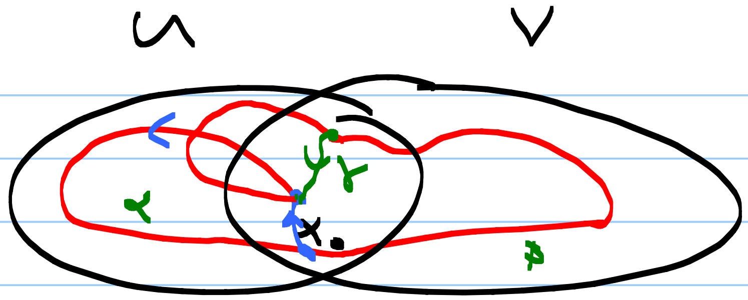
Moreover, the kernel N of Φ is generated by all elements of the form

$$(i_{U\#} \circ j_{U\#}(\omega))(i_{V\#} \circ j_{V\#}(\omega^{-1})),$$ where $\omega \in \pi_1(U \cap V)$.

Notation: $\pi_1(X) \simeq \pi_1(U) \underset{\pi_1(U\cap V)}{*} \pi_1(V)$

$$\begin{tikzcd}
 & \pi_1(U) \arrow[dr, "i_{U\#}"] & \\
\pi_1(U \cap V) \arrow[ur, "j_{U\#}"] \arrow[dr, "j_{V\#}"'] & & \pi_1(U \cup V) = \pi_1(X) \\
 & \pi_1(V) \arrow[ur, "i_{V\#}"'] &
\end{tikzcd}$$

Idea:



$\alpha \beta \gamma$

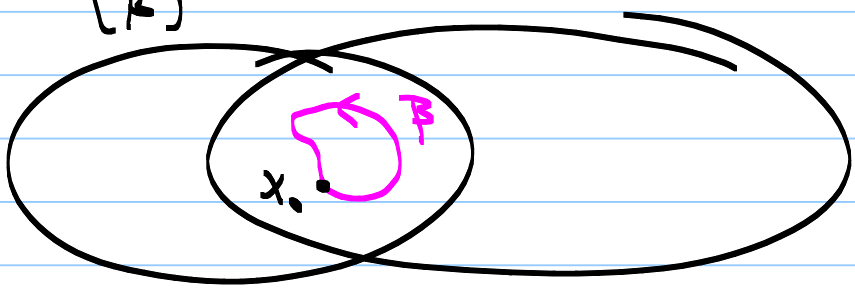
$\alpha, \gamma \in \pi_1(U, x_0), \quad \beta \in \pi_1(V, x_0).$

$\pi_1(U, x_0) \xrightarrow{i_{U\#}} \pi_1(X, x_0) \qquad i_U : U \hookrightarrow X$

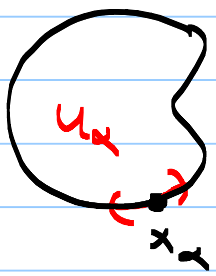
$\pi_1(V, x_0) \xrightarrow[i_{V\#}]{} \pi_1(X, x_0) \qquad i_V : V \hookrightarrow X$

$\pi_1(U, x_0) * \pi_1(V, x_0) \longrightarrow \pi_1(X, x_0)$

$[\beta] \qquad [\beta]$



Examples: 1) $X = \bigvee_\alpha X_\alpha$, where each X_α is path connected and $x_\alpha \in X_\alpha$ so that there is some open subset $x_\alpha \in U_\alpha \subseteq X_\alpha$ which deformation retracts onto $\{x_\alpha\}$.

X_α 

$X = \bigvee_\alpha X_\alpha \doteq \dot{\bigcup_\alpha} X_\alpha \Big/ x_\alpha \sim x_\beta$

$U = \bigvee_\alpha U_\alpha$ deformation retracts onto $\{x_0\}$, where $x_0 = [x_\alpha]$.

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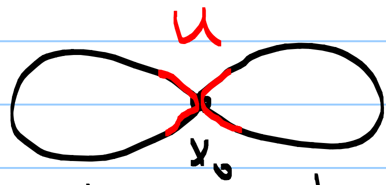
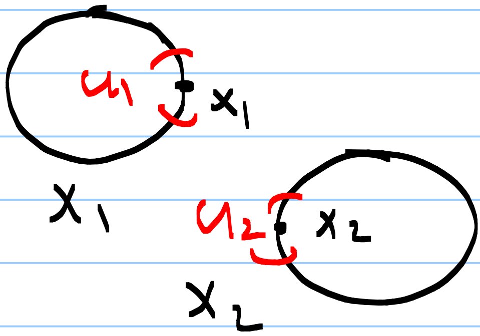
$$\pi_1(X) \doteq \ast_\alpha \pi_1(X_\alpha)$$

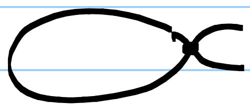
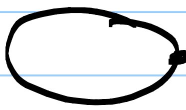
$X_\alpha' = X_\alpha \cup U$ open subset of X, which deformation retracts onto X_α.

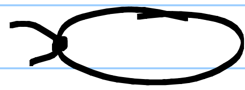
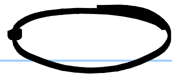
Example: $S^1 \vee S^1$



$X_1 = S^1$ $\qquad$ $S^1 = X_2$



$X_1' =$  $\simeq$  $= X_1$

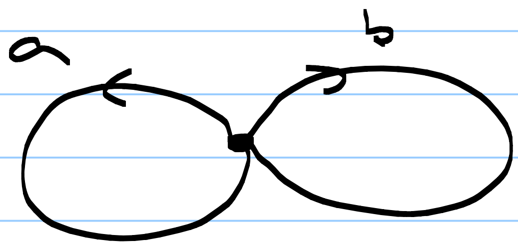
$X_2' =$  $\simeq$  X_2

$$\pi_1(S^1 \vee S^1) \simeq \pi_1(X_1') \ast \pi_1(X_2') \simeq \underbrace{\pi_1(S^1)}_{\mathbb{Z}} \ast \underbrace{\pi_1(S^1)}_{\mathbb{Z}}$$

$\pi_1(X_1' \cap X_2') = (e)$

$$\simeq \mathbb{Z} \ast \mathbb{Z}$$

$$= F_2 = \langle a, b \mid \text{—} \rangle$$



$a, b, a^2 b, a^{-1} b a^3, \ldots$

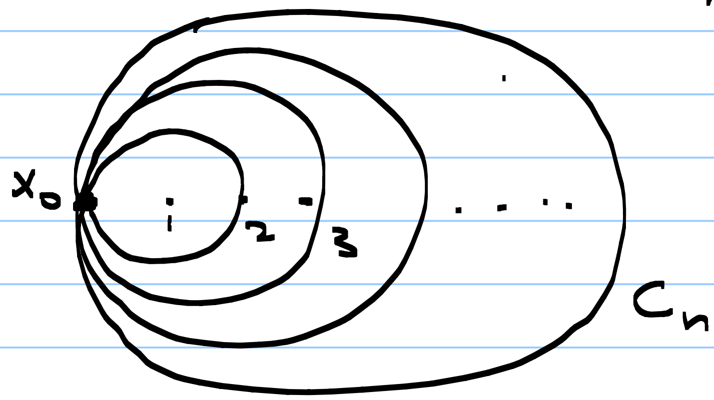
Similarly, $\pi_1(\bigvee_n S^1) \simeq \ast_n \pi_1(S^1) = \ast_n \mathbb{Z} = F_n$.

In general if $X = \bigvee_{\alpha \in \Lambda} S^1_\alpha$, then, $\pi_1(X)$ is free group on the Λ.

$\pi_1(X) = \langle g_\alpha, \alpha \in \Lambda \mid - \rangle$.

$\Lambda = \mathbb{N} = \{1, 2, 3, \dots\}$, $\pi_1(\bigvee_n S^1_n) = \underset{n \in \mathbb{N}}{*} \mathbb{Z} = F_\infty$

$X \subset \mathbb{R}^2$:



Circles of radius n

$C_n : \{(x,y) \in \mathbb{R}^2 \mid (x-n)^2 + y^2 = n^2\}$

$$\begin{tikzcd} \coprod_n S^1_n \arrow[r, "\varphi"] \arrow[d] & X \\ \coprod_n S^1_n/\sim = \bigvee_n S^1_n \arrow[ur, dotted, "\tilde{\varphi}"] & \end{tikzcd}$$

$\varphi|_{S^1_n} : S^1_n \longrightarrow C_n$ homeomorphism

$\varphi(x_0^n) = x_0$, where $x_0^n \in S^1_n$ is a point.

$$\pi_1(\bigvee_n S^1_n) \xrightarrow[f_{n_0\#}]{} \pi_1(S^1_{n_0})$$

$f_{n_0} : \bigvee_n S^1_n \longrightarrow S^1_{n_0}$, $i_{n_0} : S^1_{n_0} \longrightarrow \bigvee_n S^1_n$

$f_{n_0} \circ i_{n_0} : S^1_{n_0} \longrightarrow S^1_{n_0}$, $f_{n_0} \circ i_{n_0} = \mathrm{id}_{S^1_{n_0}}$

$$\begin{tikzcd} \pi_1(S^1_{n_0}) \arrow[r, "i_{n_0\#}"] \arrow[d, phantom, "\wr"] & \pi_1(\bigvee_n S^1_n) \arrow[r, "f_{n_0\#}"] & \pi_1(S^1_{n_0}) \arrow[d, phantom, "\wr"] \\ \mathbb{Z} \arrow[rr, "\mathrm{id}"'] & & \mathbb{Z} \end{tikzcd}$$

$f_n \circ i_m : S^1 \longrightarrow S^1$ is constant, if $m \neq n$.

Thus $(f_n \circ i_m)_\# : \mathbb{Z} \to \mathbb{Z}$ is the trivial homomorphism if $m \neq n$.

$$\begin{array}{ccccc} \pi_1(S^1) & \xrightarrow{i_{m\#}} & \pi_1(\bigvee_n S^1) & \xrightarrow{f_{n\#}} & \pi_1(S^1) \\ \| & & & & \| \\ \mathbb{Z} & & & & \mathbb{Z} \end{array}$$

If $m_1, m_2, \ldots, m_k$ are different integers then

$$\pi_1\left(\bigvee_{i=1}^k S^1_{m_i}\right) \xrightarrow{i_\#} \pi_1\left(\bigvee_n S^1\right) \xrightarrow{P} \pi_1\left(\bigvee_{i=1}^k S_{m_i}\right)$$

i is the inclusion map and P is the map which identity on each $S^1_{m_i}$ and contraction on all other S^1_n's.

Since $P \circ i : S^1_{m_1} \vee \cdots \vee S^1_{m_k} \longrightarrow S^1_{m_1} \vee \cdots \vee S^1_{m_k}$ is the identity map and thus the homomorphism

$$(P \circ i)_\# : \pi_1\left(\bigvee_{i=1}^k S_{m_i}\right) \longrightarrow \pi_1\left(\bigvee_{i=1}^k S_{m_i}\right)$$ is

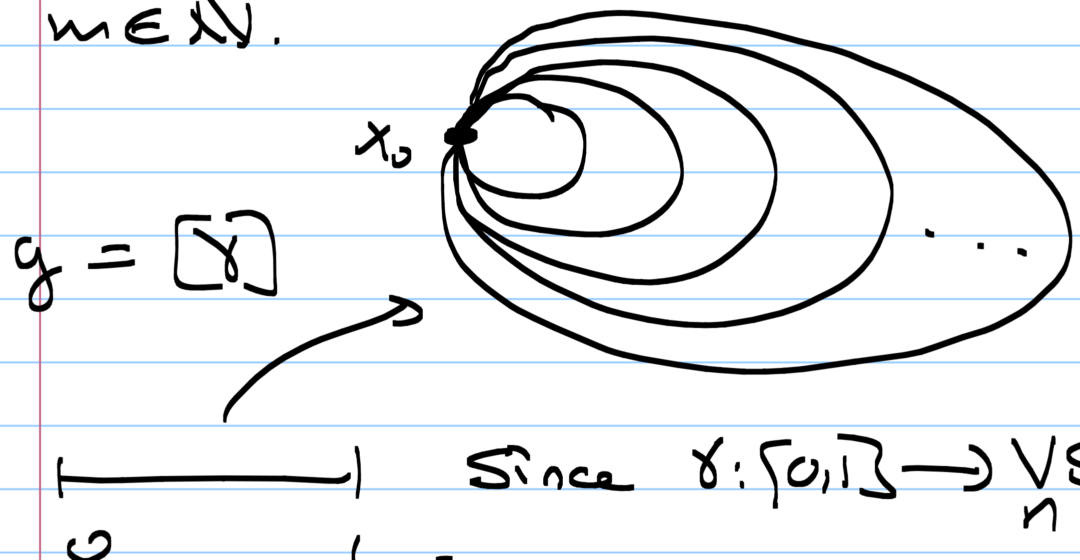
is identity. Hence $F_k \simeq i_\#\left(\pi_1\left(\bigvee_{i=1}^k S_{m_i}\right)\right)$ is a subgroup of $\pi_1(\bigvee_n S^1_n)$ $(\simeq F_\infty)$

Claim: Any element $g \in \pi_1(\bigvee_n S^1_n)$ maps to only finitely many non-trivial elements under projections:

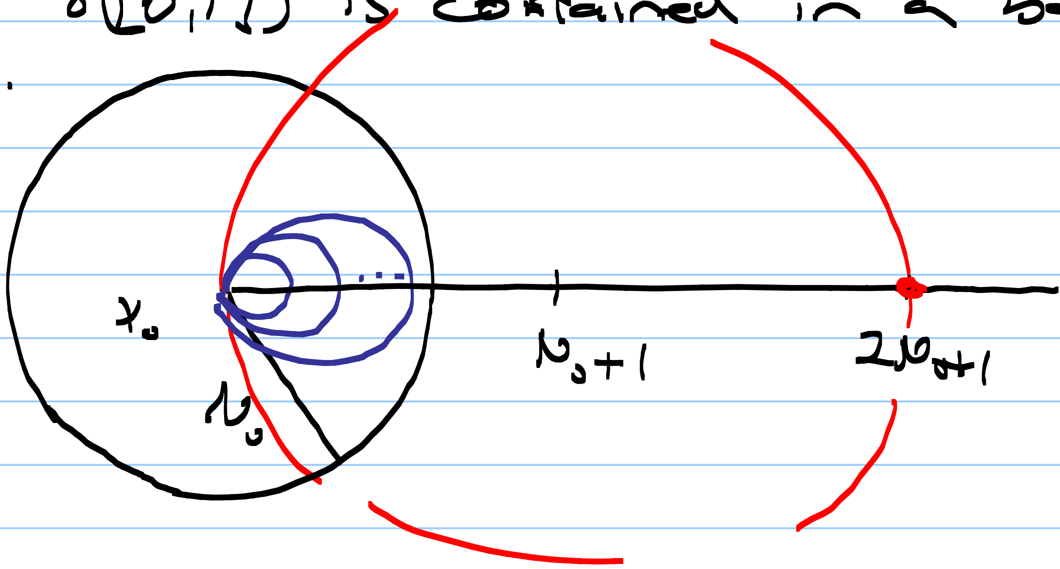
$$f_{m_\#}: \pi_1(\bigvee_n S^1_n) \longrightarrow \pi_1(S^1_m) \simeq \mathbb{Z}$$

$f_{m_\#}(g) = 0$ for all but finitely many $m \in \mathbb{N}$.

$g = [\gamma]$



Since $\gamma: [0,1] \longrightarrow \bigvee_n S^1 \subseteq \mathbb{R}^2$ is continuous and $[0,1]$ is compact $\gamma([0,1])$ is contained in a ball $B(0, N_0)$.



$(\gamma[0,1])$ does not contain $2n+1$ of $n \geq N_0$

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$[0,1] \xrightarrow{\gamma} \bigvee_n S^1_n \longrightarrow S^1_m$ is not onto if $m \geq n_0$ and thus on the π_1-level

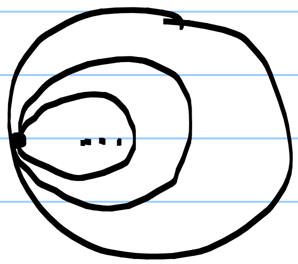
$$\begin{array}{ccc}\pi_1(\bigvee_n S^1_n) & \longrightarrow & \pi_1(S^1_m)\\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\cong$}\\ [g] & \longmapsto & 0\in\mathbb{Z}\end{array}$$

Hence, g is a word in $\alpha_1^{\mp}, \dots, \alpha_{n_0}^{\mp}$.

$$g = \alpha_1^{3}\alpha_2^{-5}\alpha_4^{2}\alpha_5^{7}\cdots\alpha_{n_0}^{-4}$$

Conclusion: $\pi_1(\bigvee_n S^1_n)$ is free group on $\mathbb{N}$.

Example: $Y = \{(x,y)\in\mathbb{R}^2 \mid \underbrace{(x-\frac{1}{n^2})^2+y^2=\frac{1}{n^4}}_{C_n}, n\in\mathbb{N}\}$



C_n

$$\sum_{n=1}^{\infty}\frac{2\pi}{n^2} = \frac{\pi^2}{6}\cdot 2\pi = \frac{\pi^3}{3}$$

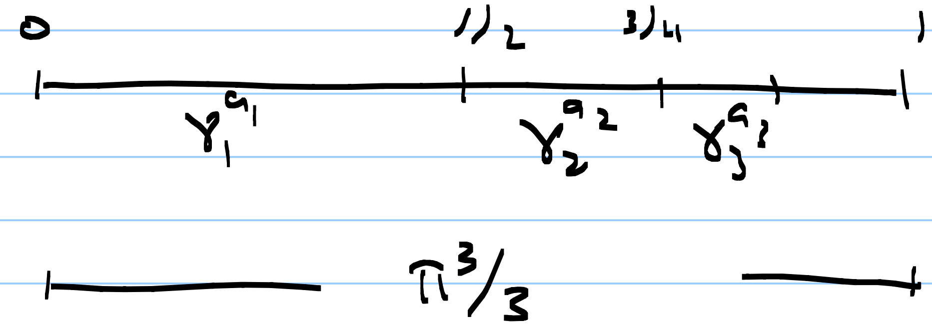
$\pi_1(Y)$ is uncountable!

To show that $\pi_1(Y)$ is uncountable we find an injective map from $[0,1]$ to $\pi_1(Y)$:

$x\in[0,1]$, $x = 0.\underset{1}{3}\underset{2}{5}\underset{3}{0}8\,2\,1\underset{n}{6}9\,0\,0\,3\,2 = 0.a_1a_2\cdots$

Map x to the loops that goes around C_n as many times as the n^{th} decimal.
If $[\gamma_i]$ represents the loop that goes around the ith circle once then x corresponds to

$$x \longmapsto \gamma_1^{a_1}\gamma_2^{a_2}\cdots\cdots\gamma_n^{a_n}\cdots\cdots$$



$\Rightarrow$ continuous loop.

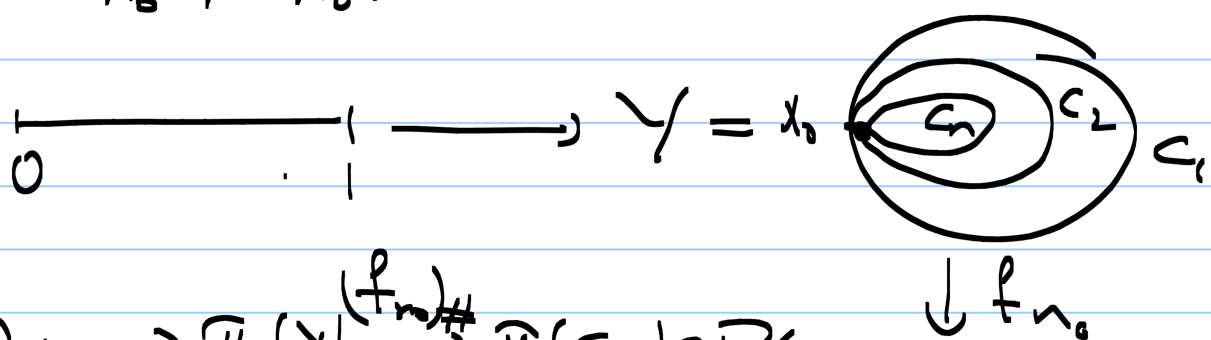
So we get a map $[0,1] \longrightarrow \pi_1(Y)$

If $x, y \in [0,1]$ and $x \neq y$ then there is some $n_0 \in \mathbb{N}$ so that

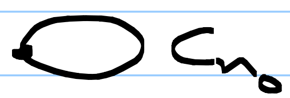
$x = 0.a_1a_2\cdots a_{n_0}a_{n_0+1}\cdots$, $y = 0.b_1b_2\cdots b_{n_0}b_{n_0+1}\cdots$

and $a_i = b_i$ for $i < n_0$ and

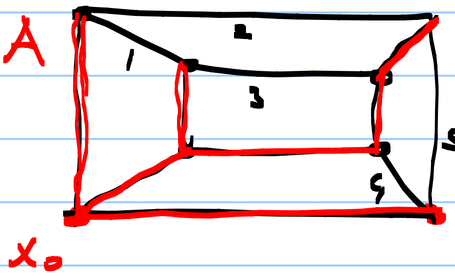
$a_{n_0} \neq b_{n_0}$.



$$[0,1] \longrightarrow \pi_1(Y) \xrightarrow{(f_{n_0})_\#} \pi_1(C_{n_0}) \simeq \mathbb{Z}$$
$$x \longmapsto a_{n_0}$$
$$y \longmapsto b_{n_0}$$



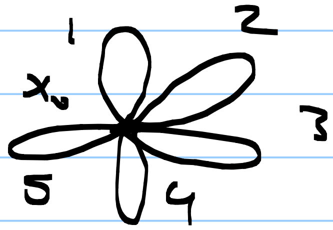
Example: X :



$A \subseteq X$ is contractible (to the point x_0)

X 1-dim'l CW-complex
$A \subseteq X$ subcomplex

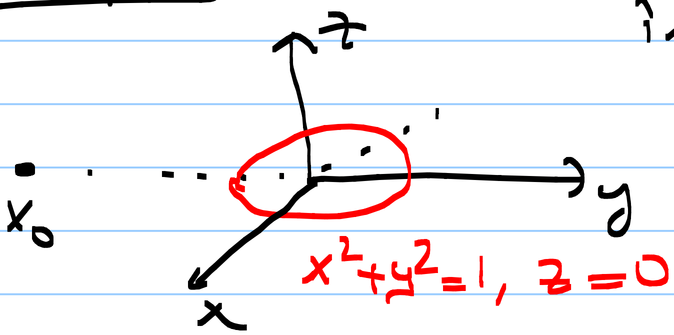
Since A is contractible X is homotopy equivalent to X/A.



Hence, X/A is homeomorphic to $\bigvee_5 S^1$.

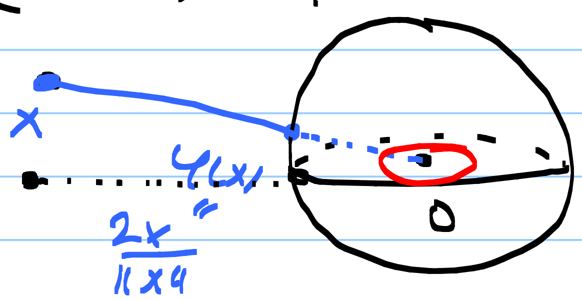
So, $\pi_1(X) \simeq \pi_1(X/A) \cong \pi_1(\bigvee_5 S^1) \cong F_5$

Example: $\mathbb{R}^3 \setminus S^1$, $S^1 \subseteq \mathbb{R}^3$ the unit circle in $\mathbb{R}^2 \times \{0\}$.



$\pi_1(\mathbb{R}^3 \setminus S^1) = ?$

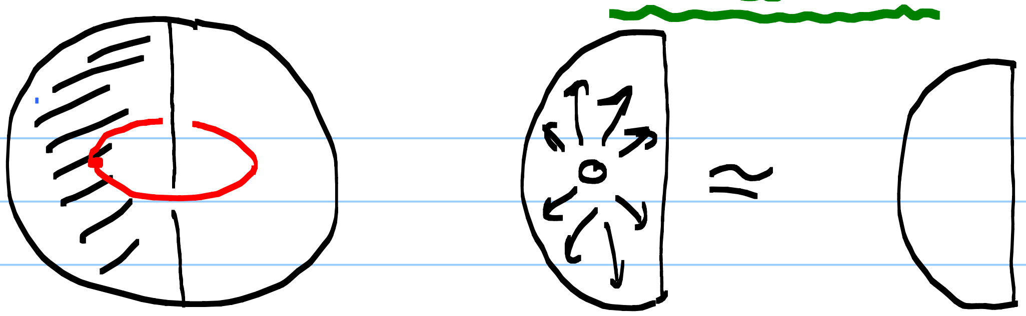
$(\mathbb{R}^3 \setminus S^1)$ deformation retracts onto $D^3 \setminus S^1$.



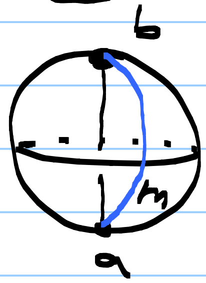
$D^3 = B[0,2]$

So, $\pi_1(\mathbb{R}^3 \setminus S^1) \simeq \pi_1(D^3 \setminus S^1)$.

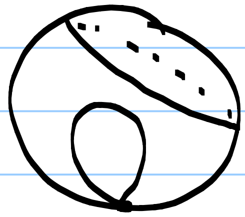
Video 23



Hence, $D^3 \setminus S^1$ deformation retracts onto $S^2 \cup [a,b]$



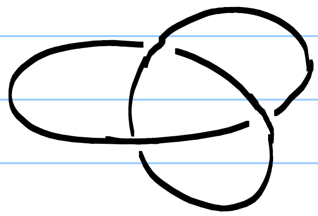
$$\pi_1(D^3 \setminus S^1) \simeq \pi_1(S^2 \cup [a,b])$$



$$S^2 \cup [a,b] \big/ _m = S^2 \vee S^1$$

$$\pi_1(S^2 \vee S^1) = \underset{\substack{\| \\ (e)}}{\pi_1(S^2)} \ast \underset{\substack{\cong \\ \mathbb{Z}}}{\pi_1(S^1)} \simeq \mathbb{Z}$$

<u>Remark:</u> $K \subseteq S^3$ knot (embedded circle)

 $\pi_1(S^3 \setminus K)$ knot group

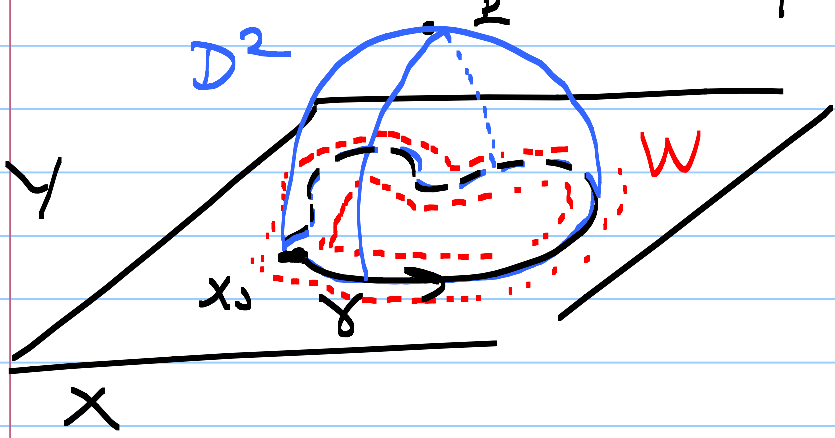
<u>Proposition:</u> Let (X, x_0) be a based topological space and $\gamma : [0,1] \longrightarrow X$ be a loop representing the element $[\gamma]$ in $\pi_1(X, x_0)$. Let Y be the attaching space defined by

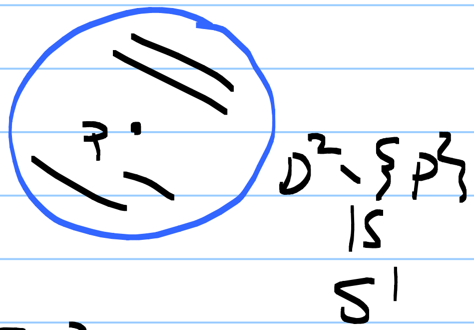
$$Y \doteq X \,\dot\cup\, D^2 \big/ _{\gamma(t) \sim (\cos 2\pi t, \sin 2\pi t),\ t \in [0,1]}$$

where $(\cos 2\pi t, \sin 2\pi t)$ parametrizes the boundary $S^1 = \partial D^2$. Then $\pi_1(Y, x_0)$ is isomorphic to

$$\pi_1(Y, x_0) = \pi_1(X, x_0) / N_{[\gamma]}, \text{ where } N_{[\gamma]} \triangleleft \pi_1(X, x_0)$$

is the normal closure of $[\gamma]$ in $\pi_1(X, x_0)$.





<u>Proof</u>: $Y = U \cup V$, $U = Y \setminus \{p\}$ and

V is a open neighborhood of D^2 in Y.

$V = D^2 \cup W$. We assume W deformation retracts onto γ. Hence, V deformation retracts to the point $\{p\}$, so it is contractible.
Note that U deformation retracts onto X.
Moreover, $U \cap V = V \setminus \{p\}$ and it deformation retracts onto W which deformation retracts onto γ.

Now Apply Zariski-van Kampen's Theorem to this setting

$$\begin{array}{ccccc} & & \pi_1(U) = \pi_1(X) & & \\ & \nearrow & & \searrow & \\ \pi_1(U \cap V) & & & & \pi_1(U \cup V) = \pi_1(Y) \\ \text{is } \mathbb{Z} = \langle [\gamma] \rangle & \searrow & & \nearrow & \\ & & \pi_1(V) = (e) & & \end{array}$$

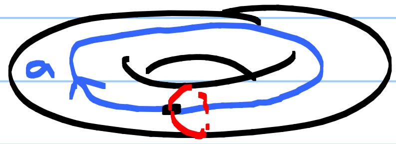
So $\pi_1(Y) = \overset{\overset{\pi_1(X)}{\|}}{\pi_1(U)} * \overset{\overset{\{e\}}{\|}}{\pi_1(V)} \Big/ N_{[\gamma]}$

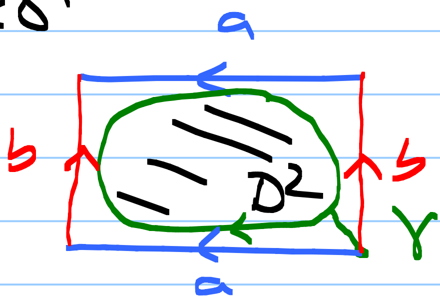
$= \pi_1(X)/N_{[\gamma]}$ $\qquad \Big(N_{[\gamma]} = \bigcap_{\substack{N \trianglelefteq G \\ [\gamma]\in N}} N \Big)$

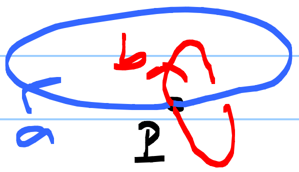
$\pi_1(X) = \langle g_\alpha \mid r_i \rangle$

$\pi_1(Y) = \langle g_\alpha \mid r_i, [\gamma] \rangle$

<u>Examples</u> 1) $T^2 = S^1 \times S^1$





X: 

$\simeq S^1 \vee S^1$, $\pi_1(X,p) = \langle a, b \mid - \rangle$

$Y = X \cup D^2$, $\pi_1(Y) = \langle a, b \mid [\gamma] \rangle$

$aba^{-1}b^{-1} = e$

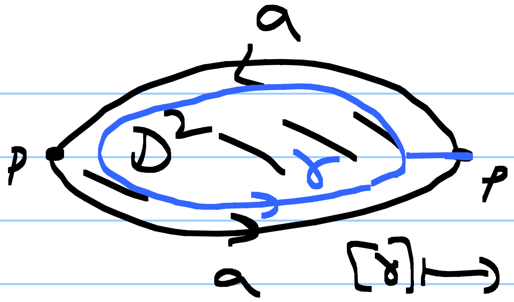
$ab = ba$

$= \langle a, b \mid aba^{-1}b^{-1} \rangle$

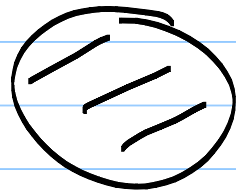
$=$ the free abelian group of rank two

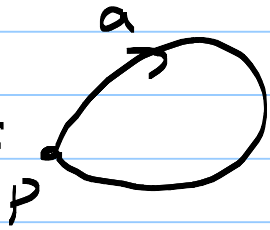
$= \underset{\langle a \rangle}{\underset{\|}{\mathbb{Z}}} \oplus \underset{\langle b \rangle}{\underset{\|}{\mathbb{Z}}}$

5) $\mathbb{RP}^2$



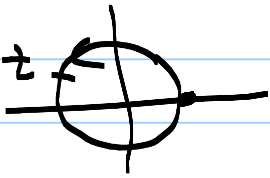
$[\gamma] \mapsto [a]^2$



X: 

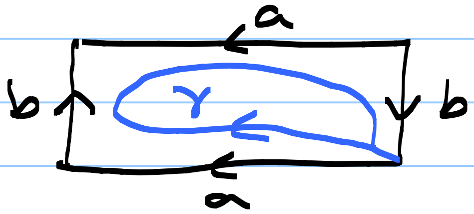
$$\underset{\partial D^2}{\partial D^2 = S^1} \longrightarrow \underset{a \cup \{p\}}{S^1}, \quad z \mapsto z^2$$

$\pi_1(X) = \langle a | - \rangle \cong \mathbb{Z}$



$$\pi_1(\mathbb{RP}^2) = \pi_1(X \cup D^2) = \langle a | a^2 \rangle \simeq \mathbb{Z}_2.$$

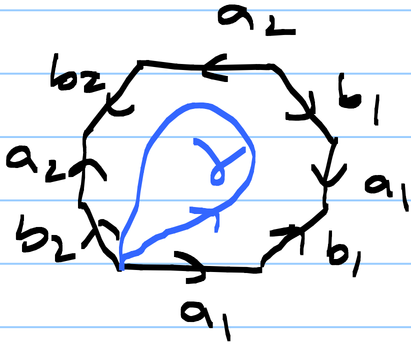
6) KB: Klein Bottle



$$\pi_1(KB) = \langle a, b | aba^{-1}b \rangle = \langle a, b | aba^{-1} = b^{-1} \rangle$$

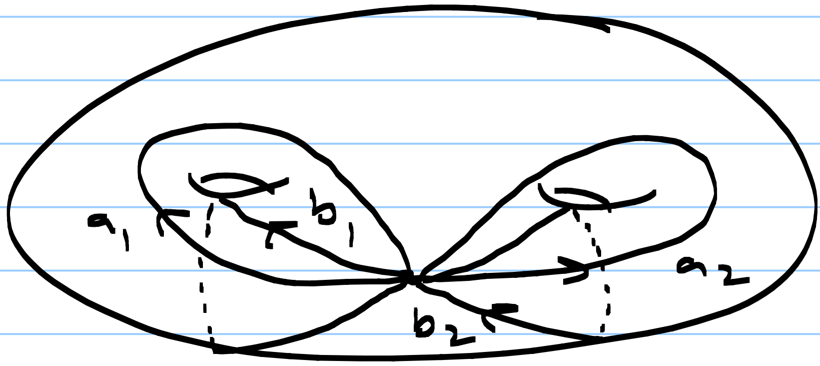
$\gamma \mapsto aba^{-1}b$

7) $T^2 \# T^2$

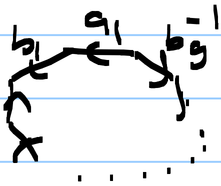


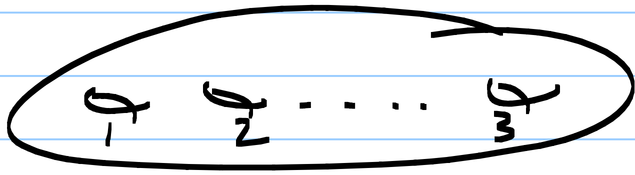
$$\gamma \mapsto a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} = [a_1, b_1][a_2, b_2]$$

$$\underset{\Sigma_2}{\pi_1(T^2 \# T^2)} = \langle a_1, b_1, a_2, b_2 \,|\, [a_1, b_1][a_2, b_2] \rangle$$



8) <u>Exercise:</u> $\underbrace{T^2 \# T^2 \# \cdots \# T^2}_{g\text{-copies}} = \Sigma_g$





$a_1 b_1 a_1^{-1} b_1^{-1} \cdots\cdots a_g b_g a_g^{-1} b_g^{-1}$

9) $\pi_1(\mathbb{RP}^2 \vee \mathbb{RP}^2) = \pi_1(\mathbb{RP}^2) * \pi_1(\mathbb{RP}^2)$
$= \mathbb{Z}_2 * \mathbb{Z}_2$

10) $X_n = e_0 \vee e_1 \vee e_2 \qquad S^1 = \partial D^2 \longrightarrow S^1 = e_0 \vee e^1$
$z \longmapsto z^n$

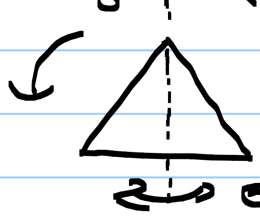
$\pi_1(X_n) = \langle a \mid a^n \rangle \simeq \mathbb{Z}/_{n\mathbb{Z}} \simeq \mathbb{Z}_n$

$\pi_1(X_2 \vee X_3) = \pi_1(X_2) * \pi_1(X_3) \simeq \mathbb{Z}_2 * \mathbb{Z}_3$
$\simeq PSL(2, \mathbb{Z})$

X_n :

11) $S_3 = \langle \sigma, \tau \mid \sigma^2, \tau^3, \underbrace{\sigma\tau\sigma = \tau^{-1}}_{\sigma\tau\sigma\tau} \rangle$

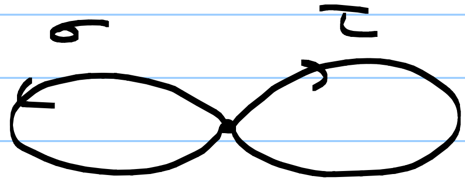
τ



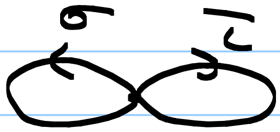
σ : reflection

$\tau : \frac{2\pi}{3}$ radian rotation

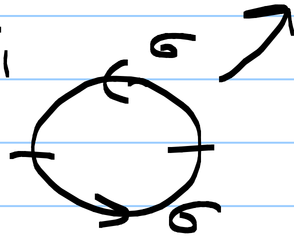
$X = e^0 \cup e^1_1 \cup e^1_2 \cup e^2_1 \cup e^2_2 \cup e^2_3$



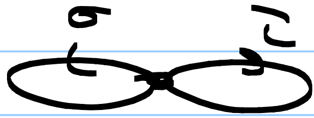
$e^2_1 : S^1 \longrightarrow$

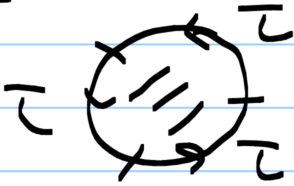


$\begin{matrix} S^1 \\ \parallel \\ \partial e^2_1 \end{matrix}$

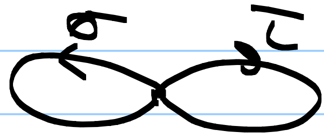


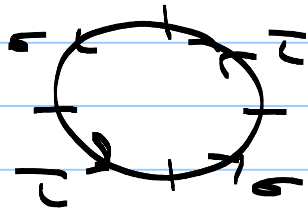
$e^2_2 : S^1 \longrightarrow$





$e^2_3 : S^1 \longrightarrow$





Exercise: Find a space X so that $\pi_1(X) = (\mathbb{Q}, +)$.

$\langle 1 \rangle = \mathbb{Z} \rightsquigarrow g_1 = 1 \in \mathbb{Q}$

$\langle 1/2 \rangle = \mathbb{Z}_{1/2} = \langle \ldots, -1/2, 0, 1/2, 1, 3/2, \ldots \rangle \rightsquigarrow g_2 = 1/2$

$\langle 1/n \rangle = \mathbb{Z}_{1/n} = \langle \ldots, -\frac{1}{n}, 0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots \rangle \rightsquigarrow g_n = 1/n$

$g_1 = g_2^2, \quad g_1 g_2 = g_2 g_1, \ldots$

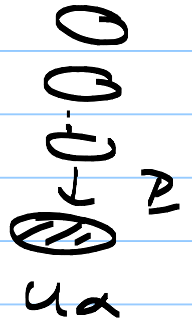
$\mathbb{Q} = \lim_{n \to \infty} \mathbb{Z}_{1/n}$

$$\begin{tikzcd}
 & \mathbb{Z}_{1/2} \arrow[dl] & \\
\mathbb{Z}_{1/6} & & \mathbb{Z} \arrow[ul] \arrow[dl] \\
 & \mathbb{Z}_{1/3} \arrow[ul] &
\end{tikzcd}$$

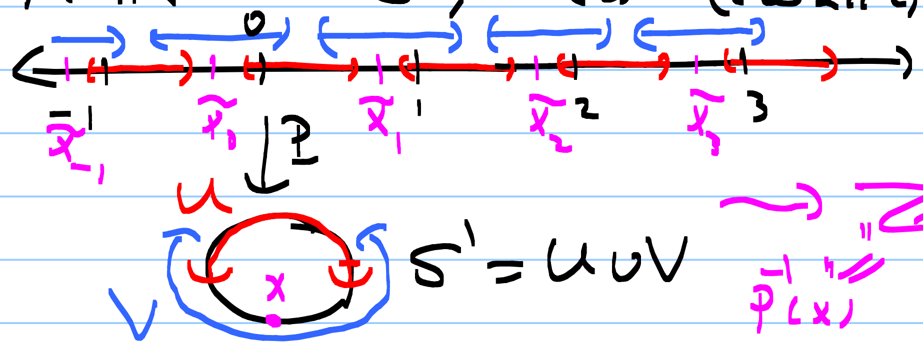
$$\begin{tikzcd}
 & 2g_2 \arrow[dl] & \\
6g_6 & & g_1 \arrow[ul] \arrow[dl] \\
 & 3g_3 \arrow[ul] &
\end{tikzcd}$$

Covering Spaces:

Definition: A covering space of a space X is a map $p: \tilde{X} \to X$, where $\tilde{X}$ is another space so that there is an open cover $\{U_\alpha\}_{\alpha \in A}$ of X, where each $p^{-1}(U_\alpha)$ is a disjoint union of open subsets of $\tilde{X}$ each of which is mapped homeomorphically onto U_α via p.



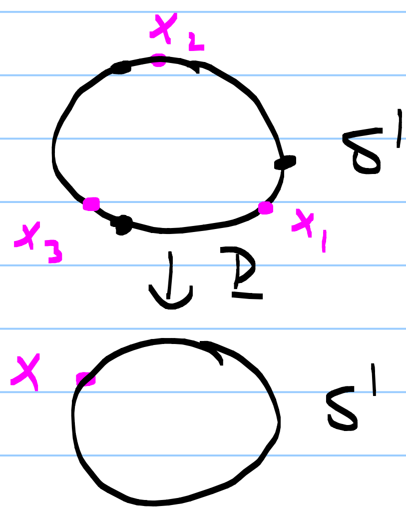
Example 1) $p: \mathbb{R} \to S^1$, $p(t) = (\cos 2\pi t, \sin 2\pi t)$



$S^1 = U \cup V$ $\rightsquigarrow$ $\mathbb{Z}$-cover

$p^{-1}(x) \text{ "=" } \mathbb{Z}$

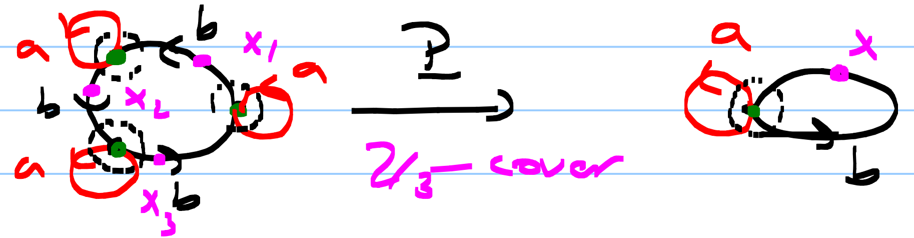
2)



$$z \mapsto z^3$$

$p^{-1}(x) \text{ "=" } \mathbb{Z}_3$ - cover

3) $\tilde{X}$



Video 25

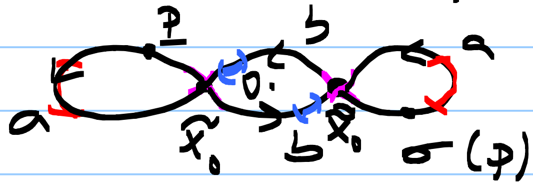
4) $P : S^n \longrightarrow \mathbb{RP}^n = S^n/_{p\sim -p}$, $p \in S^n$

$p = (x_1, \dots, x_{n+1}) \in S^n$

P is 2–1 map. $P(p) = [p] = [x_1 : \cdots : x_{n+1}]$.

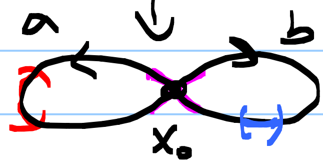
Double cover or equivalently a $\mathbb{Z}_2$-cover.

5) $\widetilde{X}$



σ : π-radian rotation about O.

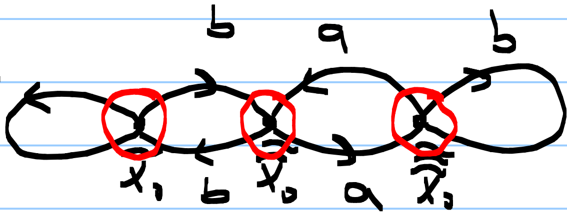
X :



, $X = \widetilde{X}/_{p\sim\sigma(p)}$, $p \in X$

$\mathbb{Z}_2$-cover

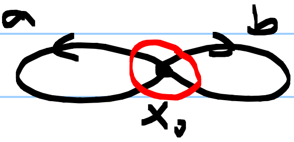
6) $\widetilde{X}$



This is a 3-fold covering.

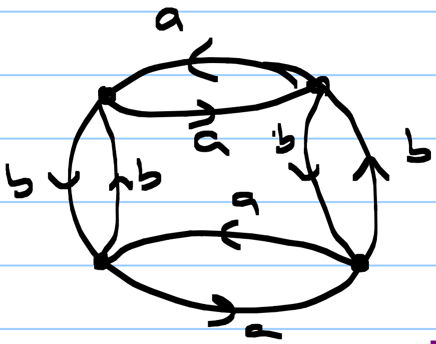
$\downarrow P$

X

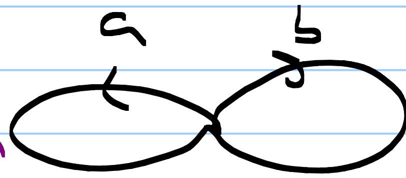


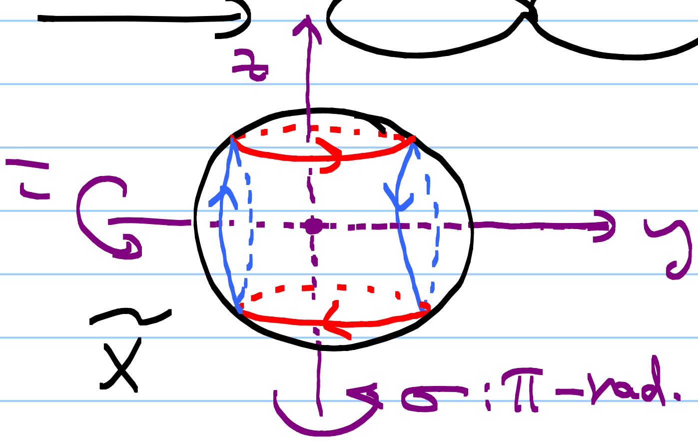
(<u>not</u> a regular cover)

7)



$\longrightarrow$





$\widetilde{X}$

$\mathbb{Z}_2 \times \mathbb{Z}_2$-cover

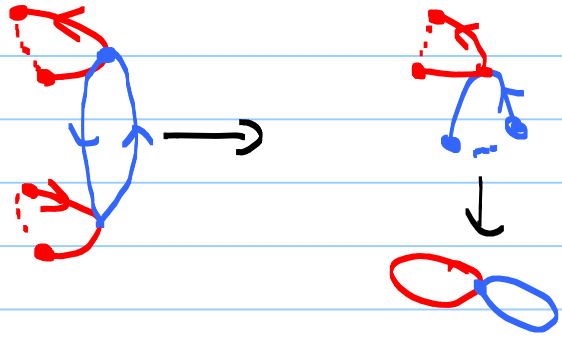
$\langle\sigma\rangle \quad \langle\tau\rangle$

$\sigma \circ \tau = \tau \circ \sigma$

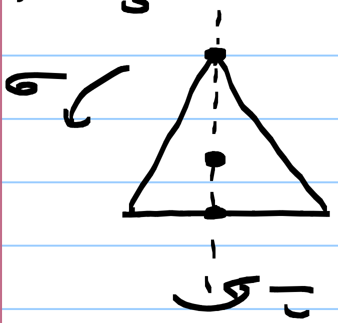
σ : π-rad. rot.

$\sigma(x,y,z) = (-x,-y,z)$ and $\tau(x,y,z) = (-x,y,-z)$ for all $(x,y,z) \in S^2$.

$\tilde{X}/\mathbb{Z}_2 \times \mathbb{Z}_2 = ?$



8) S_3 – covering



$S_3 = \langle \sigma, \tau \mid \sigma^3, \tau^2, \tau\sigma\tau\sigma \rangle$

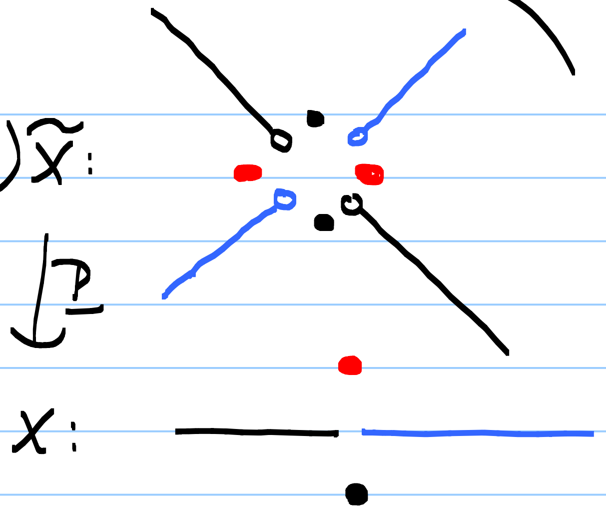
σ : $2\pi/3$ rad. rotation

$F_7 \triangleleft F_2$ (normal subgroup)

of index 6 with $F_2/F_7 \simeq S_3$.

9) $\tilde{X}$:

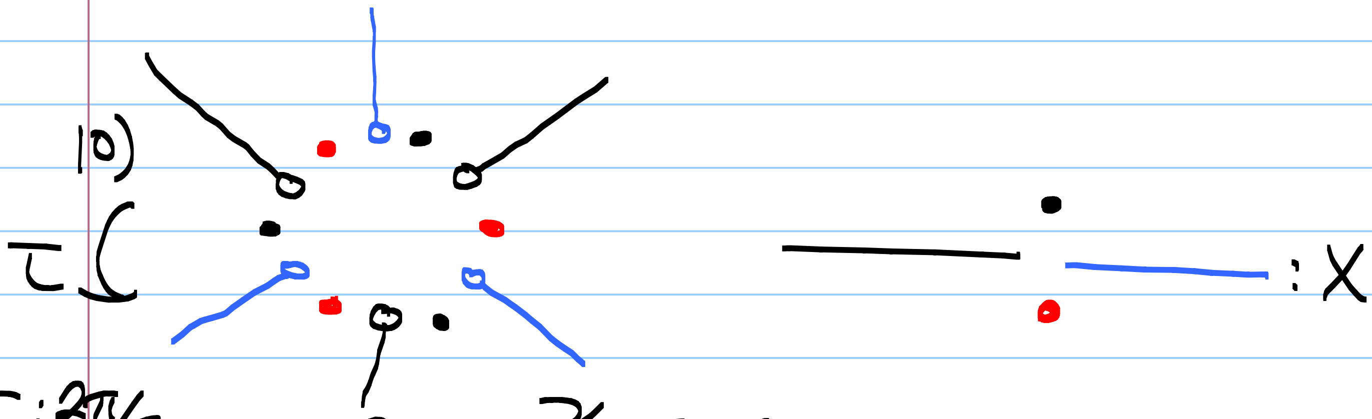
σ: π-radian rotation



$\mathbb{Z}_2$-cover

$\tilde{X}/\sigma \simeq X$

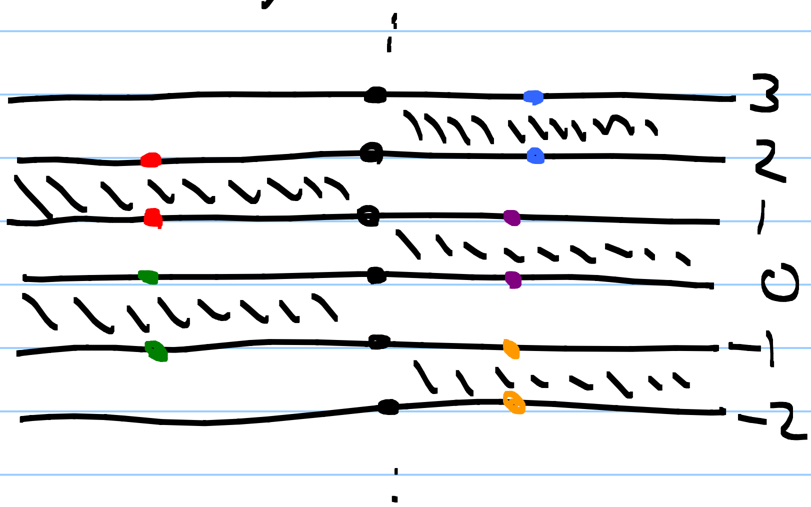
10)



τ: $2\pi/3$ rotation

$\tilde{X} \xrightarrow{\mathbb{Z}_3\text{-cover}}$

11) $\tilde{X} = \mathbb{R} \times \mathbb{Z}/\sim$



$(x,n) \sim (x,n+1)$ if and only if either $(x>0$ and n is even$)$ or $(x<0$ and n is odd$)$.

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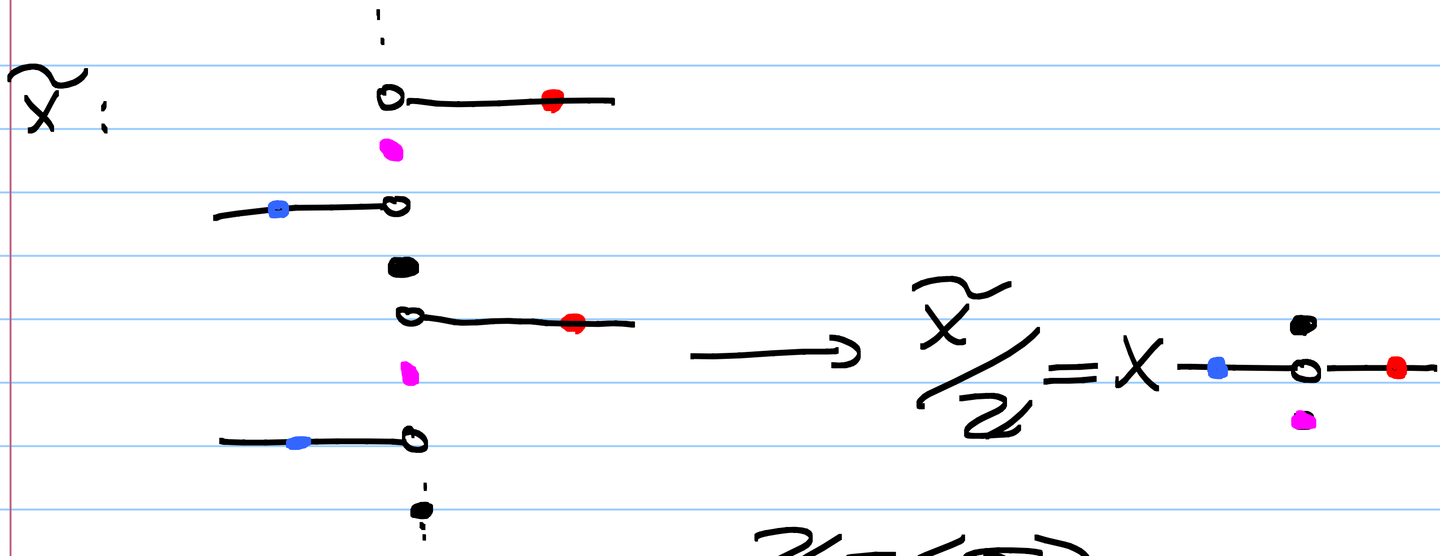


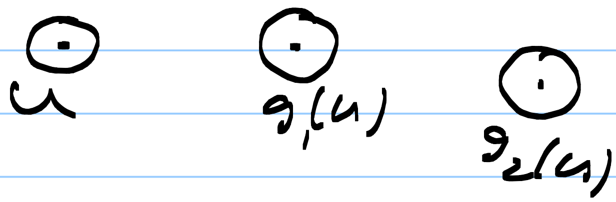
$\mathbb{Z} = \langle \sigma \rangle$

Regular $\mathbb{Z}$-covering.

Definition: A group G is said to act properly discontinuous on a space X if for every $g \in G$ and $x \in X$ there is an open set $x \in U \subseteq X$ so that

$$g_1(U) \cap g_2(U) = \emptyset \text{ whenever } g_1 \neq g_2.$$

$(G \leq \mathrm{Homeo}(X))$



Proposition: If a group $G \leq \mathrm{Homeo}(X)$ acts freely and properly discontinuous on X then the quotient map

$$p: X \longrightarrow X/G, \quad x \sim g(x),\ g \in G,\ x \in X$$

is a covering space.

Such covering spaces are called regular.

Lifting Properties:

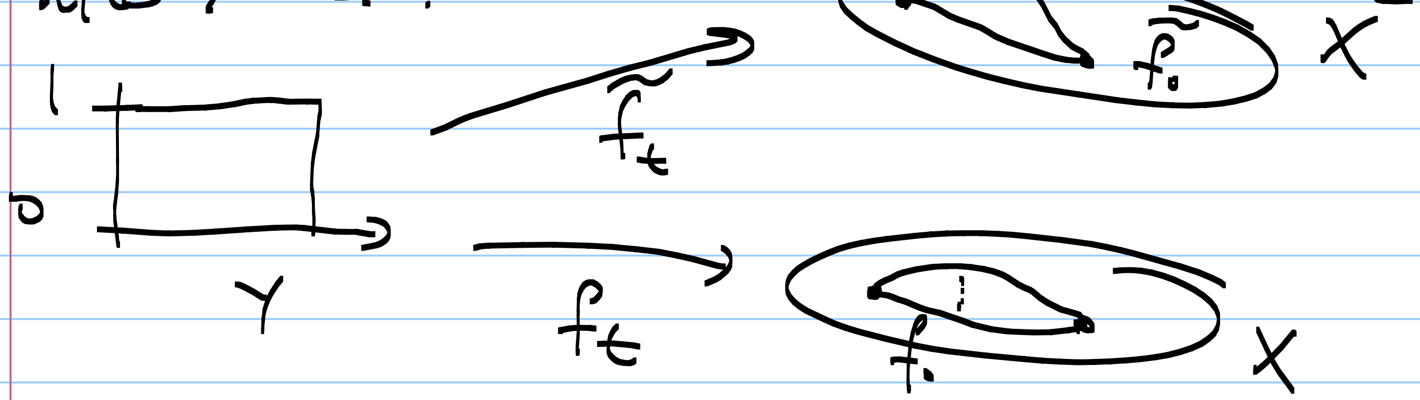
Let $p: \tilde{X} \to X$ be a covering space and $f: Y \to X$ is any map. A lifting of $f: Y \to X$ is a map $\tilde{f}: Y \to \tilde{X}$ so that $f = p \circ \tilde{f}$:

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd} \qquad f = p \circ \tilde{f}$$

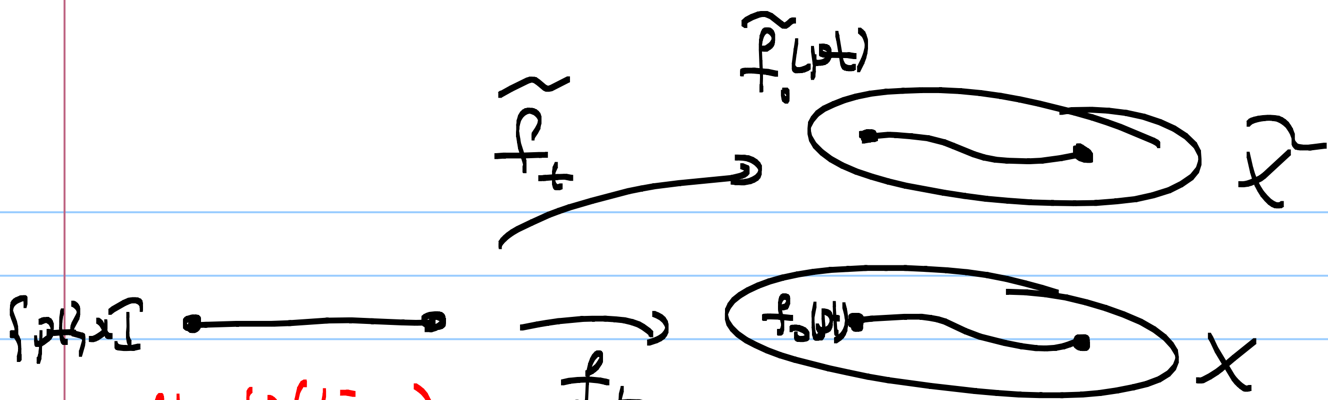
Proposition: (Homotopy lifting)

Given a covering space $p: \tilde{X} \to X$, a homotopy $f_t: Y \to X$ and a map $\tilde{f}_0: Y \to \tilde{X}$ lifting $f_0: Y \to X$, then there is a unique homotopy $\tilde{f}_t: Y \to \tilde{X}$ of $\tilde{f}_0$ that lifts f_t.

Proof This is Part (c) of the proof that $\pi_1(S^1) \simeq \mathbb{Z}$.



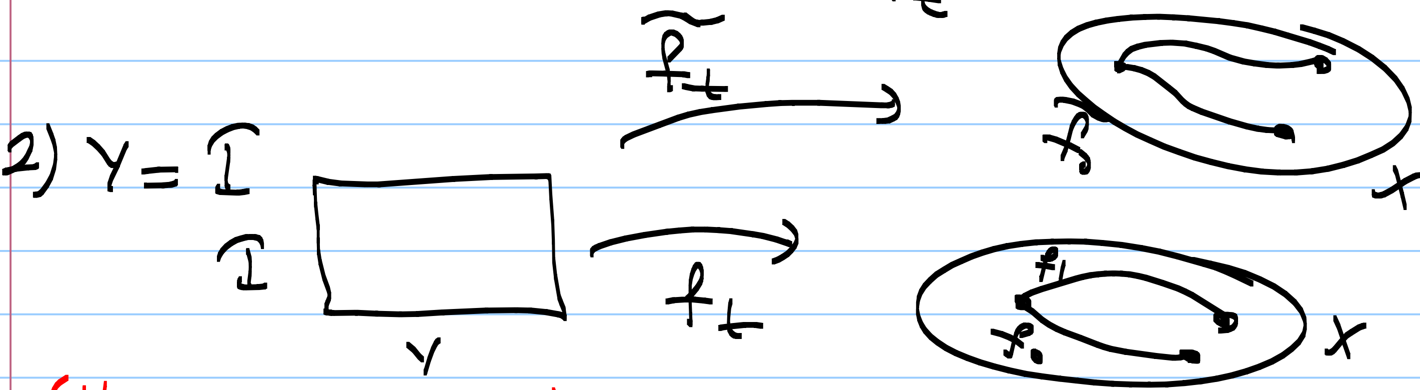
Special Cases: 1) $Y = \{pt\}$, $Y \times I = I$



(path lifting)

f_t

2) $Y = I$



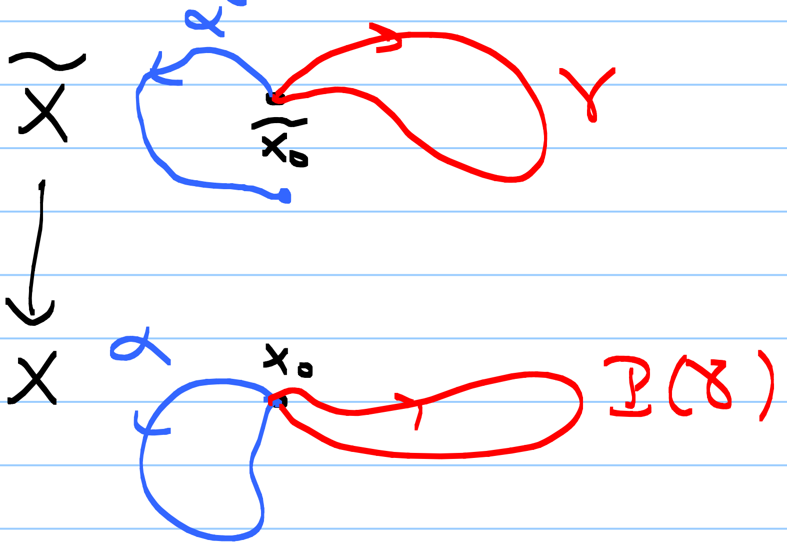
(Homotopy lifting)

Proposition: Let $p : (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ be a covering space. Then the homomorphism

$$p_{\#} : \pi_1(\tilde{X}, \tilde{x}_0) \longrightarrow \pi_1(X, x_0) \quad \text{induced by } p,$$

is injective. The image subgroup $p_{\#}(\pi_1(\tilde{X}, \tilde{x}_0))$ consists of loops in X based at x_0, whose lifts to $\tilde{X}$ starting at $\tilde{x}_0$ are loops.

Proof:



Proof: Let $\tilde{f}_0 : I \longrightarrow \tilde{X}$ be a loop based at $\tilde{x}_0$, which represents a class in the kernel of the homomorphism $p_\#: \pi_1(\tilde{X}, \tilde{x}_0) \longrightarrow \pi_1(X, x_0)$.

To show that $p_\#$ is injective we must show $\tilde{f}_0$ is homotopic to the constant path at $\tilde{x}_0$.

Then $\tilde{f}_0$ is the unique lifting of $f_0 \doteq p \circ \tilde{f}_0 : I \to X$, a loop in X based at x_0. By assumption f_0 is homotopic to a constant. Hence, there is a homotopy f_t joining f_0 to the constant loop at x_0.

By the previous proposition there is a unique homotopy $\tilde{f}_t$ of $\tilde{f}_0$ to some $\tilde{f}_1$ so that

$$p \circ \tilde{f}_t = f_t \quad \text{for all } t.$$

In particular, $p \circ \tilde{f}_1(s) = f_1(s) = x_0$, for all $s \in [0,1]$. Hence, $\tilde{f}_1(s) \in p^{-1}(x_0)$, for all $s \in [0,1]$. Since $p^{-1}(x_0)$ is a discrete set and $\tilde{f}_1(0) = \tilde{x}_0$ we see that $\tilde{f}_1(s) = \tilde{x}_0$, for all $s \in [0,1]$.

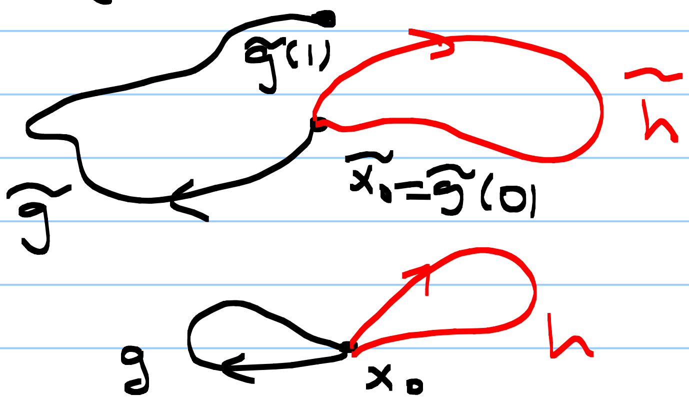
In particular, $\tilde{f}_1$ is a constant loop homotopic to $\tilde{f}_0$. This shows that $p_\#$ is an injective group homomorphism.

The proof of the second statement is left as an exercise.

If $p: \tilde{X} \to X$ is a covering then for any $x \in X$ the cardinality of $p^{-1}(x)$ is called the "number of sheets" of the covering above x.

Proposition: The number of sheets $|p^{-1}(x_0)|$ of a covering $p:(\tilde{X},\tilde{x}_0)\to(X,x_0)$ of path connected spaces is equal to the index of the subgroup $p_\#(\pi_1(\tilde{X},\tilde{x}_0))$ in $\pi_1(X,x_0)$.

Proof: Let g be a loop at x_0 and $\tilde{g}$ be the lift to $\tilde{X}$ starting at $\tilde{x}_0$. If $[h]\in H=p_\#(\pi_1(\tilde{X},\tilde{x}_0))$ and $\tilde{h}$ is lift of h then the lift $\widetilde{hg}$ of hg starting at $\tilde{x}_0$ has the same end point with $\tilde{g}$.

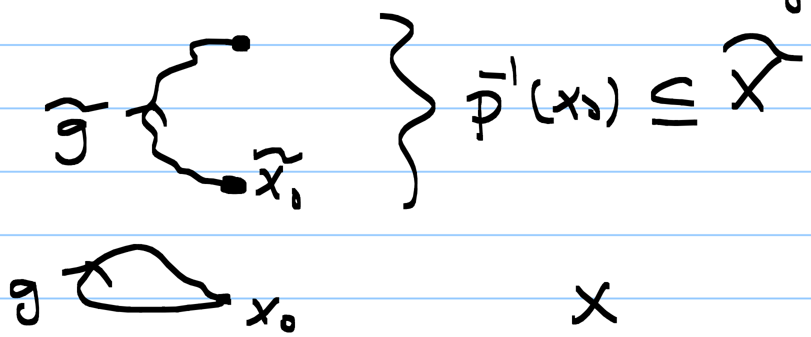


So we get a well defined function

$$\Phi:\{H[g]\mid [g]\in\pi_1(X,x_0)\}\longrightarrow p^{-1}\{x_0\}$$ by

sending the coset $H[g]$ to the end point $\tilde{g}(1)$.

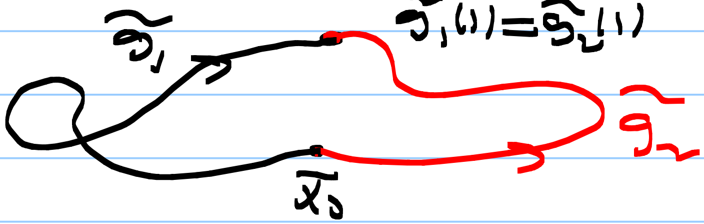
Since $\tilde{X}$ is connected Φ is surjective.

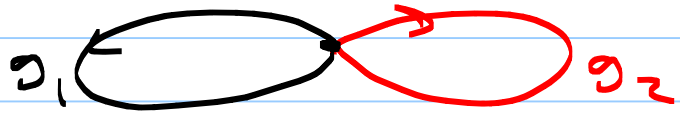


To show that Φ is injective assume that

$\Phi(H[g_1]) = \Phi(H[g_2])$, for some $[g_i] \in \pi_1(X,x_0)$, $i=1,2$.

Then if $\tilde{g}_1$ and $\tilde{g}_2$ are the lifts of g_1 and g_2 starting at $\tilde{x}_0$ then $\tilde{g}_1(1) = \tilde{g}_2(1)$.





Hence $\tilde{g}_1 \bar{\tilde{g}}_2$ is a loop at $\tilde{x}_0$. Hence, $[g_1 \bar{g}_2]$ is contained in the subgroup $H = p_\#(\pi_1(\tilde{X},\tilde{x}_0))$.

$[g_1\bar{g}_2] = [g_1][g_2]^{-1} \in H \Rightarrow H[g_1] = H[g_2]$.

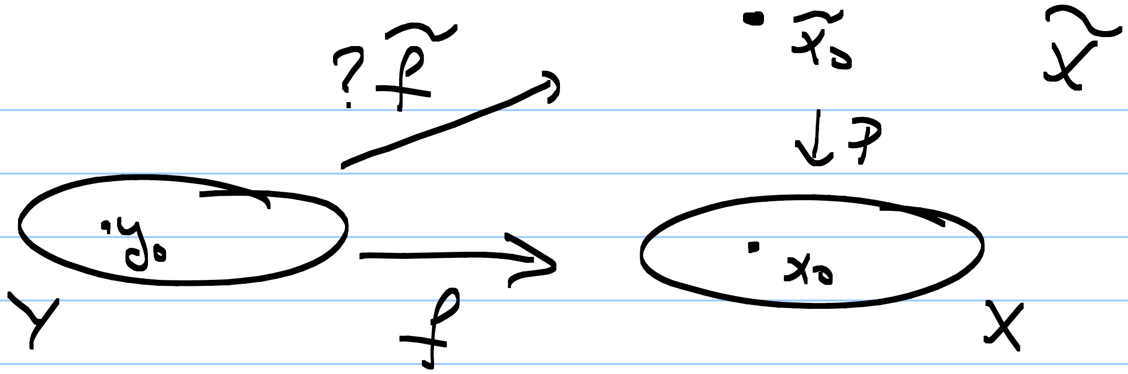
This finishes the proof. $\blacksquare$

Proposition: (Lifting Criterion)

Let $p:(\tilde{X},\tilde{x}_0) \to (X,x_0)$ be a covering space, $f:(Y,y_0) \to (X,x_0)$ a map, where $f(y_0)=x_0$ and Y is path connected and locally path connected. Then f has a lift

$\tilde{f}:(Y,y_0) \to (\tilde{X},\tilde{x}_0)$ if and only if
$f_\#(\pi_1(Y,y_0)) \subseteq p_\#(\pi_1(\tilde{X},\tilde{x}_0))$.

Proof:

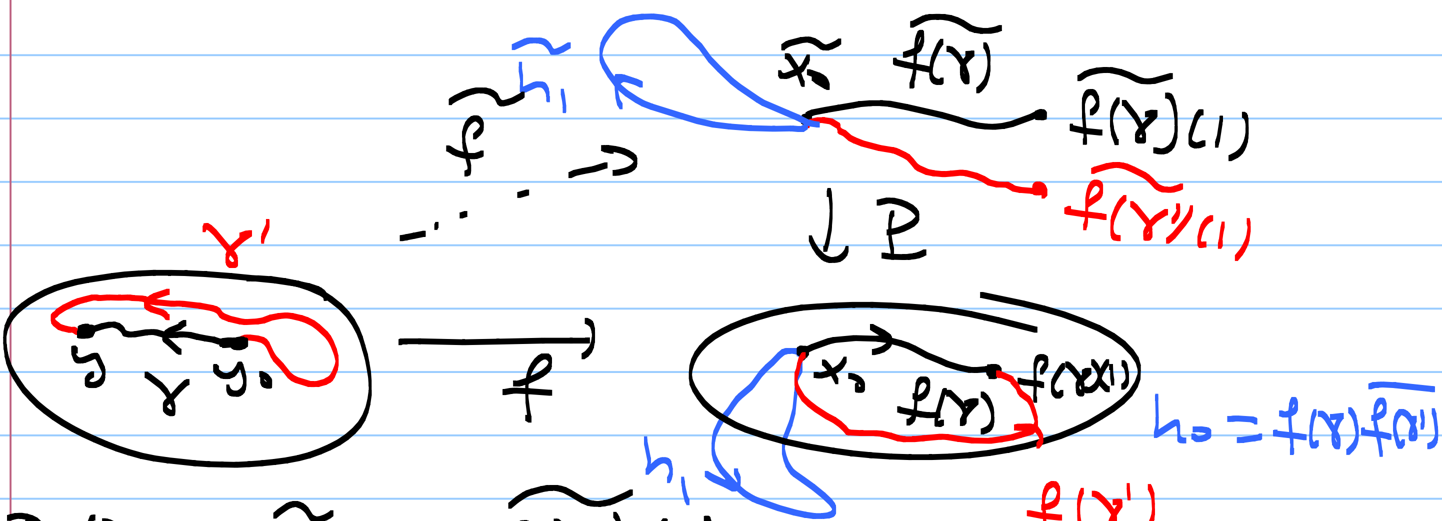


$p\circ\tilde f = f$

($\Longrightarrow$) This direction is clear since if there is a lift $\tilde f$ then $p_\#\circ\tilde f_\# = f_\#$ and thus

$$f_\#(\pi_1(Y,y_0)) = p_\#(\tilde f_\#(\pi_1(Y,y_0)))$$
$$\subseteq p_\#(\pi_1(\tilde X,\tilde x_0))$$

($\Longleftarrow$) Now assume that $f_\#(\pi_1(Y,y_0)) \subseteq p_\#(\pi_1(\tilde X,\tilde x_0))$.



Define $\tilde f(y) = \widetilde{f(\gamma)}(1)$.

Well-definedness of $\tilde f$: If γ' is another path from y_0 to y then we must show that $\widetilde{f(\gamma')}(1) = \widetilde{f(\gamma)}(1)$.

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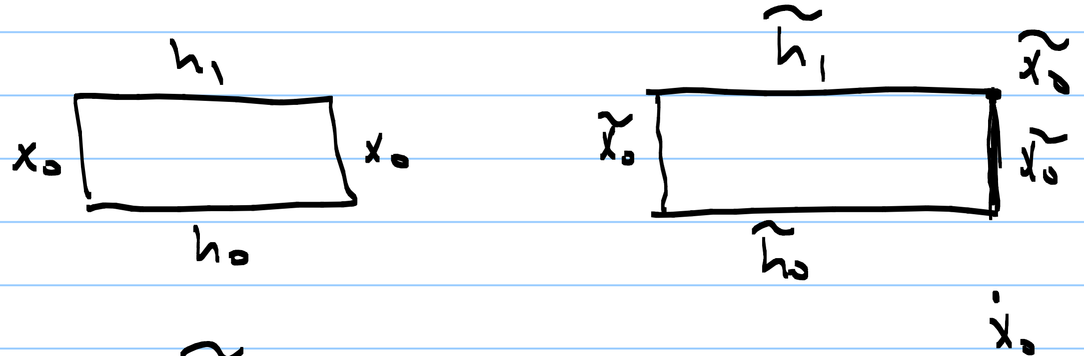
Now $f(\gamma') \cdot \overline{f(\gamma)}$ is a loop at x_0, representing a class

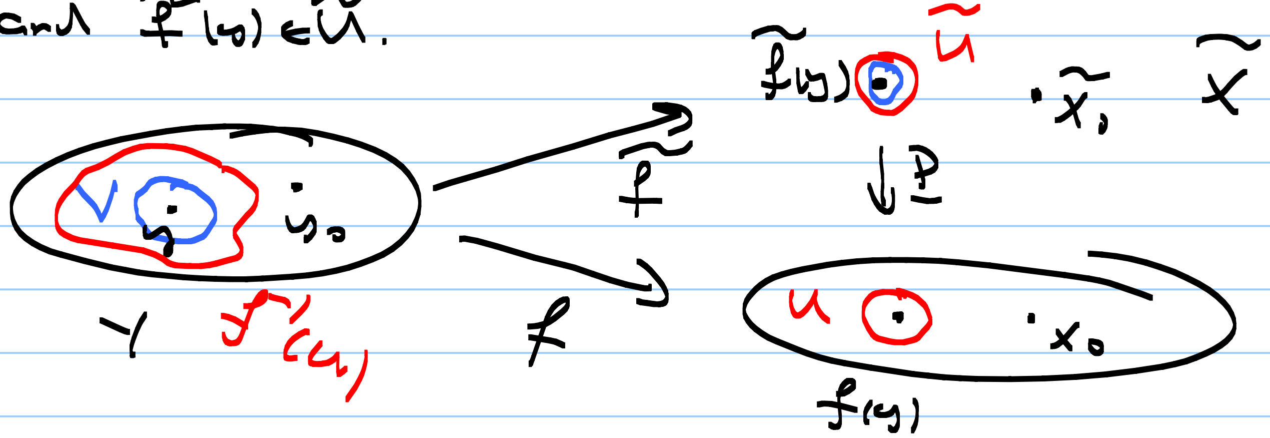
$$[h_0] \in f_\#(\pi_1(Y, y_0)) \subseteq p_\#(\pi_1(\tilde{X}, \tilde{x}_0)).$$

Hence there is a homotopy h_t of h_0 to a loop h_1 that has a lift $\tilde{h}_1$ in $\tilde{X}$ which is a loop based at $\tilde{x}_0$. By the homotopy lifting property h_t has a lift $\tilde{h}_t$. Since h_1 is a loop at $\tilde{x}_0$ so is $\tilde{h}_0$.



Hence, $\tilde{f}$ is well defined.

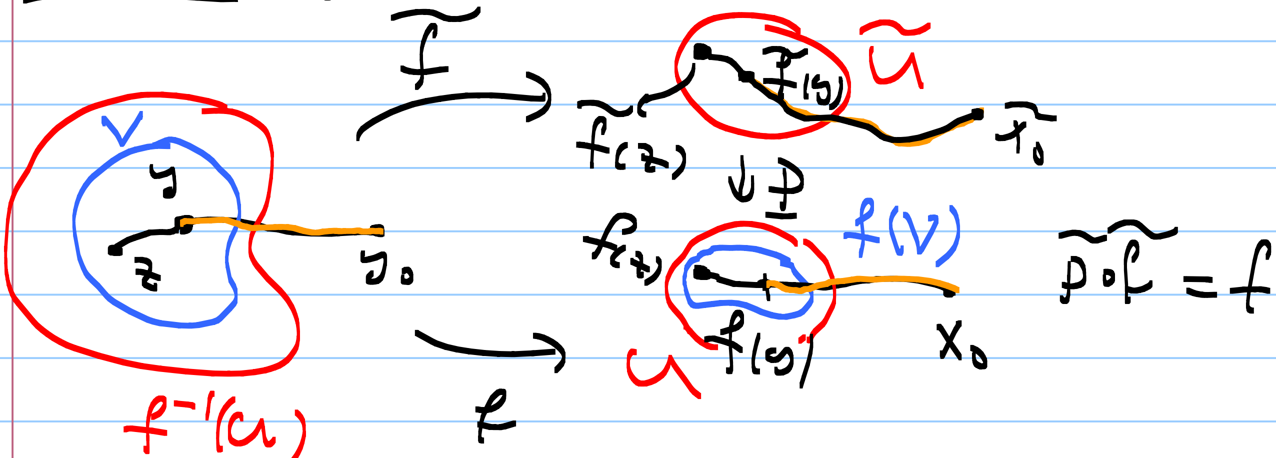
<u>$\tilde{f}$ is continuous:</u> Let $y \in Y$ and $U \subseteq X$ an open subset containing $f(y)$ such that there is some $\tilde{U} \subseteq \tilde{X}$ with $p: \tilde{U} \to U$ a homeomorphism and $\tilde{f}(y) \in \tilde{U}$.



<u>must find:</u> An open subset V in Y with $y \in V$ and $\tilde{f}(V) \subseteq \tilde{U}$.

Now since Y is locally path connected choose an open subset V so that $y \in V$, V is path connected and $f(V) \subseteq U$.

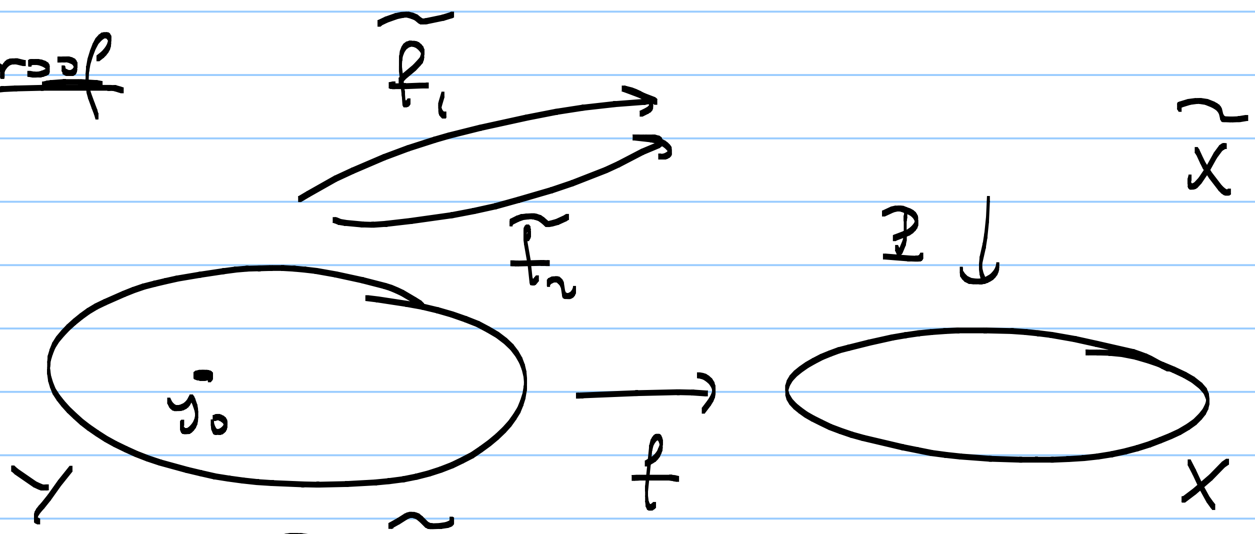
<u>Claim:</u> $\tilde{f}(V) \subseteq \tilde{U}$.



So, $\tilde{f}(V) \subseteq \tilde{U}$. Hence, $\tilde{f}$ is continuous. ∎

<u>Proposition:</u> Given a covering space $p: \tilde{X} \to X$ and a map $f: Y \to X$ with two lifts $\tilde{f}_1$ and $\tilde{f}_2$ from Y to $\tilde{X}$ that agree at one point y_0 of Y, then if Y is connected these two lifts agree on all of Y.

<u>Proof</u>



$p \circ \tilde{f}_i = f$, $i = 1, 2$.

$f_1(y_0) = f_2(y_0)$ is given.

<u>must show:</u> $f_1 = f_2$ or $f_1(y) = f_2(y)$ for all $y \in Y$.

Let A be the set of points in Y on which the two lifts agree:

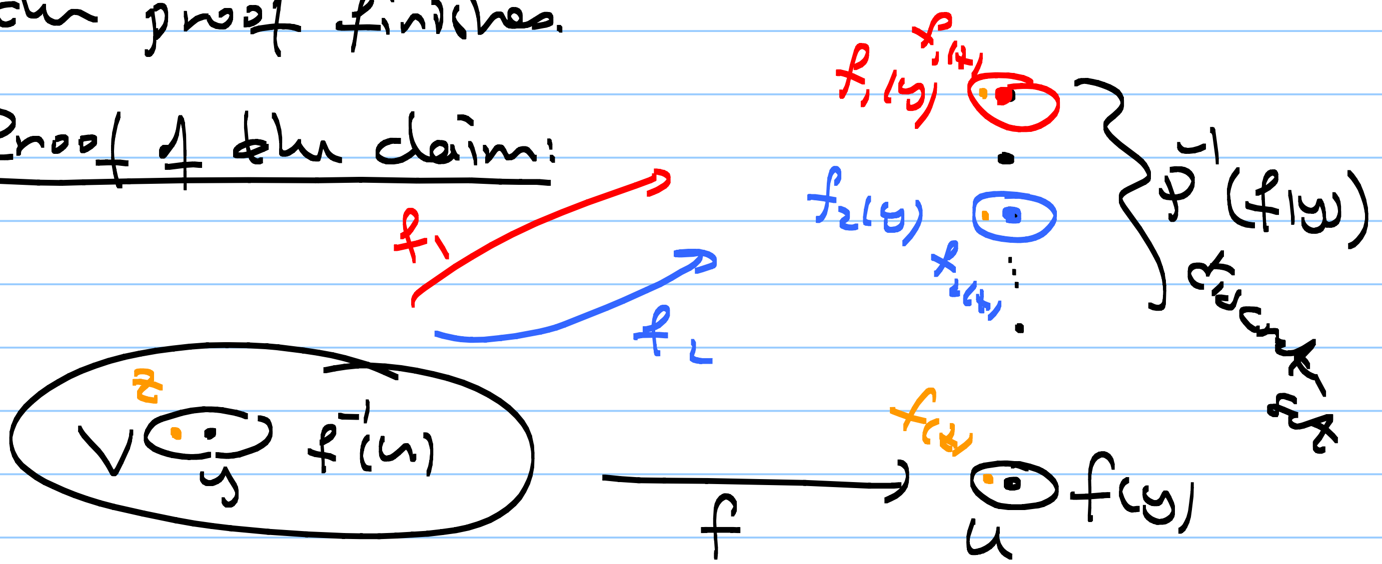
$$A = \{ y \in Y \mid f_1(y) = f_2(y) \}.$$

Clearly, $y_0 \in A$ and thus $A \neq \emptyset$.

<u>Claim:</u> A is both open and closed.

Note that the claim implies that $A = Y$ so that the proof finishes.

<u>Proof of the claim:</u>



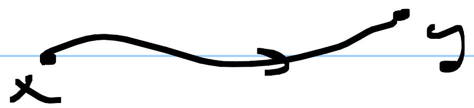
If $f_1(y) \neq f_2(y)$ then there is an open set $V = f^{-1}(U)$ so that $f_1(V) \cap f_2(V) = \emptyset$. In particular $f_1(z) \neq f_2(z)$ for all $z \in V$.

Hence, the set $Y \setminus A$ is open. Thus A is closed.
A similar argument shows that A is also open. ■

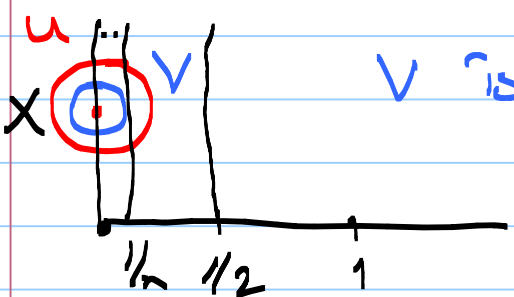
Classification of Covering Spaces:

Let X be topological space, which is path connected, locally path connected and semilocally simply connected.

i) path connected:



ii) locally path connected: $x \in X$, $x \in U$ open subset. Then there is another open subset V s.t. $x \in V \subseteq U$ and V is path connected.



V is never path connected.

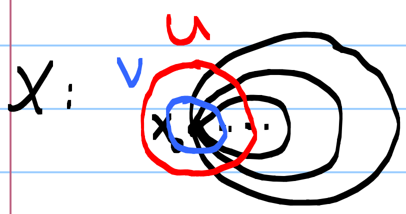
X is path connected but not locally path connected.

iii) Semi locally simply connected: For every $x \in X$ and open subset $x \in U$ there is some open subset $x \in V \subseteq U$ so that the homomorphism

$$\pi_1(V, x) \longrightarrow \pi_1(X, x)$$

is trivial.

$$X = \bigvee_{n=1}^{\infty} S^1_{1/n}$$



$$\pi_1(V, x_0) \longrightarrow \pi_1(X, x_0) \text{ is injective}$$

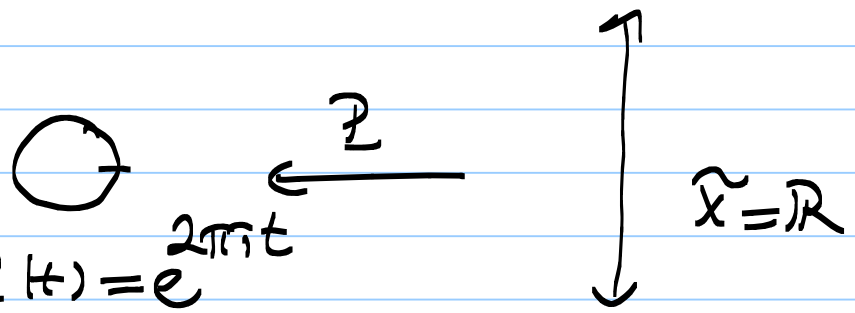
$\pi_1(V, x_0) \supseteq F_\infty$

Theorem: Let X be a path connected, locally path connected and semilocally simply connected topological space. Then X has a universal covering

$$p : \tilde{X} \longrightarrow X \text{, i.e., a simply connected}$$

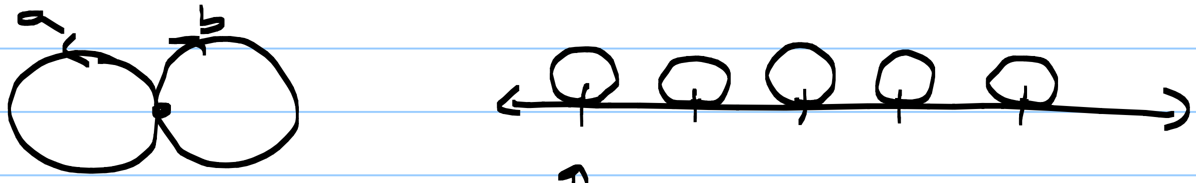
covering space.

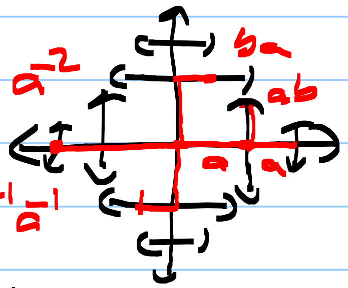
Remark: 1) $X = S^1$



$$p(t) = e^{2\pi i t}$$
$$= (\cos 2\pi t, \sin 2\pi t)$$

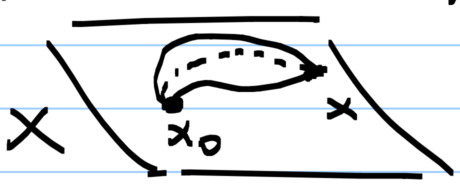
2) $X = S^1 \vee S^1$





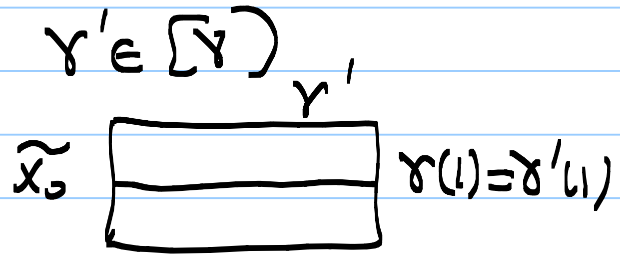
$\pi_1(X) = F_2 = \langle a, b \mid - \rangle \quad a, ab, b, abab, \ldots$

Observation: If X is a simply connected space then, there is a one to one correspondence between points of X as homotopy classes of paths starting at fixed point, where homotopies fixed the end points.

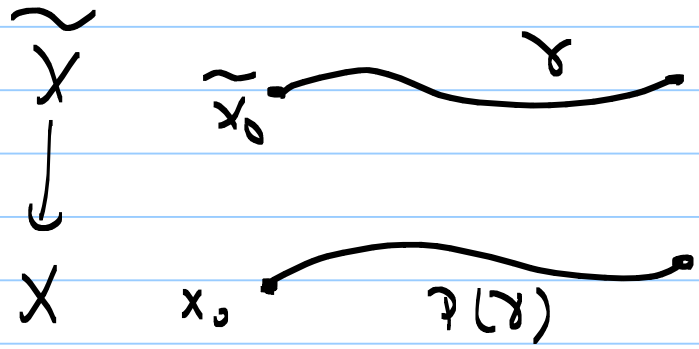


Idea: We may regard a simply connected space as the homotopy classes of paths in X starting at fixed point.

$$\tilde{X} = \{[\gamma] \mid \gamma(0) = \tilde{x}_0\}$$

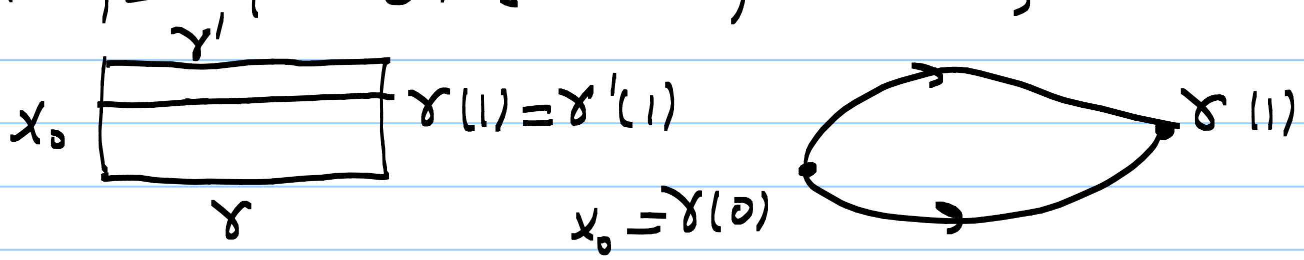


Any path in $\tilde{X}$ gives a path γ in X.



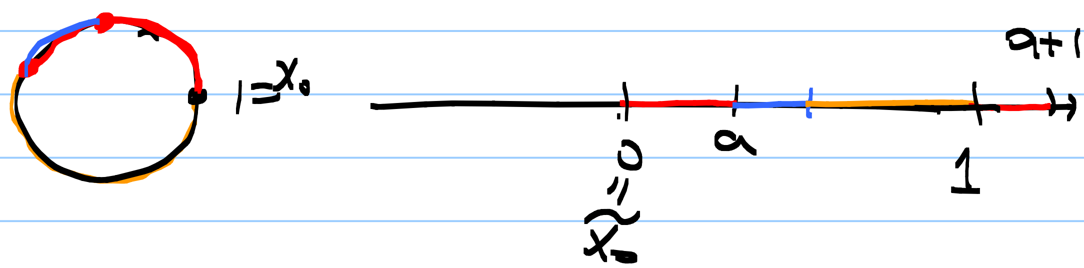
So let $\tilde{X}$ be the homotopy classes of path in X starting at a fixed point x_0, where homotopies fixes the end points of the paths at each level.

$$\tilde{X} = \{[\gamma] \mid \gamma : [0,1] \to X,\ \gamma(0) = x_0\}$$



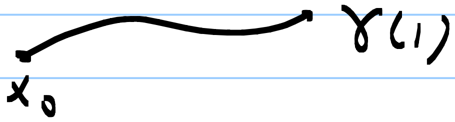
Example:

$(X, x_0) = (S^1, 1)$



How to define $P: \tilde{X} \longrightarrow X$.

$[\gamma] \in \tilde{X}$, $P([\gamma]) = \gamma(1)$



Clearly P is surjective, because X is path connected and thus for any $x \in X$ there is a path $\gamma: [0,1] \to X$ with $\gamma(0) = x_0$ and $\gamma(1) = x$.

Now we must put a topology on $\tilde{X}$:

Let $\mathcal{U}$ denote the collection of path connected open subsets $U \subseteq X$ s.t.

$$\pi_1(U) \longrightarrow \pi_1(X) \text{ is trivial.}$$

Note that if $V \subseteq U$ another open subset then

$$\pi_1(V) \longrightarrow \pi_1(U) \longrightarrow \pi_1(X) \text{ is also trivial.}$$

Hence, $\mathcal{U}$ is a basis for the topology on X.

i) $x \in X$, $x \in W$ open. Then by assumption (s.l.s.c.) there is some $U \subseteq \mathcal{U}$ s.t.
$$x \in U \subseteq W.$$

ii) $x \in X$, $x \in U_1 \cap U_2$, $U_i \in \mathcal{U}$. Then $U_1 \cap U_2 \in \mathcal{U}$.

Next we'll describe a basis for a topology on $\tilde{X}$:
Given $U \in \mathcal{U}$ and a path γ in X from x_0 to a point in U, let

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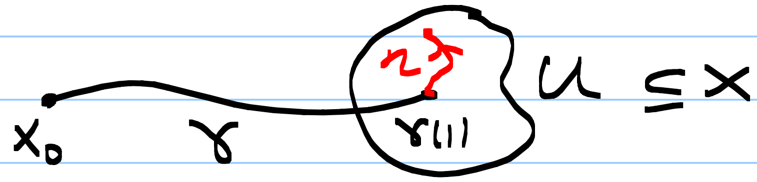
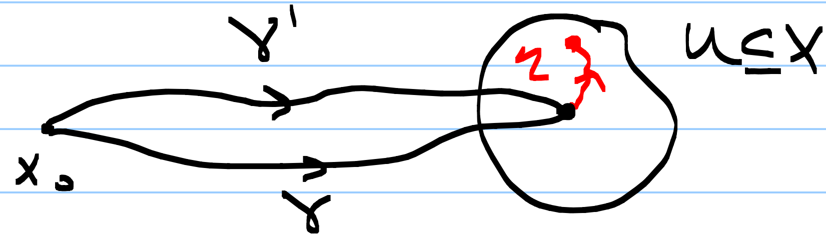
$U_{[\gamma]} = \{ [\gamma\cdot\eta] \mid \eta \text{ is a path in } U \text{ with } \eta(0)=\gamma(1) \}$



<u>Observations</u> 1) $U_{[\gamma]}$ depends only on the homotopy class of γ. In other words, if $\gamma' \in [\gamma]$ then

$U_{[\gamma']} = U_{[\gamma]}$.



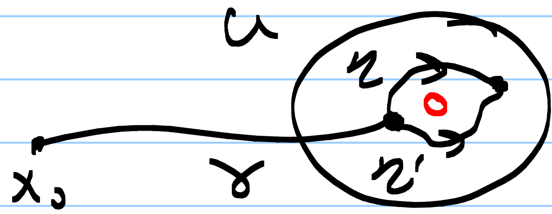
$[\gamma\cdot\eta] = [\gamma'\cdot\eta]$

2) Since U is path connected the map

$p: U_{[\gamma]} \longrightarrow U$, $[\gamma\cdot\eta] \longmapsto \eta(1)$, is onto.

3) Moreover, p is injective.

Assume $[\gamma\cdot\eta]$ and $[\gamma\cdot\eta']$ are in $U_{[\gamma]}$ so that $p([\gamma\cdot\eta]) = \eta(1) = \eta'(1) = p([\gamma\cdot\eta'])$.



Since $\pi_1(U) \longrightarrow \pi_1(X)$ is trivial the paths $[\eta]$ and $[\eta']$ are homotopic in X and thus $\gamma\cdot\eta$ and $\gamma\cdot\eta'$ are homotopic via a homotopy fixing the end points.

Conclusion: $P: U_{[\gamma]} \to U$ is a bijection.

Any point $[\gamma] \in \tilde{X}$ satisfies $[\gamma] \in U_{[\gamma]}$.

Now using these we may put a topology on $\tilde{X}$.

Claim: The collection

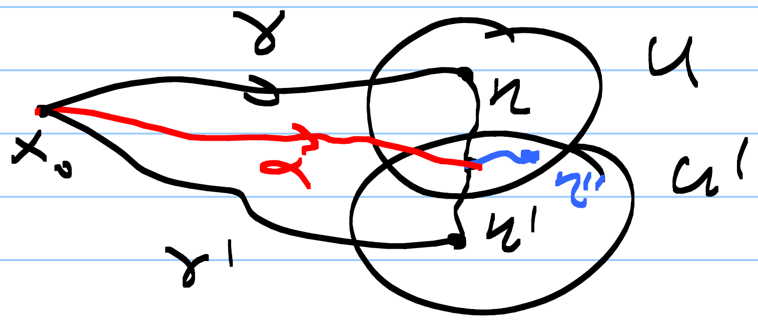
$\tilde{\mathcal{U}} = \{ U_{[\gamma]} \mid U \in \mathcal{U}, [\gamma] \in \tilde{X} \}$ is a basis

for a topology on $\tilde{X}$.

In this case each bijection $P: U_{[\gamma]} \to U$ becomes a homeomorphism.

We already showed that any $[\gamma] \in \tilde{X}$ lies in $U_{[\gamma]}$. Now let

$$[\alpha] \in U_{[\gamma]} \cap U'_{[\gamma']}.$$



$[\alpha] = [\gamma \cdot \eta] = [\gamma' \cdot \eta']$

Then $[\alpha] \in (U \cap U')_{[\alpha]} \underset{?}{\subseteq} U_{[\gamma]} \cap U_{[\gamma']}$ because

$\underline{[\alpha \cdot \eta'']} = \underline{[\gamma \cdot \eta \cdot \eta'']} = \underline{[\gamma' \cdot \eta' \cdot \eta'']}$

In particular, each restriction $P: U_{[\gamma]} \to U$

is a homeomorphism.

We must show: 1) P is continuous

2) P is covering map

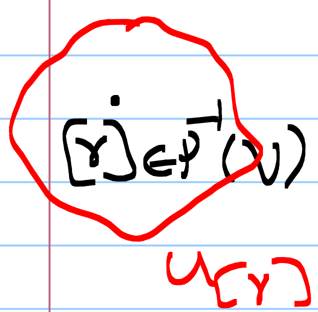
3) $\tilde{X}$ is simply connected.

P is continuous: Take any $x \in X$ and $x \in V \subseteq X$ an open subset. Then by the s.l.s.c. assumption there is an open subset U s.t. $x \in V \subseteq U \subseteq X$ with $U \in \tilde{U}$. Now $\exists$ $[\gamma] \in \tilde{X}$ s.t.

$P([\gamma]) = x$, i.e., $\gamma : [0,1] \to X$, $\gamma(0) = x_0$, $\gamma(1) = x$

$P : U_{[\gamma]} \to U$ is a bijection and thus

$P(U_{[\gamma]}) = U \subseteq V$. Hence, P is continuous.



2) P is a covering map: Take any $x \in X$. Choose some $U \subseteq \mathcal{U}$ with $x \in U$. Then for any $[\gamma] \in \tilde{X}$ with $P([\gamma]) = \gamma(1) = x$, the open

set $U_{[\gamma]}$ maps homeomorphically onto U.

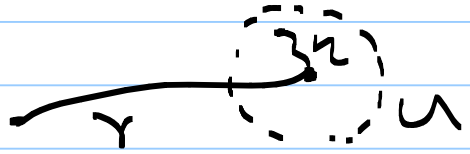
We must show:

a) $P^{-1}(U) = \bigcup_{\gamma(1)\in U} U_{[\gamma]}$

b) If $U_{[\gamma]} \cap U_{[\gamma']} \neq \emptyset$ then $U_{[\gamma]} = U_{[\gamma']}$.

Proof of a) Let $[\gamma] \in P^{-1}(U)$. Then $P([\gamma]) = \gamma(1) \in U$.

In particular, $[\gamma] \in U_{[\gamma]}$ and $P(U_{[\gamma]}) = U$.

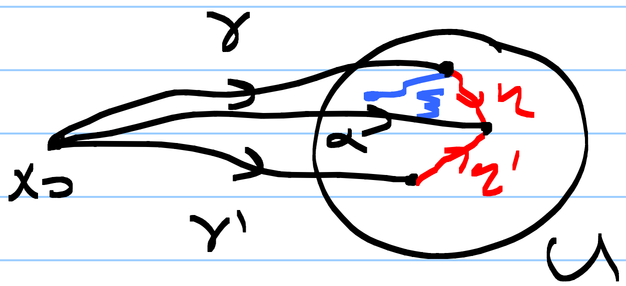


Hence, $P^{-1}(U) = \bigcup_{\gamma(1)\in U} U_{[\gamma]}$.

Proof of b) Assume that $U_{[\gamma]} \cap U_{[\gamma']} \neq \emptyset$ and

$\alpha \in U_{[\gamma]} \cap U_{[\gamma']}$. Then there are paths η and η' in U so that

$\gamma(1) = \eta(0)$, $\gamma'(1) = \eta'(0)$ and $[\gamma\cdot\eta] = [\alpha] = [\gamma'\cdot\eta']$.



So $[\gamma\cdot\eta\cdot\bar{\eta}'\cdot\bar{\gamma}'] = e$ in $\pi_1(X, x_0)$.

Hence, $[\gamma] = [\gamma'\cdot\eta'\cdot\bar{\eta}]$. Thus, $[\gamma\cdot\xi] = [\gamma'\cdot\eta'\cdot\bar{\eta}\cdot\xi]$ for any path ξ in U with $\gamma(1) = \xi(0)$.

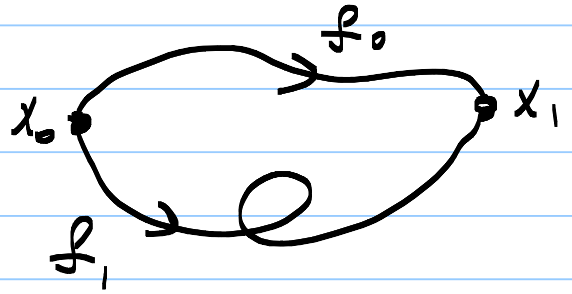
However, $[\gamma\cdot\xi]$ is a typical element of $U_{[\gamma]}$ and $[\gamma'\cdot(\eta'\cdot\bar{\eta}\cdot\xi)]$ is in $U_{[\gamma']}$. Thus, $U_{[\gamma]} \subseteq U_{[\gamma']}$.

Similarly, $U_{[\gamma']} \subseteq U_{[\gamma]}$ and thus $U_{[\gamma]} = U_{[\gamma']}$.

In the above proof we used the following fact:

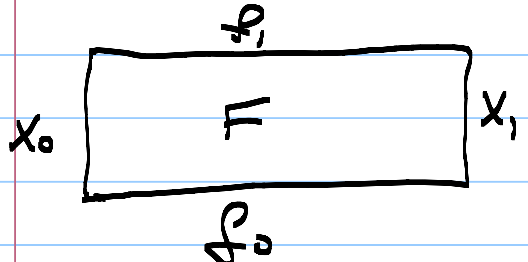
<u>Fact:</u> For any two path f_0 and f_1 in any space from x_0 to x_1, we have

$f_0 \sim f_1$ (rel $\{0,1\}$) if and only if $[f_0 \cdot \bar{f_1}] = e$ in $\pi_1(X, x_0)$.



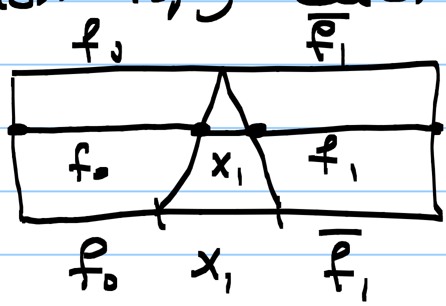
<u>Proof of the fact:</u>

($\Rightarrow$) Assume that $f_0 \sim f_1$, rel $\{0,1\}$.

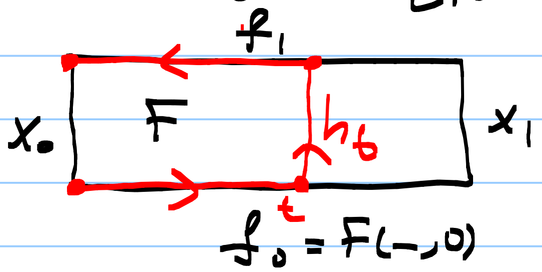


<u>must show:</u> $[f_0 \cdot \bar{f_1}] = e$ in $\pi_1(X, x_0)$.

First note that $[f_0 \cdot x_1^* \cdot \bar{f_1}] = [f_0 \cdot \bar{f_1}]$ in $\pi_1(X, x_0)$, where x_1^* is the constant path at x_1. To see this just consider the homotopy described by the diagram below:



Now we show $[f_0 \cdot x_1^* \cdot \overline{f_1}] = e$ in $\pi_1(X, x_0)$.



$h_t : I \longrightarrow X$, $t \in [0,1]$

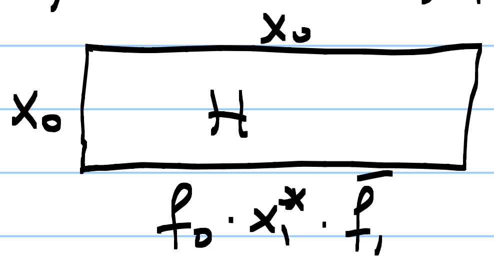
$h_1 = f_0 \cdot x_1^* \cdot f_1$, $h_0 = x_0^*$

$$h_t(s) = \begin{cases} F(4s, 0), & 0 \le s \le t/4 \\ F\left(t, \frac{4-t-4s}{4-2t}\right), & t/4 \le s \le 1 - t/4 \\ F(4-4s, 1), & 1 - t/4 \le s \le 1. \end{cases}$$

$$h_1(s) = \begin{cases} F(4s, 0), & 0 \le s \le 1/4 \\ F\left(1, \frac{3-4s}{2}\right), & 1/4 \le s \le 3/4 \\ F(4-4s, 1), & 3/4 \le s \le 1 \end{cases}$$

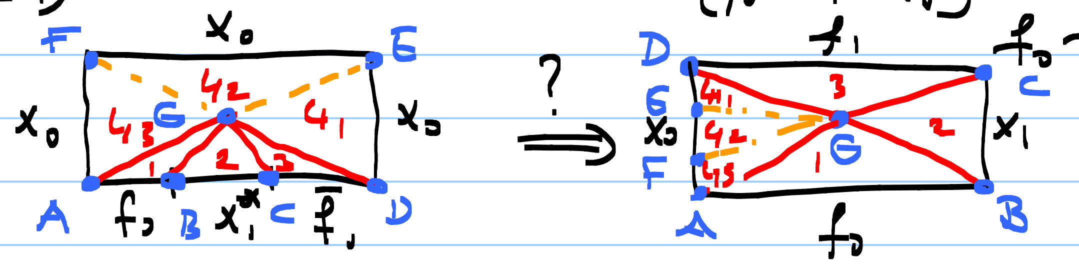
$$= \begin{cases} f_0(4s), & 0 \le s \le 1/4 \\ x_1, & 1/4 \le s \le 3/4 \\ f_1(4-4s), & 3/4 \le s \le 1. \end{cases} \sim f_0 \cdot x_1^* \cdot \overline{f_1}$$

Also, $h_0(s) = x_0$, for all s.



, so that $[f_0 \cdot x_1^* \cdot \overline{f_1}] = e$ in $\pi_1(X, x_0)$.

($\Leftarrow$) Now assume that $[f_0 \cdot x_1^* \cdot \overline{f_1}] = e$. Must show $f_0 \sim f_1$ rel $\{0,1\}$



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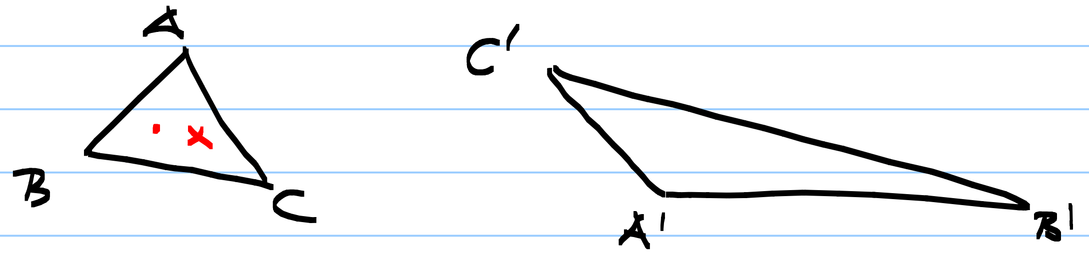
Now we may write a homeomorphism φ which is linear on each triangle mapping every vertex in the first rectangle to the vertex in the second rectangle with the same name.

$F: I\times I \to X \Rightarrow F\circ\varphi^{-1}: I\times I \to X$



$x = t_1 A + t_2 B + t_3 C \longmapsto \varphi(x) = t_1 A' + t_2 B' + t_3 C'$

$t_1, t_2, t_3 \geq 0, \ \sum t_i = 0$

(t_1, t_2, t_3) is called the barycentric coordinates of x.

This finishes the proof of that $P: \tilde{X} \to X$ is a covering map.

Now we need to prove that $\tilde{X}$ is simply connected.

$\tilde{X} = \{[\gamma] \mid \gamma: [0,1] \to X, \ \gamma(0) = x_0\}$

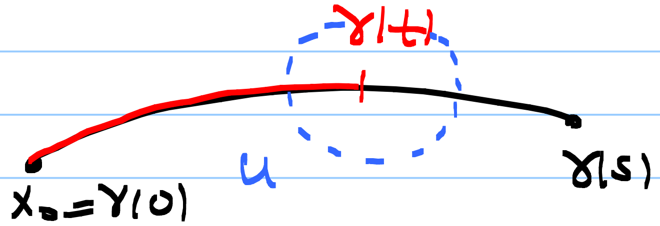
$x_0^*: [0,1] \to X$ the constant path at x_0.

$[x_0^*] \in \tilde{X}$ can be considered as a base point.

<u>$\tilde{X}$ is connected:</u> Take any point $[\gamma] \in \tilde{X}$.

So, $\gamma: [0,1] \to X$ is so that $\gamma(0) = x_0$.
For any $t \in [0,1]$ define the path

$$\gamma_t : [0,1] \to X, \quad \gamma_t(s) = \begin{cases} \gamma(s), & 0 \leq s \leq t \\ \gamma(t), & t \leq s \leq 1. \end{cases}$$



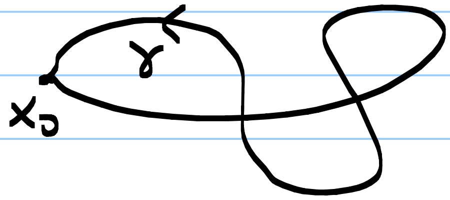
Then we get a path in $\tilde{X}$: $t \mapsto [\gamma_t]$

γ_0 = constant path at x_0
$\gamma_1 = \gamma$

So the path $t \mapsto [\gamma_t]$ in $\tilde{X}$ joins $[x_0^*]$ to $[\gamma]$. This path is continuous by the local description of topology on $\tilde{X}$ and thus $\tilde{X}$ is path connected.

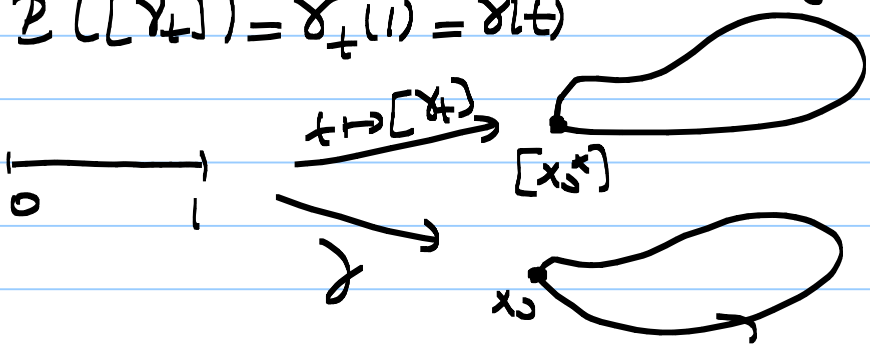
<u>$\tilde{X}$ is simply connected</u>: Since the homomorphism

$P_\# : \pi_1(\tilde{X}, [x_0^*]) \to \pi_1(X, x_0)$ is injective, it is enough to show that $\operatorname{Im} P_\#$ is trivial. Let $[\gamma]$ be in the image of $P_\#$. So γ is a loop at x_0.



Then the path $t \mapsto [\gamma_t]$ is a lift of γ to the covering $\tilde{X} \to X$, because

$P([\gamma_t]) = \gamma_t(1) = \gamma(t)$



$$\begin{array}{c} \tilde{X} \\ \downarrow P \\ X \end{array}$$

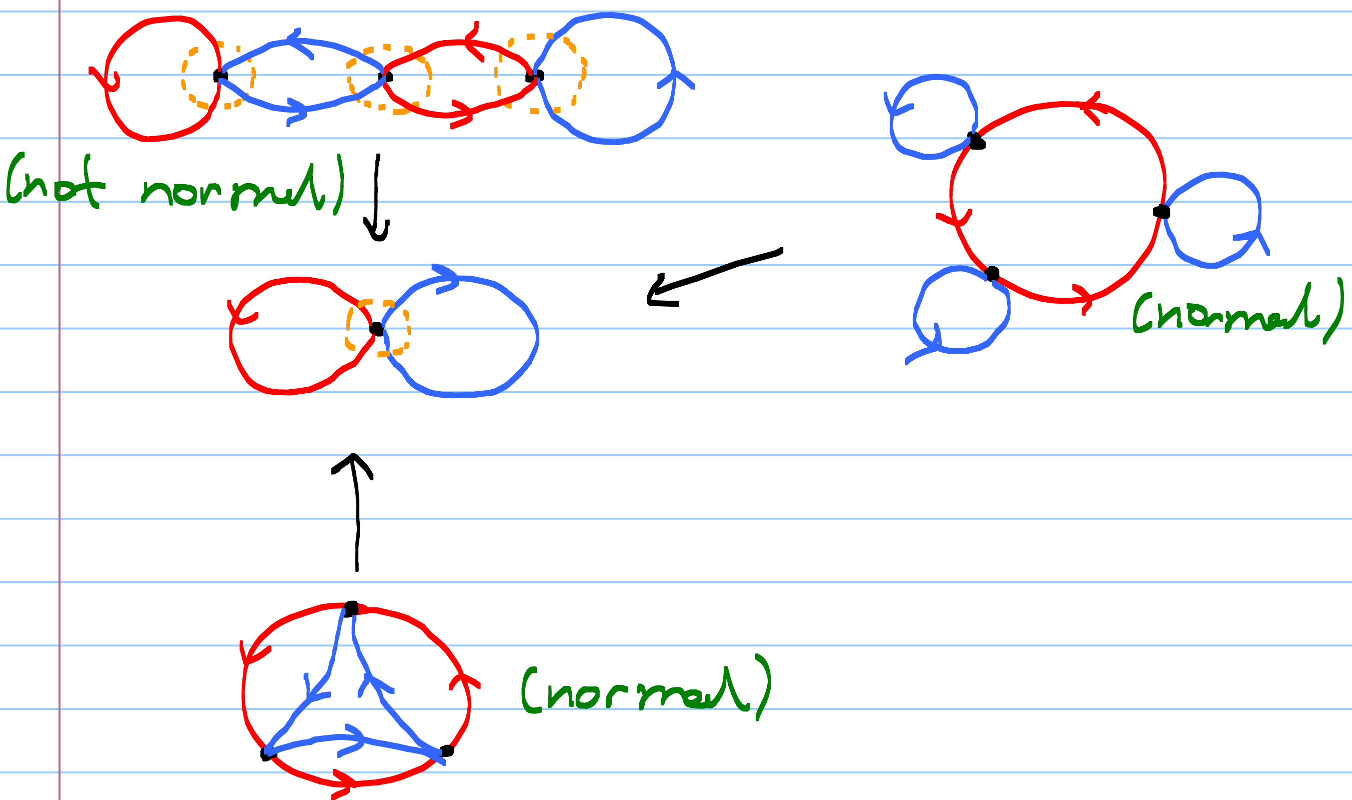
Since $t \longmapsto [\gamma_t]$ is a loop at $[x_0^*]$ the end points of this path must be the same.

$t=0 \Rightarrow x_0^*$, $\quad t=1 \Rightarrow [\gamma]$

$\Rightarrow [x_0^*] = [\gamma] \quad \Rightarrow [\gamma] = e \in \pi_1(X, x_0)$.

So we've proved that any topological space (path connected, l.c.p., s.l.s.c) has a universal covering space.

Remark: In practice we construct covering spaces as follows: Note that for a simply connected space any connected covering of it is itself because its fundamental group is trivial and thus any covering is 1-fold.



Proposition: Suppose X is path connected, locally path connected, and semilocally simply connected. Then for

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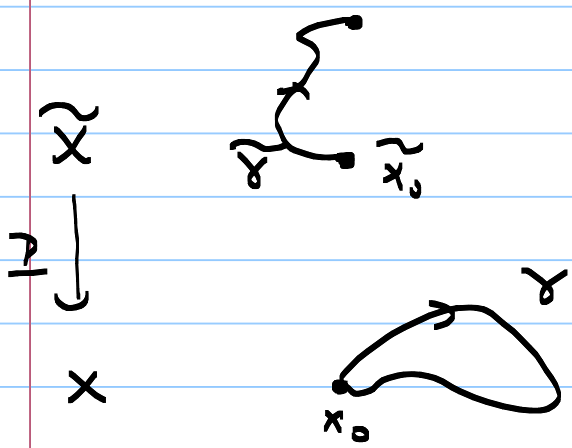
every subgroup $H \leq \pi_1(X, x_0)$ there is a covering space $P: X_H \to X$ such that

$$P_{\#}(\pi_1(X_H, \tilde{x}_0)) = H, \text{ for a suitable chosen}$$

base point $\tilde{x}_0 \in X_H$.

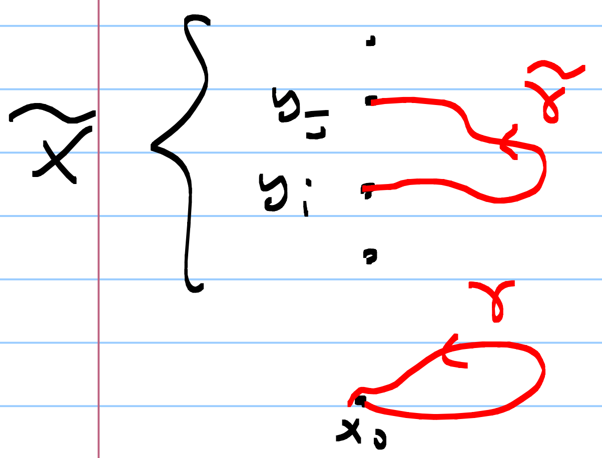
Proof: The main observation is the following. A loop γ lifts to a loop in the covering space if and only if $[\gamma]$ belongs to $H = P_{\#}(\pi_1(X, \tilde{x}_0))$.

In particular, if $\tilde{X}$ is simply connected then no lift of a loop γ with $[\gamma] \neq e$ is a loop in $\tilde{X}$.



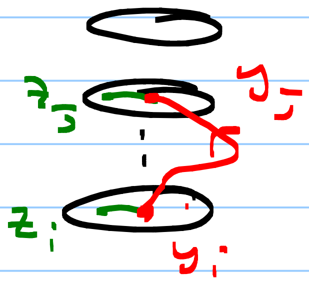
X_H will be constructed as a quotient of $\tilde{X}$:

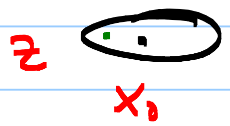
$\tilde{X} \to X$ universal cover



If $y_i, y_j \in P^{-1}(x_0)$ then we say $y_i \sim y_j$ if and only if the image of the homotopy class $\tilde{\gamma}$ in $\pi_1(X, x_0)$ is in H.

$y_i \sim y_j \iff [\gamma] \in H \le \pi_1(X, x_0)$.





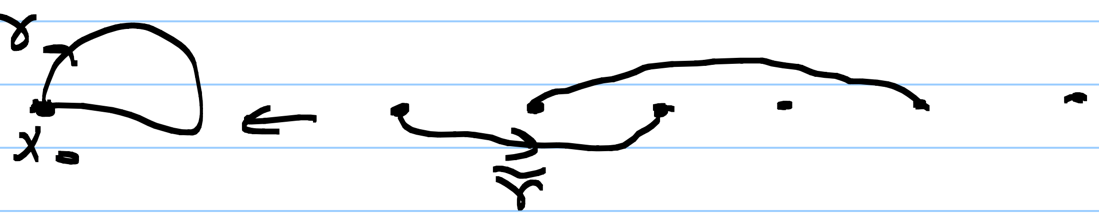
$U \subseteq X$ basic neighborhood for the universal cover.

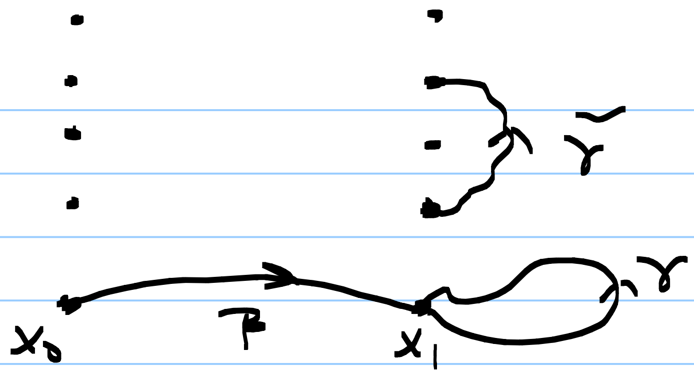
So let X_H be the quotient space $\tilde{X}/\sim$, where $\sim$ is as defined above. Note that for any basic neighborhood U in X two components $\tilde{U}_i$ and $\tilde{U}_j$ of $p^{-1}(U)$ are either identified by a homeomorphism or no points of $\tilde{U}_i$ and $\tilde{U}_j$ are identified.

$$\begin{tikzcd} \tilde{U}_i \arrow[r, "p_{\tilde{U}_i}"] \arrow[rr, bend right, "p^{-1}_{\tilde{U}_j} \circ p_{\tilde{U}_i}"'] & U, & \tilde{U}_j \arrow[r, "p_{\tilde{U}_j}"] & U \end{tikzcd}$$

Note that $X_H \longrightarrow X$ is still a covering space.

$p_\#(\pi_1(X_H, y_0)) = H$ because we identified the end points of any lift of $[\gamma] \in \pi_1(X, x_0)$ if and only if $[\gamma] \in H$.





$[\beta\cdot\gamma\cdot\bar\beta]\in H$?

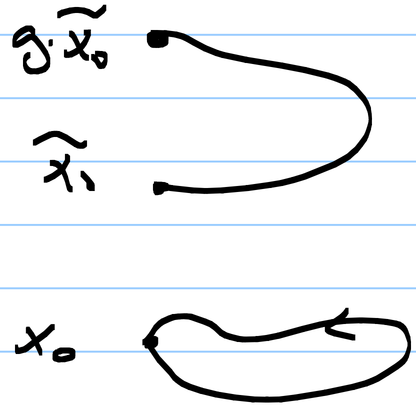
This finishes the proof. ∎

<u>Remark:</u> $G=\pi_1(X,x_0)$, $\tilde X \to X$ universal cover.

Then G acts on $\tilde X$ and $\tilde X/G = X$.

Moreover, if $H\le G$, then $X_H=\tilde X/H$.

$G=\pi_1(X,x_0)$



<u>Definition:</u> Let $p_1:\tilde X_1\to X$ and $p_2:\tilde X_2\to X$ be two covering spaces. We'll say that they are isomorphic if there is a homeomorphism $f:\tilde X_1\to\tilde X_2$ so that $p_1=p_2\circ f$. $(\iff f^{-1}\circ p_1=p_2)$

$$\begin{tikzcd}
\tilde X_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & \tilde X_2 \arrow[dl, "p_2"] \\
& X &
\end{tikzcd}$$

Proposition: If X is path connected and locally path connected, then two path connected covering spaces $p_1 : \tilde{X}_1 \to X$ and $p_2 : \tilde{X}_2 \to X$ are isomorphic via an isomorphism $f : \tilde{X}_1 \to \tilde{X}_2$ taking a base point $\tilde{x}_1 \in p_1^{-1}(x_0)$ to a basepoint $\tilde{x}_2 \in p_2^{-1}(x_0)$ if and only if

$$p_{1\#}(\pi_1(\tilde{X}_1, \tilde{x}_1)) = p_{2\#}(\pi_1(\tilde{X}_2, \tilde{x}_2)).$$

$$\begin{tikzcd}
\tilde{X}_1 \ \tilde{x}_1 \arrow[rr, bend left, "f"] \arrow[dr, "p_1"'] & & \tilde{x}_2 \ \tilde{X}_2 \arrow[dl, "p_2"] \\
& X \ x_0 &
\end{tikzcd}$$

Proof: ($\Longrightarrow$) Suppose there is an isomorphism $f : (\tilde{X}_1, \tilde{x}_1) \longrightarrow (\tilde{X}_2, \tilde{x}_2)$ so that $p_1 = p_2 \circ f$.

Hence, $p_{1\#}(\pi_1(\tilde{X}_1, \tilde{x}_1)) = (p_{2\#} \circ f_\#)(\pi_1(\tilde{X}_1, \tilde{x}_1))$

$$= p_{2\#}(\pi_1(\tilde{X}_2, \tilde{x}_2))$$

($\Longleftarrow$) Conversely if $p_1(\pi_1(\tilde{X}, \tilde{x}_1)) = p_2(\tilde{X}_2, \tilde{x}_2)$ then by the lifting criterion the map

$$\begin{tikzcd}
& (\tilde{X}_2, \tilde{x}_2) \arrow[d, "p_2"] \\
(\tilde{X}_1, \tilde{x}_1) \arrow[r, "p_1"'] \arrow[ur, dotted, "\exists!\, f"] & (X, x_0)
\end{tikzcd}$$

there is a unique $f : (\tilde{X}_1, \tilde{x}_1) \to (X_2, \tilde{x}_2)$ so that $p_2 \circ f = p_1$.

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Similarly, there is unique $g:(\tilde{X}_2,\tilde{x}_2)\to(\tilde{X}_1,\tilde{x}_1)$ such that $p_1\circ g = p_2$.

$$\begin{tikzcd}
(\tilde{X}_1,\tilde{x}_1) \arrow[r, "f"] \arrow[d, "p_1"'] \arrow[rr, bend right, "\mathrm{id}_{\tilde{X}_1}"] & (\tilde{X}_2,\tilde{x}_2) \arrow[r, "g"] \arrow[dl, "p_2"'] \arrow[dr, "p_2"] & (\tilde{X}_1,\tilde{x}_1) \arrow[d, "p_1"] \\
(X,x_0) \arrow[rr, equal, "\mathrm{id}"'] & & (X,x_0)
\end{tikzcd}$$

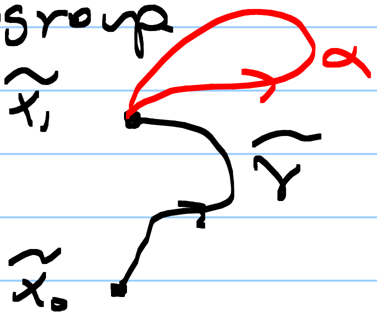
$$\begin{tikzcd}
y \arrow[r, mapsto] \arrow[d, mapsto] & f(y) \arrow[r, mapsto] \arrow[dl, mapsto] \arrow[dr, mapsto] & g(f(y)) \arrow[d, mapsto] \\
x \arrow[rr, mapsto] & & x
\end{tikzcd}$$

$\Rightarrow f\circ g = \mathrm{id}_{\tilde{X}_1}$. Similarly, $g\circ f = \mathrm{id}_{\tilde{X}_2}$. Hence, f and g are homeomorphisms. ■

Now we are ready to state the classification theorem.

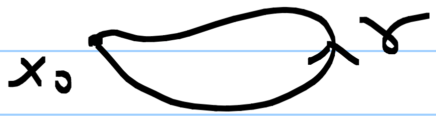
__Theorem:__ Let X be path connected, locally path connected and semilocally simply connected. Then there is a bijection between the set of base point-preserving isomorphism classes of path-connected covering spaces
$$p:(\tilde{X},\tilde{x}_0)\longrightarrow(X,x_0)$$
and the set of subgroups of $\pi_1(X,x_0)$, obtained by associating the subgroup $p_\#(\pi_1(\tilde{X},\tilde{x}_0))$ to the covering space $(\tilde{X},\tilde{x}_0)$. If the base points are ignored, this correspondence gives a bijection between isomorphism classes of path-connected covering spaces $p:\tilde{X}\to X$ and conjugacy classes of subgroups of $\pi_1(X,x_0)$.

Proof: Note that we just need to prove the last statement. For a covering space $p:(\tilde{X},\tilde{x}_0)\to(X,x_0)$ changing the basepoint within $p^{-1}(x_0)$ corresponds exactly to changing the $p_\#(\pi_1(\tilde{X},\tilde{x}_0))$ to a conjugate subgroup



$$H = p_\#(\pi_1(X,\tilde{x}_0))$$

$$[\alpha]\in\pi_1(\tilde{X}_1,\tilde{x}_1)$$



$$H_1 = p_\#(\pi_1(X,\tilde{x}_1))$$

$$\pi_1(X,\tilde{x}_0) = \{[\tilde{\gamma}.\alpha.\bar{\tilde{\gamma}}] \mid [\alpha]\in\pi_1(\tilde{X},\tilde{x}_1)\}.$$

$p_\# \downarrow \qquad \downarrow$

$$H \qquad \{[\gamma].p_\#([\alpha])[\gamma]^{-1} \mid [\alpha]\in\pi_1(\tilde{X}_1,\tilde{x}_1)\}$$

$$\| $$

$$[\gamma]H_1[\gamma]^{-1}$$

$$\Rightarrow H = [\gamma]H_1[\gamma]^{-1}.$$

The converse is left as an exercise!

Remark: Let $H=(e)\leq\pi_1(X,x_0)$. Any conjugate of H is itself. Thus the simply connected covering space we constructed earlier is unique upto isomorphism. Hence, we may call the universal covering space.

Deck Transformation and Group Actions:

For a covering space $p:\tilde{X}\to X$ the isomorphisms $\tilde{X}\to\tilde{X}$ are called deck transformations or covering transformations.

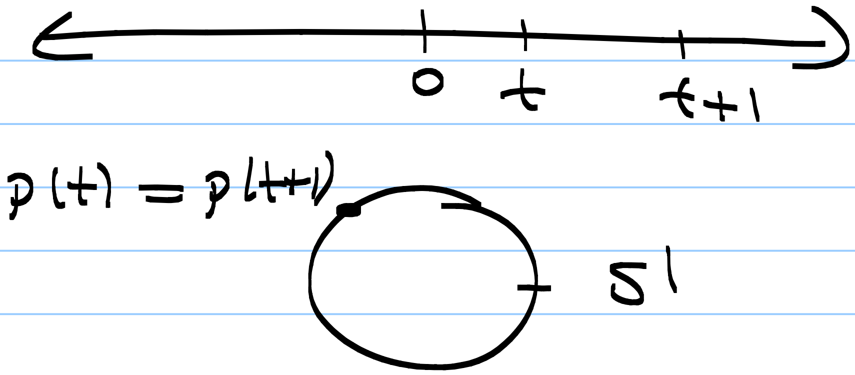
$$\begin{tikzcd} \tilde{X} \arrow[rr, "f"] \arrow[dr, "p"'] & & \tilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd} \qquad \begin{tikzcd} \tilde{x} \arrow[rr, bend left] \arrow[dr, bend right] & & f(\tilde{x}) \arrow[dl, bend left] \\ & x & \end{tikzcd}$$

$p\circ f = p$

$$\begin{tikzcd} \tilde{X} \arrow[r, "f"] \arrow[d, "p"'] & \tilde{X} \arrow[d, "p"] \\ X \arrow[r, "id"'] & X \end{tikzcd}$$

Hence, deck transformations are just the lifts of the $id: X\to X$ to $\tilde{X}$.

Example: $\mathbb{R}\to S^1$, $p(t)=(\cos 2\pi t, \sin 2\pi t)$



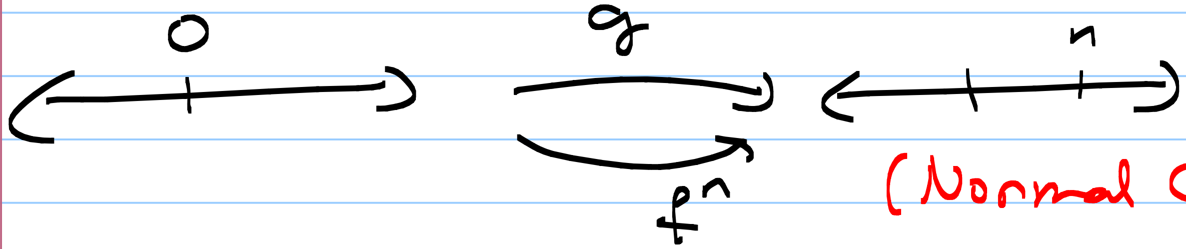
Hence, $f(t)=t+1$, $t\in\mathbb{R}$, $f:\mathbb{R}\to\mathbb{R}$ is a covering transformation.

$f^n:\mathbb{R}\to\mathbb{R}$, $f^n(t)=f(f(\cdots(f))\cdots)(t)=t+n$

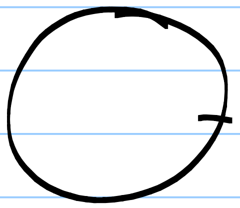
is a covering transformation for all n.

Let $G(\tilde{X})$ denote the group of deck transformations.

Then for the above covering $G(\mathbb{R}) \simeq \mathbb{Z}$.



(Normal Cover!)

$p^{-1}(1) = \mathbb{Z} \subseteq \mathbb{R}$  $p(0) = 1$

$g = f^n \implies G(\mathbb{R}) \simeq \mathbb{Z}$.

$G(\mathbb{R} \xrightarrow{p} S^1) \simeq \mathbb{Z}$

<u>Definition</u>: A covering space $p: \tilde{X} \to X$ is called normal if for each $x \in X$ and each pair of points $\tilde{x}, \tilde{x}'$ over x there is a deck transformation taking $\tilde{x}$ to $\tilde{x}'$.

$$\begin{array}{c} \cdot\ \tilde{x}' \\ \cdot\ \tilde{x} \\ \downarrow p \\ \cdot\ x \end{array} \qquad f(\tilde{x}) = \tilde{x}'$$

Video 35

Proposition: Let $\tilde{p}:(\tilde{X},\tilde{x}_1) \to (X,x_1)$ be a path connected covering space of path connected, locally path connected space X, and let H be the subgroup $p_\#(\pi_1(\tilde{X},\tilde{x}_1)) \leq \pi_1(X,x_1)$. Then:

a) This covering space is normal if and only if H is a normal subgroup of $\pi_1(X,x_1)$.

b) $G(\tilde{X})$ is isomorphic to the quotient $N(H)/H$ where $N(H)$ is the normalizer of H in $\pi_1(X,x_1)$.

In particular, $G(\tilde{X})$ is isomorphic to $\pi_1(X,x_0)/H$ if $\tilde{X}_0$ is a normal covering. Hence for the universal cover $\tilde{X}\to X$ we have $G(\tilde{X}) \simeq \pi_1(X,x_1)$.

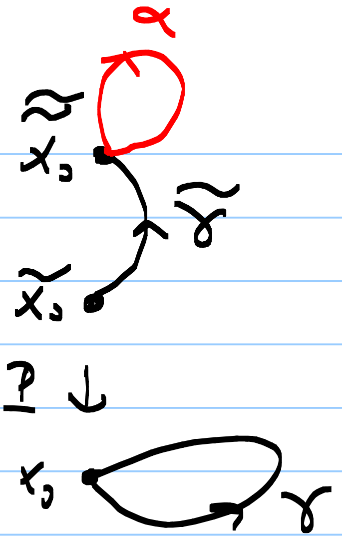
Corollary Assume the above set up. Then the cover $p:\tilde{X}\to X$ is normal if and only if for any $[\gamma]\in\pi_1(X,x_0)$ we have either all lifts of γ are loops or all lifts of γ are non-loops.

Proof of the Proposition:

a) First assume that the covering is normal. So we must prove that $H = p_\#(\pi_1(\tilde{X},\tilde{x}_1))$ is a normal subgroup of $\pi_1(X,x_0)$. It is enough to prove that

$$[\gamma]\,H\,[\gamma]^{-1} = H \quad \text{for any } [\gamma]\in\pi_1(X,x_0).$$

Let $\tilde{\gamma}$ be the unique lift of γ starting at $\tilde{x}_0$.



$$\tilde\gamma \cdot \pi_1(\tilde X, \tilde{\tilde x}_0)\, \overline{\tilde\gamma} \simeq \pi_1(\tilde X, \tilde x_0).$$

$$\big\downarrow p_\#$$

$$[\gamma]\cdot H_1 \cdot [\gamma]^{-1} = H, \text{ where}$$

$$H_1 = p_\#(\pi_1(\tilde X, \tilde{\tilde x}_0))$$

By assumption there is a deck transformation $f \in G(\tilde X)$ so that $f(\tilde x_0) = \tilde{\tilde x}_0$.

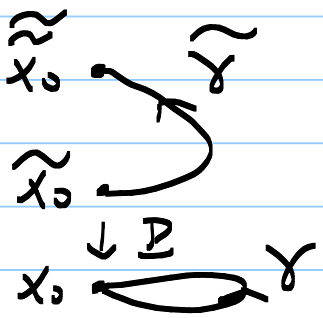
In particular, $f_\#(\pi_1(\tilde X, \tilde x_0)) = \pi_1(\tilde X, \tilde{\tilde x}_0)$, since f is a homeomorphism. Since $p = p \circ f$ we get

$$\begin{tikzcd} \tilde X \arrow[rr, "f"] \arrow[dr, "p"'] & & \tilde X \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

$$H = p_\#(\pi_1(\tilde X, \tilde x_0)) = p_\#(f_\#(\pi_1(\tilde X, \tilde x_0)))$$
$$= p_\#(\pi_1(\tilde X, \tilde{\tilde x}_0))$$
$$= H_1.$$

Hence, $H = H_1 = [\gamma] H [\gamma]^{-1}$, where $[\gamma] \in \pi_1(X, x_0)$ is any element. Thus $H \triangleleft \pi_1(X, x_0)$.

Now assume that H is normal in $\pi_1(X, x_0)$. Let $\tilde x_0$ and $\tilde{\tilde x}_1$ lie above x_0 and choose a path $\tilde\gamma$ joining $\tilde x_1$ to $\tilde{\tilde x}_0$. Let $\gamma = p(\tilde\gamma)$.



Let $H_1 = p_\#(\pi_1(\tilde X, \tilde{\tilde x}_0))$ as above.

Then $H_1 = [\gamma]^{-1} H [\gamma] = H$

Consider the diagram below:

$$\begin{tikzcd} & (\tilde{X}, \tilde{\tilde{x}}_0) \arrow[d, "p"] \\ (\tilde{X}, \tilde{x}_0) \arrow[r, "p"'] \arrow[ur, dashed, "\exists ! f"] & (X, x_0) \end{tikzcd}$$

Since $H = H$, by the lifting criterion there is a (unique) lift $f : (\tilde{X}, \tilde{x}_0) \longrightarrow (\tilde{X}, \tilde{\tilde{x}}_0)$ satisfying $p \circ f = p$.

Similarly, there is a unique $g : (\tilde{X}, \tilde{\tilde{x}}_0) \to (\tilde{X}, \tilde{x}_1)$ so that $p \circ g = p$.

Now, $g \circ f : (\tilde{X}, \tilde{x}_0) \longrightarrow (\tilde{X}, \tilde{x}_0)$ satisfies

$p \circ (g \circ f) = (p \circ g) \circ f = p \circ f = p$ and thus $g \circ f$ is the unique lift of

$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\ (\tilde{X}, \tilde{x}_0) \arrow[r, "p"'] \arrow[ur, "g \circ f"] & (X, x_0) \end{tikzcd}$$

However, $\mathrm{id}_{\tilde{X}}$ is also a lift of $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ taking $\tilde{x}_0$ to $\tilde{x}_0$. Now by the uniqueness of the lift we see that

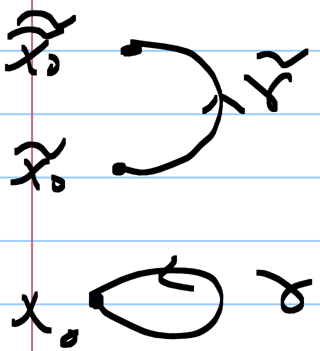
$$g \circ f = \mathrm{id}_{\tilde{X}}.$$

Similarly, $f \circ g = \mathrm{id}_{\tilde{X}}$, so that f is a homeomorphism and thus a deck transformation. Therefore, $p : \tilde{X} \to X$ is a normal cover.

This finishes the proof of (a).

b) To prove that $N(H)/_{H} \simeq G(\tilde{X})$ we construct a group homomorphism $\varphi : N(H) \to G(\tilde{X})$ which is onto and $\ker\varphi = H$.

Indeed, we will use the above arguments: Namely, if $[\gamma] \in N(H)$ then $[\gamma]^{-1} H [\gamma] = H$. Let $\tilde{\gamma}$ be the unique lift of γ starting at $\tilde{x}_0$.



$$H = p_{\#}(\pi_1(\tilde{X}, \tilde{x}_0)),$$

$$H_1 = p_{\#}(\pi_1(\tilde{X}, \hat{\tilde{x}}_0)).$$

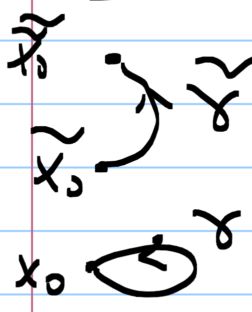
$$H = [\gamma] H_1 [\gamma]^{-1}$$

Hence, there is a unique deck transformation $f : \tilde{X} \to \tilde{X}$ taking $\tilde{x}_0$ to $\hat{\tilde{x}}_0$.

Now we define $\varphi : N(H) \to G(\tilde{X})$ as

$\varphi([\gamma]) = f.$

<u>φ is onto</u>: Let $f \in G(\tilde{X})$, and set $\hat{\tilde{x}}_0 = f(\tilde{x}_0)$.

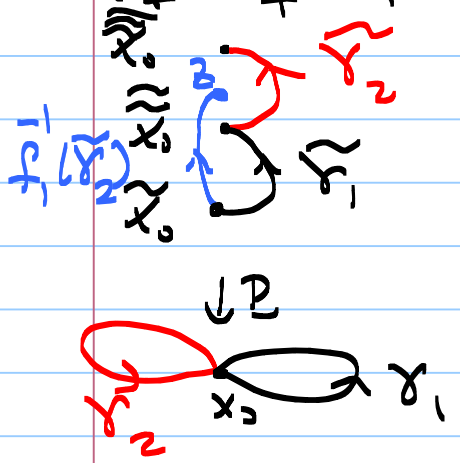


Choose a path $\tilde{\gamma}$ joining $\tilde{x}_0$ to $\hat{\tilde{x}}_0$. Then $\gamma = p \circ \tilde{\gamma}$ is a loop at x_0 and by the definition of φ

$\varphi([\gamma]) = f.$

<u>φ is a group homomorphism</u>: Let $[\gamma_1]$ and $[\gamma_2]$ be two elements in $N(H)$. Let γ_1 be the unique

lift of γ_1 starting at $\tilde{x}_0$ and $\tilde{\gamma}_2$ be the lift of γ_2 starting at $\tilde{\tilde{x}}_0 = \tilde{\gamma}_1(\tilde{x}_0)$. Then $\tilde{\gamma}_1 \cdot \tilde{\gamma}_2$ is the lift of $\gamma_1 \cdot \gamma_2$ starting at $\tilde{x}_0$.



Let $\varphi([\gamma_1]) = f_1$ and $\varphi([\gamma_1][\gamma_2]) = f$. So, $f_1(\tilde{x}_0) = \tilde{x}_1$ and $f(\tilde{x}_0) = \tilde{\tilde{\tilde{x}}}_0$.

Also, note that $f_1^{-1}(\tilde{\gamma}_2)$ is the lift of γ_2 starting at $\tilde{x}_0$.

Let $\varphi([\gamma_2]) = f_2$, then $f_2(\tilde{x}_0) = z$. Note that $f_1^{-1}(\tilde{\tilde{\tilde{x}}}_0) = z$.

Now, $(\varphi([\gamma_1]) \circ \varphi([\gamma_2]))(\tilde{x}_0) = (f_1 \circ f_2)(\tilde{x}_0)$

$$= f_1(f_2(\tilde{x}_0))$$

$$= f_1(z)$$

$$= f_1(f_1^{-1}(\tilde{\tilde{\tilde{x}}}_0))$$

$$= \tilde{\tilde{\tilde{x}}}_0$$

$$= \varphi([\gamma_1] \cdot [\gamma_2])(\tilde{x}_0).$$

Since a deck transformation is uniquely determined by its image at a single point we see that

$$\varphi([\gamma_1] \cdot [\gamma_2]) = \varphi([\gamma_1]) \circ \varphi([\gamma_2]).$$

Hence, φ is a homomorphism.

$\ker\varphi = H$: Let $\varphi([\gamma]) = \mathrm{id}_{\tilde{X}}$. So, if

$f = \varphi([\gamma])$ then $f(\tilde{x}_0) = \tilde{x}_0$. Then the unique lift $\tilde{\gamma}$ of γ starting at $\tilde{x}_0$ is a loop. Hence, $[\tilde{\gamma}] \in \pi_1(\tilde{X}, \tilde{x}_0)$ so that

$$[\gamma] = P([\tilde{\gamma}]) \in P_{\#}(\pi_1(\tilde{X}, \tilde{x}_0)) = H.$$

Hence, $\ker\varphi \subseteq H$. The other direction $H \subseteq \ker\varphi$ can be similarly.

For the last statement if the cover is normal then $N(H) = \pi_1(X, x_0)$ and thus

$$G(\tilde{X}) \simeq N(H)/H = \pi_1(X, x_0)/H.$$

Finally, if $\tilde{X}$ is the universal covering then $H = (e)$ so that

$$G(\tilde{X}) \simeq \pi_1(X, x_0). \quad \blacksquare$$

Proof of the Corollary:

Let $P: \tilde{X} \to X$ be a normal covering and $[\gamma] \in \pi_1(X, x_0)$. Let $\tilde{\gamma}$ be the lift of γ starting at $\tilde{x}_0$.
If $\hat{x}_0 \in P^{-1}(x_0)$ is another point above x_0 then $g(\tilde{\gamma})$ is the lift of γ starting at $\hat{x}_0$, where $g \in G(\tilde{X})$ with $g(\tilde{x}_0) = \hat{x}_0$.

$g(\tilde{x}_0) = \hat{x}_0 \rightarrow g(\tilde{\gamma})$

$\tilde{x}_0 \rightarrow \tilde{\gamma}$

$x_0 \; \gamma$

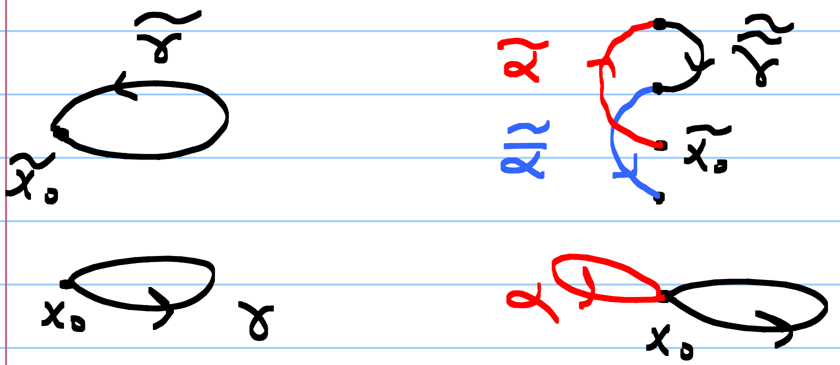
Since the two lifts $\tilde{\gamma}$ and $g(\tilde{\gamma})$ are homeomorphic (via g) they are both loop or both non loop.

This finishes the proof of one direction.

Now assume the covering and thus the subgroup H is **not** normal. Then there is some $[\gamma] \in H$ and $[\alpha] \in \pi_1(X, x_0)$ so that

$$[\alpha \gamma \bar{\alpha}] \notin H.$$

Hence, the lift $\tilde{\gamma}$ of γ starting at $\tilde{x}_0$ is a loop, whereas the lift $\tilde{\alpha}.\tilde{\gamma}.\tilde{\bar{\alpha}}$ is not a loop at $\tilde{x}_0$.



This implies that the lift $\tilde{\tilde{\gamma}}$ of γ starting at $\tilde{\alpha}(1)$ cannot be a loop, since in that case the lift of $\bar{\alpha}$ at $\tilde{\alpha}(1)$, would take the point $\tilde{\alpha}(1)$ back to point $\tilde{\alpha}(0) = \tilde{x}_0$, which would imply that the lift of $\alpha\gamma\bar{\alpha}$ at $\tilde{x}_0$ is a loop, a contradiction. This finishes the proof. ∎

Group Actions of Spaces:

Let G be a group and Y a topological space. We say that G acts on Y via homeomorphisms if there is a homomorphism

$$\varphi : G \longrightarrow \mathrm{Homeo}(Y),$$ the group of homeomorphisms of Y.

Video 37

Note that if $\tilde{X} \to X$ is a covering then $G(\tilde{X})$ is a subgroup of $\mathrm{Homeo}(\tilde{X})$.

If $\varphi : G \to \mathrm{Homeo}(Y)$ is injective then we say that the action is faithful.

Property (*): Each point $y \in Y$ has a neighborhood U such that all the images $g(U)$ for varying $g \in G$ are disjoint. In other words,

$$g_1(U) \cap g_2(U) \neq \emptyset \text{ implies } g_1 = g_2.$$

Let G acts on a space Y and the action satisfies the property (*).

$y \in Y$, $y \in U$ (*), $g_1(U) \cap g_2(U) \neq \emptyset \Rightarrow g_1 = g_2$.

Thus if $g_1 \neq g_2$ then $g_1(U) \cap g_2(U) = \emptyset$.

Let $X = Y/G \doteq Y/\sim$ $\quad y_1 \sim y_2 \Leftrightarrow y_1 = g(y_2)$ for some $g \in G$.

X is the space of G-orbits of Y.

Proposition: Assume the above setup. Then,

a) The quotient map $p : Y \to Y/G = X$, $p(y) = G(y)$ ($G(y) = \{g(y) \mid g \in G\}$) is a normal covering space.

b) G is the Deck transformation group of this covering space if Y is path connected.

c) G is isomorphic to $\pi_1(Y/G)/p_\#(\pi_1(Y))$ if Y is path connected and locally path connected.

<u>Proof</u>: $p: Y \longrightarrow Y/G = X$

Let $x = G(y)$. Choose an open subset $U \subseteq Y$ st. $y \in U$ and it satisfies the property (*).

If $V = p(U)$ then $p^{-1}(V) = \bigcup_{g \in G} g(U)$, which is a disjoint union of open subsets. In particular, $p^{-1}(V)$ is open and $p|_{g(U)} : g(U) \to V$ is a homeomorphism.

Hence, $p: Y \to Y/G = X$ is a covering space.

Note also that (assuming the action is faithful) the cover $p: Y \to X$ is a $|G|$-fold cover.

Hence, the group isomorphisms of $p: Y \to X$, $G(Y)$ has cardinality at most $|G|$.

On the other hand, every $g \in G$ defines an isomorphism of this cover:

$g \in G$

$$\begin{tikzcd} Y \arrow[rr, "\varphi(g)"] \arrow[dr, "p"'] & & Y \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

$p \circ \varphi(g) = p$

$\bullet\ y$
$\bullet\ g_1(y)$
$\bullet$

$$[y] = G(y) = \{ g(y) \mid g \in G \}$$

By definition G acts transitively on each orbit. Hence, the covering is normal.

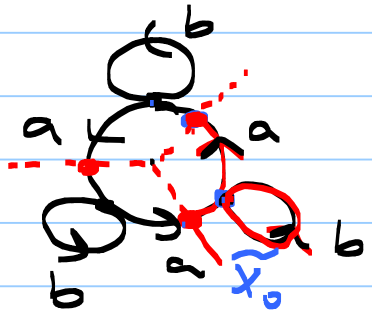
In particular, $G \simeq G(Y)$ the group Deck transformations.

For part (c), $G(Y) \simeq N(H)/_{H}$, where

$H = p_{\#}(\pi_1(Y, \tilde{x}_0))$. Since the cover is normal

$N(H) = \pi_1(X, x_0)$. So, $G(Y) \simeq \pi_1(X, x_0) / p_{\#}(\pi_1(Y, \tilde{x}_0))$. ∎

<u>Example:</u>

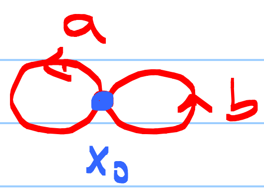
$Y =$ 

$\mathbb{Z}_3 = \langle \sigma \rangle$

$\sigma : \mathbb{R}^2 \to \mathbb{R}^2$

σ : $2\pi/_3$ radian counterclockwise rotation.

$$\sigma = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad \theta = 2\pi/_3.$$

$Y/_{\mathbb{Z}_3} =$ 

$Y \to Y/_{\mathbb{Z}_3}$ regular $\mathbb{Z}_3$ covering.

$\pi_1(Y/\mathbb{Z}_3) = F_2 = \langle a, b\rangle$

Y is homotopy equivalent to $\bigvee_4 S^1$ and thus $\pi_1(Y) \simeq F_4$.

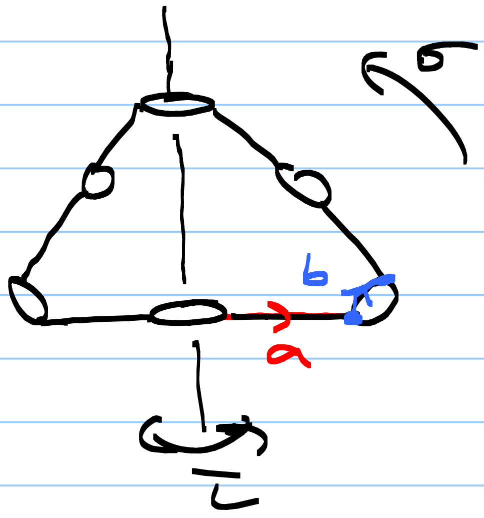
$p_\#(\pi_1(Y)) = \langle a, b, a b a^{-1}, a^2 b a^{-2}\rangle$

$p_\#(\pi_1(Y))$ is a normal subgroup of F_2 with

$F_2 / p_\#(\pi_1(Y)) \simeq \mathbb{Z}_3$.

2)

Y

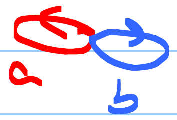


$G = S_3 = \langle \sigma, \tau \mid \sigma^3, \tau^2, \tau\sigma\tau\sigma\rangle$

σ: $2\pi/3$ radian rotation

τ: reflection w.r.t. y-axis.

$\downarrow p$

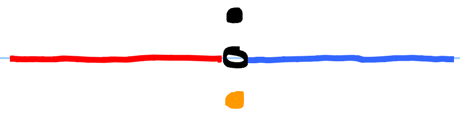
$Y/S^3 =$ 

$\pi_1(Y) \simeq F_7$

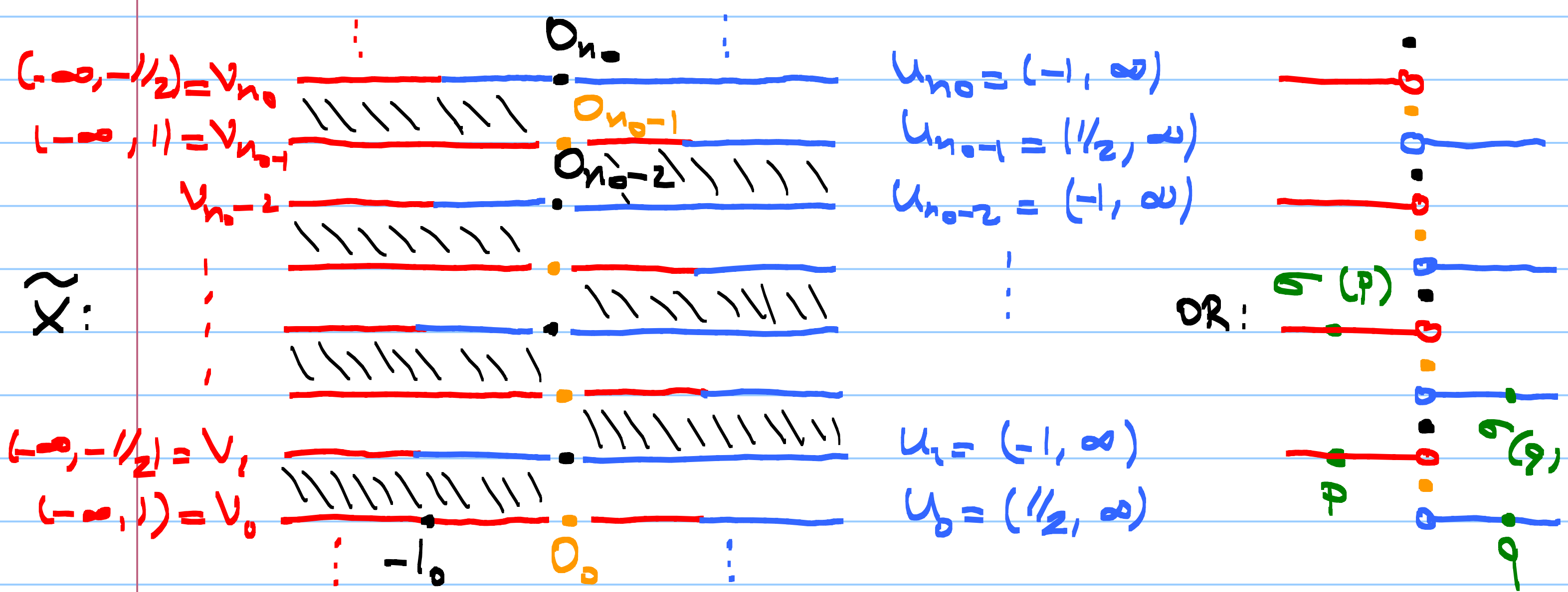
$\pi_1(X) = F_2 = \langle a, b \mid - \rangle$

$F_7 \simeq p_\#(\pi_1(Y)) \overset{6}{\triangleleft} \pi_1(X) \simeq F_2$ and the normal quotient is isomorphic to S_3.

Video 38 and 45

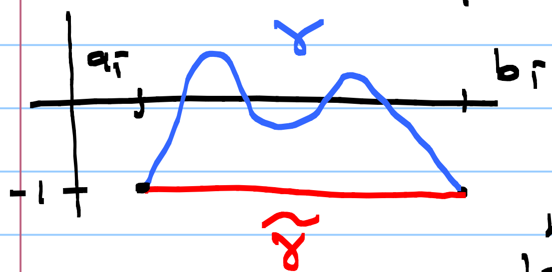
3) X :  the real line with double origin.

The universal cover $\tilde{X}$ of X is the following space $\tilde{X} = \mathbb{R}\times\mathbb{Z}/\sim$, where $(x,n)\sim(x,n+1)$ if and only if ($x<0$ and n is even) or ($x>0$ and n is odd).



Let $\gamma:[0,1]\to\tilde{X}$ be a loop at $-1_0=\{-1\}\times\{0\}$. Each $\mathbb{R}\times\{n\}$ is open and they cover $\tilde{X}$. Since $\gamma([0,1])$ is compact it is contained in the union of finitely many of them, say $\gamma([0,1])\subseteq\mathbb{R}\times\{0,1,\dots,n_0\}$.

$\gamma^{-1}(U_{n_0})$ is a disjoint of open intervals (if $n_0\geq 1$), whose union contain the compact subset $\gamma^{-1}(0_{n_0})$. Thus only finitely many of these intervals, say $(a_1,b_1),\dots,(a_k,b_k)$ satisfy $0_{n_0}\in\gamma((a_i,b_i))$, $i=1,\dots,k$. Clearly, $\gamma(a_i)=\gamma(b_i)=-1$, for each $i=1,\dots,k$. We replace each



$\gamma|_{[a_i,b_i]}$ with its homotopy $\tilde{\gamma}|_{[a_i,b_i]}$. Now the new loop never takes the value 0_{n_0}, i.e., it takes only negative values of $\mathbb{R}\times\{n_0\}$. In particular, we have homotoped γ to a loop, whose image lying in $\mathbb{R}\times\{0,1,\dots,n_0-1\}$.

By induction we see that we can homotope γ to a loop lying in $\mathbb{R}\times\{0\}$. Clearly, we can homotopy γ further to constant loop at $\{-1\}\times\{0\} = -1_0$.

Let $\sigma : \widetilde{X} \to \widetilde{X}$, $\sigma(x,n) = (x, n+2)$. The deck transformation group $G(\widetilde{X})$ of $p : \widetilde{X} \to X$ is the infinite cyclic group generated by σ.

In particular, $\pi_1(X)$ is infinite cyclic.

<u>Proposition</u>: Let X, Y be Hausdorff space, where X is compact and Y is connected. If $f : X \to Y$ is a map, which is locally a homeomorphism near each point, then f is a finite sheeted covering space.

Proof is again left as an exercise.

CHAPTER 2: Homology

Simplicial Homology:

Δ-complex: $\Delta^n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0, \sum t_i = 1\}$

the standard n-simplex in $\mathbb{R}^{n+1}$.

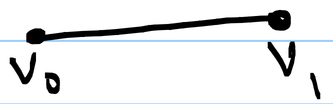
More generally, if $\{v_0, v_1, \dots, v_n\}$ is a set of vectors in $\mathbb{R}^m$ such that

$\{v_1 - v_0, v_2 - v_0, \dots, v_n - v_0\}$ is linearly independent

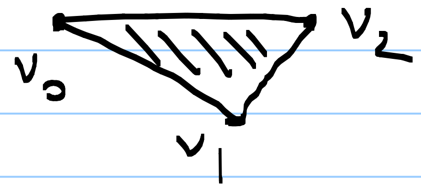
then the n-simplex determined by $\{v_0, v_1, \dots, v_n\}$ is defined by

$$[v_0, v_1, \dots, v_n] = \left\{ \sum_{i=0}^{n} t_i v_i \;\middle|\; t_i \geq 0, \sum_i t_i = 1 \right\}$$

$[v_0, v_1]$



$[v_0, v_1, v_2]$

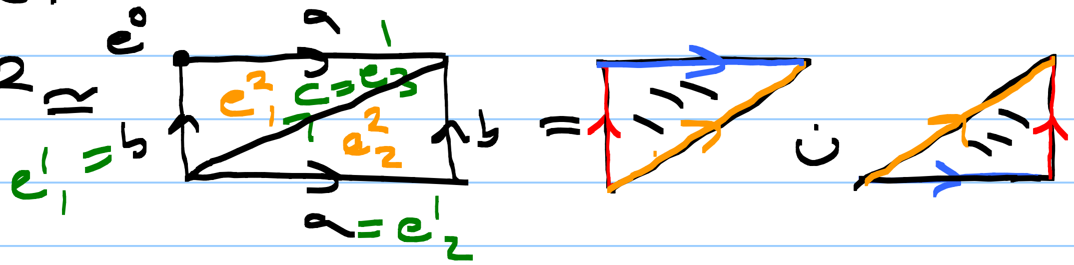


Note that $[v_0, v_1, \dots, v_n]$ is homeomorphic to Δ^n by the map

$$\Delta^n \to [v_0, v_1, \dots, v_n], \quad (t_0, \dots, t_n) \mapsto \sum_{i=0}^{n} t_i v_i.$$

A delta complex is a quotient space of some disjoint union of simplicies, where certain faces of simplicies are identified by linear isomorphisms.

<u>Example</u> $T^2 \simeq$



A face of $[v_0, v_1, \dots, v_n]$ is $[v_0, \dots, v_{i-1}, v_{i+1}, \dots v_n]$

<u>Simplicial Homology</u>: Let X be a Δ-complex.

Define $\Delta_n(X)$ as the free abelian group with basis the open n-simplices e^n_α of X.

⌜<u>Remark</u>: $\Delta^n \simeq D^n$ and thus each Δ-complex is a CW-complex.⌟

<u>Example</u> For the Δ-complex structure for T^2

$$\Delta_0(T^2) = \langle e^0_1 \rangle \simeq \mathbb{Z}$$

$$\Delta_1(T^2) = \langle e^1_1, e^1_2, e^1_3 \rangle \simeq \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$$

$$\Delta_2(T^2) = \langle e^2_1, e^2_2 \rangle \simeq \mathbb{Z} \oplus \mathbb{Z}$$

Note that elements of $\Delta_n(X)$ have the form

$\sum_\alpha n_\alpha e^n_\alpha$, where $n_\alpha = 0$ for all but finitely many α.

<u>Boundary of Simplex:</u>

i^{th} element is deleted

$$\partial([v_0, v_1, \dots, v_n]) = \sum_{i=0}^{n} (-1)^i [v_0, v_1, \dots, \hat{v_i}, \dots, v_n]$$

Any element of $\Delta_n(X)$ is called an n-chain.

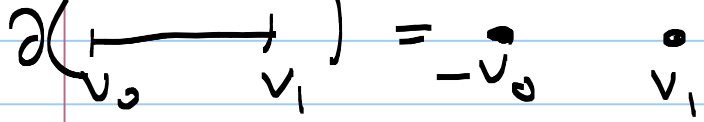
Hence boundary of an n-chain is an $n-1$-chain.

If $\sigma \in \Delta_n(X)$, say $\sigma = \sum n_\alpha e_\alpha^n$, then we define

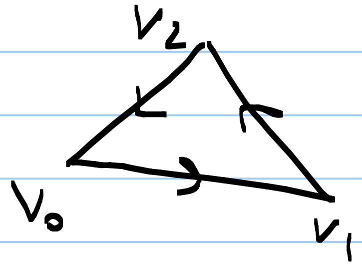
$$\partial(\sigma) = \sum_\alpha n_\alpha \partial e_\alpha^n.$$

<u>Example:</u> $\partial[v_0] = 0$.

$$\partial[v_0, v_1] = (-1)^0 [v_1] + (-1)^1 [v_0]$$
$$= [v_1] - [v_0]$$



$$\partial[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$$



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Lemma: If $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$ denotes the boundary homomorphism, then

$$\partial_{n-1} \circ \partial_n = 0, \text{ for all } n \geq 1.$$

Note that this implies $\mathrm{Im}(\partial_n) \subseteq \ker(\partial_{n-1})$

Definition: The nth Δ-homology group of a Δ-complex X is defined as the quotient group

$$H_n^\Delta(X) = \frac{\ker(\partial_{n-1})}{\mathrm{Im}(\partial_n)}.$$

Proof: It is enough to prove that

$\partial_{n-1} \circ \partial_n([v_0, \dots, v_n]) = 0$ for any n-simplex.

$$\partial_{n-1}(\partial_n[v_0, \dots, v_n]) = \partial_{n-1}\left(\sum_{i=0}^{n} (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_n]\right)$$

$$= \sum_{i=0}^{n} (-1)^i \partial_{n-1}([v_0, \dots, \hat{v}_i, \dots, v_n])$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{i-1} (-1)^i (-1)^j [v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_n]$$

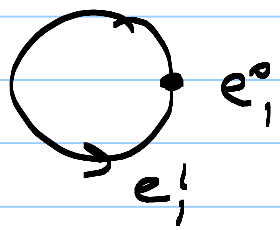
$$+ \sum_{i=0}^{n} \sum_{j=i+1}^{n} (-1)^i (-1)^{j-1} [v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_n]$$

$$= \sum_{j<i} (-1)^{i+j} [v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_n]$$
$$+ \sum_{i<j} (-1)^{i+j-1} [v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_n]$$

$$= \sum_{j<i} (-1)^{i+j} [v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_n]$$
$$+ \sum_{j<i} (-1)^{i+j-1} [v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_n]$$

$$= 0.$$

<u>Examples:</u> 1) $X = S^1 =$



$e_1^0 = [v_0]$

$e_1^1 = [v_0, v_1]$

$C_0(X) = \mathbb{Z}[e_1^0] \simeq \mathbb{Z}$

$C_1(X) = \mathbb{Z}[e_1^1] \simeq \mathbb{Z}$

$\partial e_1^1 = \partial [v_0, v_1] = [v_1] - [v_0]$

$$\begin{array}{ccccccc}
0 = C_2(X) & \longrightarrow & C_1(X) & \longrightarrow & C_0(X) & \longrightarrow & 0 \\
 & & \wr\| & & \wr\| & & \\
 & & \mathbb{Z} & & \mathbb{Z} & & \\
 & & \| & & \| & & \\
 & & \langle e_1^1 \rangle & & \langle e_1^0 \rangle & &
\end{array}$$

$[v_0, v_1] \longmapsto [v_1] - [v_0] = [v_0] - [v_0] = 0$

$$0 \xrightarrow{\partial_2} \overset{C_1}{\overset{\shortparallel}{\mathbb{Z}}} \xrightarrow{\partial_1 = 0} \overset{C_2}{\overset{\shortparallel}{\mathbb{Z}}} \xrightarrow{\partial_0} 0$$

$$H_1(S^1) = \frac{\ker \partial_1}{\operatorname{Im} \partial_2} = \frac{\mathbb{Z}}{(0)} \simeq \mathbb{Z}$$

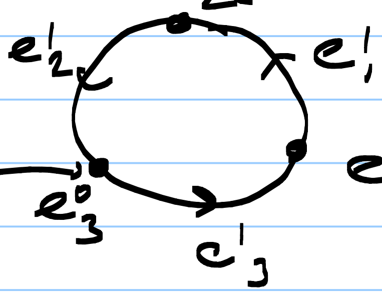
$$H_0(S^1) = \frac{\ker \partial_0}{\operatorname{Im} \partial_1} = \frac{\mathbb{Z}}{(0)} \simeq \mathbb{Z}$$

Since $C_k(S^1) = 0$ if $k \geq 2$, $H_k(S^1) = 0$.

$$H_k(S^1) \simeq \begin{cases} \mathbb{Z} & k = 0, 1 \\ 0 & k \geq 2. \end{cases}$$

$e_2^0 \leftrightarrow \sigma_2^0 \leftrightarrow (0,1,0)$

Example 2



$(0,0,1) \leftrightarrow \sigma_3^0 \leftrightarrow e_3^0$ $\qquad$ $e_1^0 \leftrightarrow \sigma_1^0 \leftrightarrow (1,0,0)$

$$\sigma_i^0 : [v_0] \longrightarrow S^1 \quad i = 1, 2, 3$$

$$\sigma_i^1 : [v_0 ; v_1] \longrightarrow S^1, \quad i = 1, 2, 3$$

$$C_k = 0 \text{ if } k \geq 2, \quad C_1 = \underset{\langle \sigma_1^1 \rangle}{\underset{\shortparallel}{\mathbb{Z}}} \oplus \underset{\langle \sigma_2^1 \rangle}{\underset{\shortparallel}{\mathbb{Z}}} \oplus \underset{\langle \sigma_3^1 \rangle}{\underset{\shortparallel}{\mathbb{Z}}}$$

$$C_0 = \underset{\langle \sigma_1^0 \rangle}{\underset{\shortparallel}{\mathbb{Z}}} \oplus \underset{\langle \sigma_2^0 \rangle}{\underset{\shortparallel}{\mathbb{Z}}} \oplus \underset{\langle \sigma_3^0 \rangle}{\underset{\shortparallel}{\mathbb{Z}}}$$

$$0 = C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

$$C_1 = \mathbb{Z}^3, \qquad C_0 = \mathbb{Z}^3$$

$\sigma_1^1 = (1,0,0)$, $\sigma_2^1 = (0,1,0)$, $\sigma_3^1 = (0,0,1)$

$\partial_1 \sigma_1^1 = (0,1,0) - (1,0,0)$

$\partial_1 \sigma_2^1 = (0,0,1) - (0,1,0)$

$\partial_1 \sigma_3^1 = (1,0,0) - (0,0,1)$

$$0 \to \mathbb{Z}^3 \xrightarrow[\partial_1]{} \mathbb{Z}^3 \xrightarrow[\partial_0]{} 0$$

$$\partial_1 = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\ker \partial_1 \simeq \mathbb{Z} = \langle \sigma_1^1 + \sigma_2^1 + \sigma_3^1 \rangle$

$\operatorname{Im} \partial_1 \simeq \langle (-1,1,0), (0,-1,1), (1,0,-1) \rangle$

$H_k(S^1) = 0$ if $k \geq 2$.

$$H_1(S^1) = \frac{\ker \partial_1}{\operatorname{Im} \partial_2} \simeq \frac{\mathbb{Z}}{(0)} \simeq \mathbb{Z}\langle \sigma_1^1 + \sigma_2^1 + \sigma_3^1 \rangle.$$

$$H_0(S^1) = \frac{\ker \partial_0}{\operatorname{Im} \partial_1} = \frac{\mathbb{Z}^3}{\operatorname{Im} \partial_1} = \frac{\langle (-1,\cancel{1},0), (0,\cancel{-1},1), (0,0,1) \rangle}{\langle (-1,\cancel{1},0), (0,\cancel{-1},1) \rangle}$$

$$\simeq \langle \overline{(0,0,1)} \rangle = \langle \overline{(0,1,0)} \rangle$$
$$= \langle \overline{(1,0,0)} \rangle$$

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Definition: The group $\ker(\partial_k : C_k(X) \to C_{k-1}(X))$ is called the group of k-cycles of X.

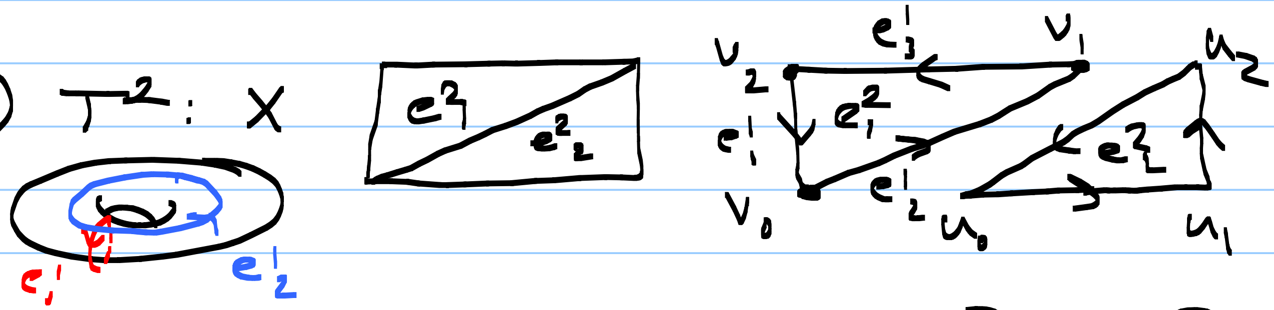
Similarly, the group $\operatorname{Im}(\partial_{k+1} : C_{k+1}(X) \to C_k(X))$ is called the group of k-boundaries of X.

Notation: $Z_k(X) = \ker \partial_k \subseteq C_k(X)$

$B_k(X) = \operatorname{Im} \partial_{k+1} \subseteq C_k(X)$

$B_k(X) \subseteq Z_k(X)$ and $H_k(X) = \dfrac{Z_k(X)}{B_k(X)}$.

3) T^2 : X



$\partial e_1^2 = \partial[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$

$\partial e_2^2 = \partial[u_0, u_1, u_2] = [u_1, u_2] - [u_0, u_2] + [u_0, u_1]$

$\partial(e_1^2 - e_2^2) = 0$.

For any 1-simplex e_i^1, $\partial e_i^1 = 0$ because there is only one 0-simplex.

$$0 = C_3(X) \xrightarrow{\partial_3} C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1 = 0} \underset{\substack{\cong \\ \mathbb{Z}}}{C_0(X)} \xrightarrow{\partial_0} 0$$

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$$0 \to C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1 = 0} C_0(X) \to 0$$

$$C_2(X) \cong \mathbb{Z}\oplus\mathbb{Z} \cong \langle \sigma_1^2, \sigma_2^2 \rangle \qquad C_1(X) \cong \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$$

$$\partial_2 \sigma_1^2 = \sigma_1^1 + \sigma_2^1 + \sigma_3^1 = \partial_2 \sigma_2^2$$

$$\sigma_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad \ker \sigma_2 = \langle \sigma_1^2 - \sigma_2^2 \rangle = \langle (1,-1) \rangle \cong \mathbb{Z}$$

$$\operatorname{Im} \sigma_2 = \langle (1,1,1) \rangle$$

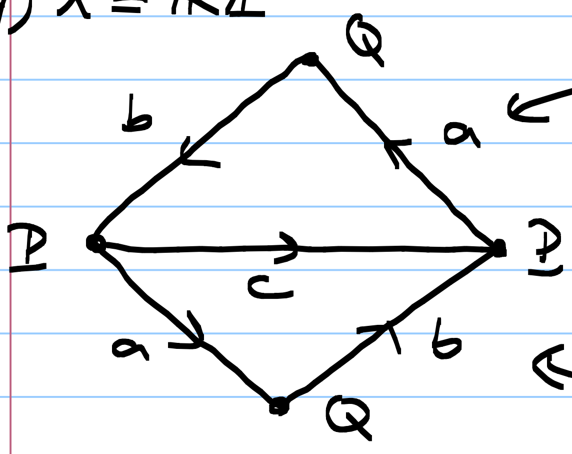
$$H_2(T^2) = \frac{\ker \sigma_2}{\operatorname{Im} \sigma_3} = \frac{\mathbb{Z}}{0} = \mathbb{Z} = \langle \sigma_1^2 - \sigma_2^2 \rangle$$

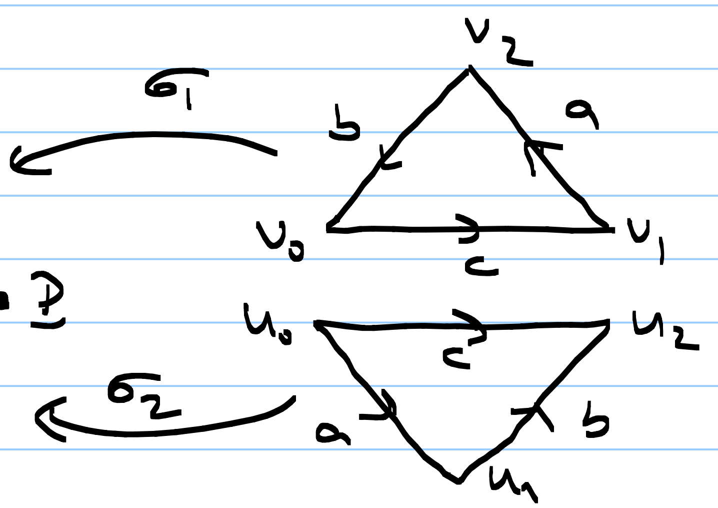
$$H_1(T^2) = \frac{\ker \sigma_1}{\operatorname{Im} \sigma_2} = \frac{\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}}{\langle (1,1,1) \rangle} = \frac{\langle (1,0,0), (0,1,0), \cancel{(1,1,1)} \rangle}{\langle \cancel{(1,1,1)} \rangle}$$

$$\cong \mathbb{Z}\oplus\mathbb{Z}$$

$$H_0(T^2) = \frac{\ker \partial_0}{\operatorname{Im} \partial_1} = \frac{\mathbb{Z}}{(0)} \cong \mathbb{Z}.$$

4) $X = \mathbb{RP}^2$





$\sigma_1 = [v_0, v_1, v_2]$, $\partial\sigma_1 = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$

$= a + b + c$

$\sigma_2 = [u_0, u_1, u_2]$, $\sigma_2 = [u_1, u_2] - [u_0, u_2] + [u_0, u_1]$

$= b - c + a$

$\partial a = Q - P$, $\partial b = P - Q$, $\partial c = P - P = 0$.

$$\begin{array}{ccccccccc}
0 & \xrightarrow{\partial_3} & C_2(X) & \xrightarrow{\partial_2} & C_1(X) & \xrightarrow{\partial_1} & C_0(X) & \xrightarrow{\partial_0} & 0 \\
 & & \| & & \| & & \| & & \\
 & & \mathbb{Z} \oplus \mathbb{Z} & & \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} & & \mathbb{Z} \oplus \mathbb{Z} & & \\
 & & \langle\sigma_1\rangle \quad \langle\sigma_2\rangle & & \langle a\rangle \quad \langle b\rangle \quad \langle c\rangle & & \langle P\rangle \quad \langle Q\rangle & & \\
 & & \begin{pmatrix}1\\0\end{pmatrix} \begin{pmatrix}0\\1\end{pmatrix} & & \begin{pmatrix}1\\0\\0\end{pmatrix} \begin{pmatrix}0\\1\\0\end{pmatrix} \begin{pmatrix}0\\0\\1\end{pmatrix} & & \begin{pmatrix}1\\0\end{pmatrix} \begin{pmatrix}0\\1\end{pmatrix} & &
\end{array}$$

$\partial_2\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}$, $\partial_2\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\1\\-1\end{pmatrix} \Rightarrow \partial_2 = \begin{pmatrix}1 & 1\\1 & 1\\1 & -1\end{pmatrix}$

$\partial_1\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix}$, $\partial_1\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix}$, $\partial_1\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

$\partial_1 = \begin{pmatrix}-1 & 1 & 0\\1 & -1 & 0\end{pmatrix}$.

rank $\partial_2 = 2$ and hence $\ker\partial_2 = 0$.

$\mathrm{Im}\,\partial_2 = \left\langle \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\-1\end{pmatrix} \right\rangle$

$\ker\partial_1 = \left\langle \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\rangle$, $\mathrm{Im}\,\partial_1 = \left\langle \begin{pmatrix}1\\-1\end{pmatrix} \right\rangle$.

$$H_2(\mathbb{RP}^2) = \frac{\ker \partial_2}{\operatorname{Im} \partial_3} = \frac{(0)}{(0)} = (0)$$

$$H_1(\mathbb{RP}^2) = \frac{\ker \partial_1}{\operatorname{Im} \partial_2} = \frac{\left\langle \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\rangle}{\left\langle \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\-1\end{pmatrix} \right\rangle} = \frac{\left\langle \cancel{\begin{pmatrix}1\\1\\1\end{pmatrix}}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\rangle}{\left\langle \cancel{\begin{pmatrix}1\\1\\1\end{pmatrix}}, \begin{pmatrix}0\\0\\2\end{pmatrix} \right\rangle}$$

$$\simeq \mathbb{Z}/2\mathbb{Z} \simeq \mathbb{Z}_2 .$$

$$H_0(\mathbb{RP}^2) = \frac{\ker \partial_0}{\operatorname{Im} \partial_1} = \frac{\left\langle \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\rangle}{\left\langle \begin{pmatrix}-1\\1\end{pmatrix} \right\rangle} = \frac{\left\langle \cancel{\begin{pmatrix}-1\\1\end{pmatrix}}, \begin{pmatrix}0\\1\end{pmatrix} \right\rangle}{\left\langle \cancel{\begin{pmatrix}-1\\1\end{pmatrix}} \right\rangle}$$

$$\simeq \mathbb{Z} .$$

$$H_k(\mathbb{RP}^2) = \begin{cases} \mathbb{Z}_2 ; & k=1 \\ \mathbb{Z} ; & k=0 \\ 0 ; & k \geq 2 \end{cases}$$

<u>Remark:</u> Although delta (or simplicial) homology is relatively easy to compute they are <u>not</u> functorial. It is functorial only under simplicial maps.

Therefore, we'll define a more universal homology theory for topological spaces.

Singular Homology:

A singular n-simplex in a space X is just a map $\sigma : \Delta^n \to X$. Let $C_n(X)$ denote the free abelian group with basis the set of all singular n-simplicies in X. Elements of $C_n(X)$ will be called n-chains in X. Hence, n-chains in X are just formal sums of the form

$$\sum_i n_i \sigma_i \ , \ n_i \in \mathbb{Z} \ , \ \sigma_i : \Delta^n \to X \text{ maps,}$$

where all but finitely many n_i are zero.

For a simplex $\sigma : \Delta^n \to X$ its boundary is defined as

$$\partial \sigma = \sum (-1)^i \sigma|_{[v_0, \ldots, \hat{v}_i, \ldots, v_n]} \ , \text{ where}$$

$\Delta^n = [v_0, \ldots, v_i, \ldots, v_n]$.

We've seen that $\partial^2 [v_0, \ldots, v_n] = 0$ and this implies $\partial^2 \sigma = 0$ for all $\sigma \in C_n(X)$.

Hence $\operatorname{Im}(\partial_{n+1} : C_{n+1}(X) \to C_n(X))$

$$\subseteq \ker(\partial_n : C_n(X) \to C_{n-1}(X)).$$

So, we may define the n^{th} singular homology of X as the quotient group

$$H_k(X) = \frac{\ker \partial_n}{\operatorname{Im} \partial_{n+1}}.$$

As before elements of $\ker \partial_n$ are called singular n-cycles in X and denoted as

$$Z_n(X) = \ker \partial_n,$$ and elements of $\operatorname{Im} \partial_{n+1}$

are called singular n-boundaries in X and denoted as

$$B_n(X) = \operatorname{Im} \partial_{n+1}.$$

So, $$H_n(X) = \frac{Z_n(X)}{B_n(X)}.$$

<u>Remark</u>: Let $\alpha = \sum n_i \sigma_i$ be an n-cycle. So

$\partial \alpha = 0$. Writing σ_i's more than once we may assume that each nonzero n_i equals ± 1.

So, $$\alpha = \sum_{\text{finite}} \epsilon_i \sigma_i \qquad \epsilon_i = \pm 1.$$

$$0 = \partial \alpha = \sum \epsilon_i \, \partial \sigma_i$$

By gluing σ_i's as dictated by the relation

$\sum_i \epsilon_i \partial \sigma_i = 0$ along the boundaries of the

simplices σ_i's we obtain a continuous map from a topological n-manifold into X where its restriction to each simplex is σ_i.

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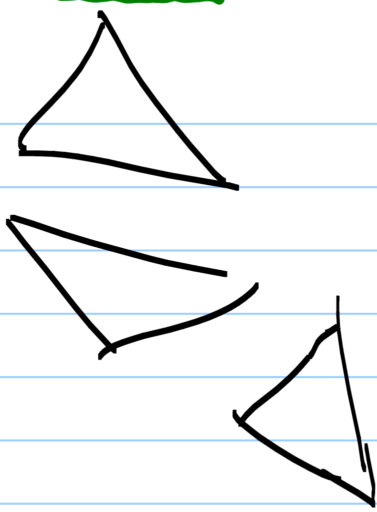
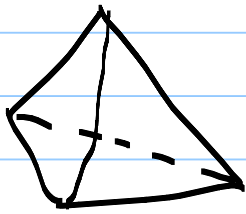
$\sigma_1 : [v_0, \ldots, v_n] \longrightarrow X$

$\sigma_2 : [u_0, \ldots, u_n] \longrightarrow X$

$\sigma_3 : [w_0, \ldots, w_n] \longrightarrow X$

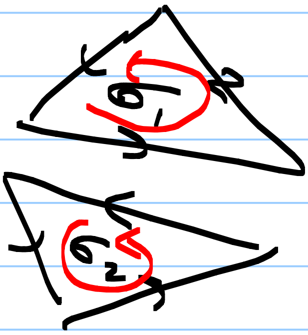




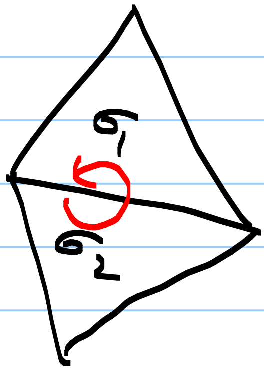
The resulting topological n-manifold is denoted as K_α.

In particular, any 1-cycle is represented by a map from S^1, the only 1-dim'l compact manifold.

Similarly, any 2-cycle is represented by a map from an closed orientable surface without boundary.



$\sigma_1 + \sigma_2$



So K_α cannot be $\mathbb{RP}^2$, KB or any other closed non orientable surface.

Proposition: Corresponding to the decomposition of a space X into its path components X_α there is an isomorphism of $H_n(X)$ with the direct sum $\bigoplus_\alpha H_n(X_\alpha)$.

Proof: $X = \dot{\bigcup_\alpha} X_\alpha$, X_α path connected.

Δ^n is pathconnected and thus any singular simplex $\sigma : \Delta^n \to X$ has image in exactly one X_α.

So $C_n(X) \simeq \bigoplus_\alpha C_n(X_\alpha)$. Moreover, ∂_n respects this decomposition: $\sigma : \Delta_n \to X_\alpha \subseteq X \Rightarrow$

$\partial\sigma : \partial\Delta_n \to X_\alpha$ and thus

$$\begin{array}{ccccc} C_n(X) & \simeq & \bigoplus_\alpha & C_n(X_\alpha) & \\ \downarrow \partial_n & & \downarrow \oplus\partial_n & \downarrow \partial_n & \\ C_{n-1}(X) & \simeq & \bigoplus_\alpha & C_{n-1}(X_\alpha) & \end{array}$$

Hence, $Z_n(X) \simeq \bigoplus_\alpha Z_n(X_\alpha)$ and $B_n(X) \simeq \bigoplus_\alpha B_n(X_\alpha)$

so that $H_n(X) = \dfrac{Z_n(X)}{B_n(X)} = \dfrac{\bigoplus_\alpha Z_n(X_\alpha)}{\bigoplus_\alpha B_n(X_\alpha)} \simeq \bigoplus_\alpha \left(\dfrac{Z_n(X_\alpha)}{B_n(X_\alpha)}\right)$

$\Rightarrow H_n(X) \simeq \bigoplus_\alpha H_n(X_\alpha)$. ∎

Proposition: If X is nonempty and path-connected then $H_0(X) \approx \mathbb{Z}$. Hence, for any space X, $H_0(X)$ is a direct sum of $\mathbb{Z}$'s, one for each path connected component of X.

Proof: By definition $H_0(X) = \dfrac{C_0(X)}{\operatorname{Im} \partial_1}$.

Define a homomorphism

$$\epsilon : C_0(X) \longrightarrow \mathbb{Z} \text{ by } \epsilon\Big(\sum_i n_i \sigma_i\Big) = \sum_i n_i .$$

ϵ is onto since $\epsilon(\sigma) = 1$, for any 0-simplex

$\sigma : [v_0] \longrightarrow X$.

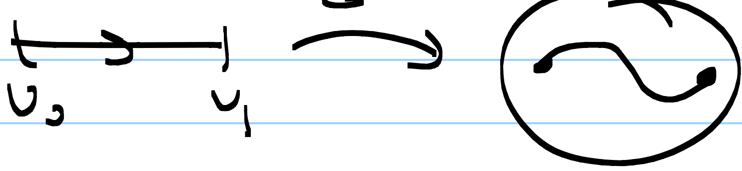
__Claim:__ $\ker \epsilon = \operatorname{Im} \partial_1$.

Note that the above claim proves the proposition:

$$H_0(X) = \frac{C_0(X)}{\operatorname{Im}\partial_1} = \frac{C_0(X)}{\ker \epsilon} \simeq \operatorname{Im} \epsilon = \mathbb{Z}.$$

Proof of the claim: __$\operatorname{Im}\partial_1 \subseteq \ker\epsilon$__

$\sigma : [v_0, v_1] \longrightarrow X$, $\partial_1 \sigma = \sigma|_{[v_1]} - \sigma|_{[v_0]}$



$$\epsilon(\partial_1(\sigma)) = \epsilon\big(\sigma|_{[v_1]} - \sigma|_{[v_0]}\big) = 1 - 1 = 0.$$

Hence, $\epsilon \circ \partial_1 = 0$ on $C_0(X)$, because it is zero at each basis element.

For the other direction, suppose that

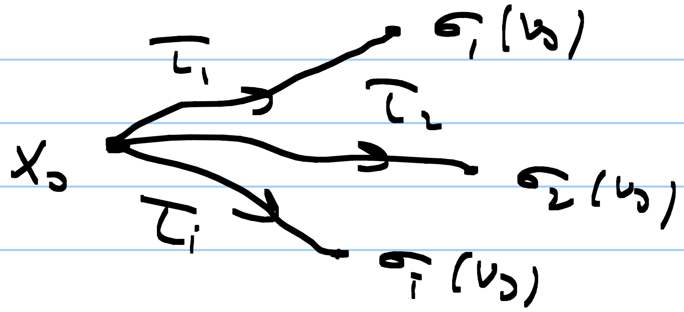
$\epsilon(\sum n_i \sigma_i) = 0$ for some 0-chain $\sum n_i \sigma_i$.

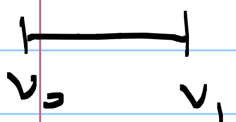
Hence $\sum n_i = 0$. The σ_i's are singular 0-simplices, which are points of X.

$\sigma_i : [v_0] \longrightarrow X$.

Choose paths $\tau_i : I \longrightarrow X$ from a base point x_0 to $\sigma_i(v_0)$ and let σ_0 be the singular 0-simplex with image x_0.

$\sigma_0 : [v_0] \longrightarrow X$, $\sigma_0(v_0) = x_0$.





We can view each τ_i as a singular 1-simplex

$\tau_i : [v_0, v_1] \longrightarrow X$, $\tau_i(v_0) = x_0$, $\tau_i(v_1) = \sigma_i(v_0)$.

So, $\partial \tau_i = \sigma_i - \sigma_0$, for all i.

Finally, $\partial(\sum_i n_i \tau_i) = \sum_i n_i \, \partial(\tau_i)$

$$\Rightarrow \partial\left(\sum_i n_i \tau_i\right) = \sum_i n_i(\sigma_i - \sigma_0)$$

$$= \sum_i n_i \sigma_i - \sum_i n_i \sigma_0$$

$$= \sum_i n_i \sigma_i - \underbrace{\left(\sum_i n_i\right)}_{=0} \sigma_0$$

$$= \sum_i n_i \sigma_i$$

Hence, $\sum_i n_i \sigma_i \in \operatorname{Im}\partial_1$. $\Rightarrow \ker \epsilon \subseteq \operatorname{Im}\partial_1$.

$\therefore \ker \epsilon = \operatorname{Im}\partial_1$. This proves the claim.

<u>Proposition</u>: If X is a point, then $H_n(X) = 0$ for all $n > 0$ and $H_0(X) \simeq \mathbb{Z}$.

<u>Proof</u>: Any n-singular n-simplex $\sigma_n : \Delta^n \to X = \{x_0\}$ is the constant map.

Thus $C_n(X) = \langle \sigma_n \rangle \simeq \mathbb{Z}$.

$$C_n(X) \xrightarrow{\partial_n} C_{n-1}(X)$$

$$\partial_n(\sigma_n) = \sum_{i=0}^{n} (-1)^i \underbrace{\sigma_n|_{[v_0, \ldots, \hat{v}_i, \ldots, v_n]}}_{\sigma_{n-1}} = \begin{cases} \sigma_{n-1} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

provided that $n \geq 1$.

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$$0 \to C_{n+1} \xrightarrow[\simeq]{\partial_{n+1}} C_n \xrightarrow[=0]{\partial_n} C_{n-1} \xrightarrow[\simeq]{} \cdots$$

$$C_{n+1} \cong \mathbb{Z}, \quad C_n \cong \mathbb{Z}, \quad C_{n-1} \cong \mathbb{Z}$$

$$H_n(X) = \frac{\ker \partial_n}{\operatorname{Im} \partial_{n+1}} = \frac{\mathbb{Z}}{\mathbb{Z}} = (0) \qquad \text{if } n > 0.$$

$$H_{n+1}(X) = \frac{\ker \partial_{n+1}}{\operatorname{Im} \partial_{n+2}} = \frac{(0)}{(0)} = (0).$$

This finishes the proof. ∎

Reduced Homology Groups: $\tilde{H}_n(X)$.

$$\cdots \to C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \to \cdots \xrightarrow{\partial_1} C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

If $n \geq 1$, $\tilde{H}_n(X) = \dfrac{\ker \partial_n}{\operatorname{Im} \partial_{n+1}} = H_n(X)$

$$H_0(X) = \frac{\ker \partial_0}{\operatorname{Im} \partial_1} = \frac{C_0(X)}{\operatorname{Im} \partial_1} \simeq \tilde{H}_0(X) \oplus \mathbb{Z}$$

Proof: $\epsilon : C_0(X) \to \mathbb{Z}$, $\epsilon(\sum_i n_i \sigma_i) = \sum n_i$.

$\ker \epsilon = \operatorname{Im} \partial_1$ if X is connected.

$X = \dot\bigcup X_\alpha$ $\quad$ $p_\alpha \in X_\alpha$

$$\begin{tikzcd}
C_0(X) = \bigoplus_\alpha C_0(X_\alpha) \arrow[r, "\epsilon"] \arrow[d] & \mathbb{Z} & 1 \\
\bigoplus_\alpha H_0(X_\alpha) \arrow[ur, "\epsilon"'] & {[p_\alpha]} \arrow[ur, mapsto] &
\end{tikzcd}$$

$$H_0(X_\alpha) \overset{\overline{\epsilon}}{\simeq} \mathbb{Z}$$

$$\tilde{H}_0(X) = \frac{\ker \epsilon}{\operatorname{Im} \partial_1} = \ker\left(\overline{\epsilon} : \bigoplus_{\alpha} \mathbb{Z}_\alpha \to \mathbb{Z}\right)$$

$$\mathbb{Z}_\alpha = \langle [P_\alpha] \rangle, \quad [P_\alpha] \mapsto 1$$

Fix some α_0.

$$= \langle [P_\alpha] - [P_{\alpha_0}] \mid \alpha \in \Lambda, \alpha \neq \alpha_0 \rangle$$

= free abelian group of rank one less than rank of $|\Lambda|$.

$$0 \to \tilde{H}_0(X) \to H_0(X) \xrightarrow{\overline{\epsilon}} \mathbb{Z} \to 0$$

$$\Rightarrow H_0(X) \simeq \tilde{H}_0(X) \oplus \mathbb{Z}.$$

Homotopy Invariance:

Let $f: X \to Y$ be a continuous map. Then f induces a homomorphism on the chain level as follows:

$f_\#: C_n(X) \to C_n(Y)$ defined by

$$f_\#\left(\sum_i n_i \sigma_i\right) = \sum_i n_i (f \circ \sigma_i).$$

$\sigma_i : \Delta^n \to X$ cont. map

$$\begin{array}{ccc} C_n(X) & \xrightarrow{f_\#} & C_n(Y) \\ \partial_n \downarrow & & \downarrow \partial_n \\ C_{n-1}(X) & \xrightarrow[f_\#]{} & C_{n-1}(Y) \end{array}$$ is commutative.

$$(\partial_n \circ f_\#)(\sigma) = \partial_n(f\circ\sigma) = \sum_i (-1)^i f\circ\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}$$

$f\circ\sigma : \Delta^n \to Y$

$\Delta^n = [v_0, \dots, v_n]$

$$f_\#(\partial_n \sigma) = f_\#\Big(\sum_i (-1)^i \sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}\Big)$$
$$= \sum_i (-1)^i (f\circ\sigma)|_{[v_0,\dots,\hat{v}_i,\dots,v_n]})$$

Hence, $f_\# \circ \partial_n = \partial_n \circ f_\#$.

If $\sum n_i \sigma_i \in Z_n(X)$, then

$$\partial_n(f_\#(\sum n_i\sigma_i)) = f_\#(\underbrace{\partial_n(\sum n_i \sigma_i)}_{=0}) = 0$$

and hence, $f_\#(\sum n_i \sigma_i) \in Z_n(Y)$.

Similarly, $f_\#(\partial_n(\sum n_i\sigma_i)) = \partial_n(f_\#(\sum n_i \sigma_i))$

implies that $f_\#(B_n(X)) \subseteq B_n(Y)$.

$$f_\#: C_n(X) \to C_n(Y)$$
$$f_\#(Z_n(X)) \subseteq Z_n(Y)$$
$$f_\#(B_n(X)) \subseteq B_n(Y)$$

$$\begin{tikzcd}
Z_n(X) \arrow[r, "f_\#"] \arrow[d] \arrow[dr] & Z_n(Y) \arrow[d] \\
Z_n(X)/B_n(X) \arrow[r, dashed, "f_*"] \arrow[d, phantom, "\shortparallel"] & Z_n(Y)/B_n(Y) \arrow[d, phantom, "\shortparallel"] \\
H_n(X) \arrow[r, "f_*"'] & H_n(Y)
\end{tikzcd}$$

So, any continuous map $f: X \to Y$ induces a homomorphism
$$f_*: H_n(X) \to H_n(Y).$$

Proposition: If $f: X \to Y$ and $g: Y \to Z$ are continuous maps then we have
$$(f \circ g)_* = f_* \circ g_*.$$

Moreover, the identity map $id_X: X \to X$ induces identity homomorphism
$$(id_X)_* = id_{H_n(X)}: H_n(X) \to H_n(X).$$
Proof is left as an exercise.

Theorem: If two maps $f, g: X \to Y$ a homotopic then $f_* = g_*: H_n(X) \to H_n(Y)$

Corollary: If $f: X \to Y$ is a homotopy equivalence that f_* is an isomorphism.

Proof of the Corollary: Let $g: Y \to X$ be a homotopy inverse for f. Then $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$. By the above theorem $(id_X)_* = (g \circ f)_* \Rightarrow id = g_* \circ f_*$, and $(id_Y)_* = (f \circ g)_* \Rightarrow id = f_* \circ g_*$.

$$
\begin{tikzcd}
H_n(X) \arrow[r, shift left, "f_*"] & H_n(Y) \arrow[l, shift left, "g_*"]
\end{tikzcd}
$$

Hence, f_* is an isomorphism.

Corollary If f is homotopic to the constant map then $f_* = 0$, if $n \geq 1$.

Proof: Assume that f is homotopic to the constant map $g: X \to Y$, $g(X) = y_0$.

Then $f_*: H_n(X) \to H_n(Y)$

is equal to the homomorphism induced by the composition

$$
\begin{tikzcd}
X \arrow[r, "g_0"] \arrow[rr, bend right, "g"'] & \{y_0\} \arrow[r, "i"] & Y
\end{tikzcd}
\qquad g = i \circ g_0
$$

$$f_* = g_* = (i \circ g_0)_* = i_* \circ g_{0*}$$

$$g_{0*} : H_n(X) \longrightarrow H_n(\{y_0\}) = (0) \text{ if } n \geq 1.$$

Hence, $f_* = 0$ if $n \geq 1$. ∎

<u>Corollary</u> If X is a contractible space then $\tilde{H}_n(X) = (0)$ for all n.

<u>Proof</u> $id : X \to X$ is homotopic to the constant map.

So $id_* : H_n(X) \longrightarrow H_n(X)$ is trivial if $n \geq 1$.

Hence, $H_n(X) = (0)$ if $n \geq 1$. $\Rightarrow \tilde{H}_n(X) = (0)$

For $n=0$, note that X is connected. So

$H_0(X) \simeq \mathbb{Z}$ and $\tilde{H}_0(X) = (0)$, because we have

$H_0(X) \simeq \tilde{H}_0(X) \oplus \mathbb{Z}$. ∎

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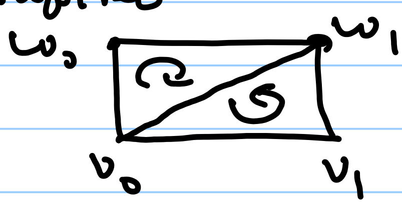
Proof of the Theorem (Homotopy Invariance):

$f, g: X \to Y$ homotopic, so there is a homotopy

$F: X \times I \to Y$, $F(x,0) = f(x)$ and $F(x,1) = g(x)$.

$\Delta^n \times I$ = union of $n+1$-simplices



<u>Ex</u> $\Delta^1 \times I = [v_0, v_1] \times I$

$= [v_0, v_1, w_1] \cup [v_0, w_0, w_1]$

In general, if $\Delta^n = [v_0, \dots, v_n]$ then we have

$$\Delta^n \times I = \bigcup_{i=0}^{n} [v_0, \dots, v_i, w_i, \dots, w_n], \text{ where}$$

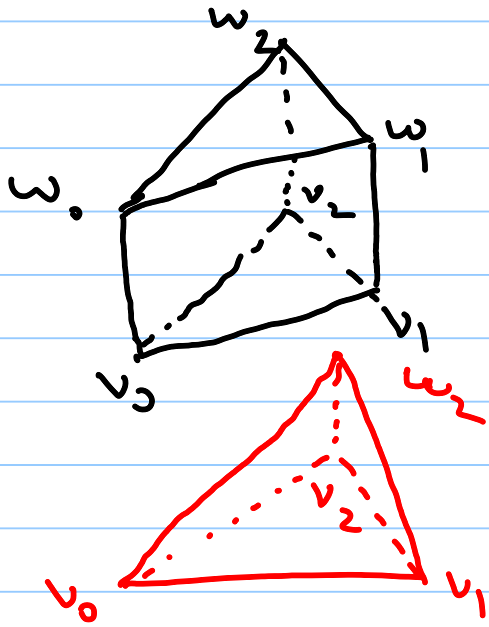
$\Delta^n \times \{0\} = [v_0, \dots, v_n]$, $\Delta^n \times \{1\} = [w_0, \dots, w_n]$.

$\Delta^2 \times I = [v_0, v_1, v_2] \times I$

$= [v_0, v_1, v_2, w_2]$

$\cup [v_0, v_1, w_1, w_2]$

$\cup [v_0, w_0, w_1, w_2]$



Using the above setup we define so called the Prism Operator

$$P : C_n(X) \longrightarrow C_{n+1}(Y) \quad \text{by}$$

$$P(\sigma) = \sum_i (-1)^i \, F \circ (\sigma \times id)|_{[v_0, \dots, v_i, w_i, \dots, w_n]}$$

$$\sigma : \Delta^n \to X, \quad \Delta^n \times I \xrightarrow[\sigma \times id]{} X \times I \xrightarrow[F]{} Y$$

<u>Claim:</u> $\partial P = g_\# - f_\# - P\partial$

$$\begin{tikzcd}
C_{n+1}(X) \arrow[r, "\partial"] \arrow[d] & C_n(X) \arrow[r, "\partial"] \arrow[d] \arrow[dl, "P"'] & C_{n-1}(X) \arrow[d] \arrow[dl, "P"'] \\
C_{n+1}(Y) \arrow[r, "\partial"'] & C_n(Y) \arrow[r, "\partial"'] & C_{n-1}(Y)
\end{tikzcd}$$

$$\partial P, \; P\partial : C_n(X) \longrightarrow C_n(Y)$$

<u>Proof of the claim:</u>

$$\partial P(\sigma) = \sum_{j \leq i} (-1)^i (-1)^j F \circ (\sigma \times id)|_{[v_0, \dots, \hat{v}_j, \dots, v_i, w_i, \dots, w_n]}$$

$$+ \sum_{j \geq i} (-1)^i (-1)^{j+1} F \circ (\sigma \times id)|_{[v_0, \dots, v_i, w_i, \dots, \hat{w}_j, \dots, w_n]}$$

The terms $i = j$ in the two summations cancel except for $F \circ (\sigma \times id)|_{[\hat{v}_0, w_0, \dots, w_n]}$

which is $g \circ \sigma = g_\#(\sigma)$, and $-F \circ (\sigma \times \mathrm{id})|_{[w_0, \dots, w_n]}$

which is $-f \circ \sigma = -f_\#(\sigma)$.

The terms with $i \neq j$ adds up to exactly

$-P\partial(\sigma)$ since,

$$P\partial(\sigma) = \sum_{i<j} (-1)^i (-1)^j F \circ (\sigma \times \mathrm{id})|_{[v_0, \dots, v_i, w_i, \dots, \hat{w}_j, \dots, w_n]}$$

$$+ \sum_{i>j} (-1)^{i-1} (-1)^j F \circ (\sigma \times \mathrm{id})|_{[v_0, \dots, \hat{v}_j, \dots, v_i, w_i, \dots, w_n]}$$

Hence, $\partial P + P\partial = g_\# - f_\#$.

Finishing the proof:

must show $f_\# = g_\# : H_n(X) \to H_n(Y)$.

Let $\sum n_i \sigma_i \in Z_n(X)$. Then

$$f_\#\left(\sum n_i \sigma_i\right) - g_\#\left(\sum n_i \sigma_i\right) = (\partial P + P\partial)\left(\sum n_i \sigma_i\right)$$

$$= \underbrace{\partial\left(P\left(\sum n_i \sigma_i\right)\right)}_{\in B_n(Y)} + \partial\Big(\underbrace{\partial\left(\sum n_i \sigma_i\right)}_{= 0}\Big)$$

So $f_\#(\alpha) = g_\#(\alpha)$, where $\alpha = \left[\sum n_i \sigma_i\right] \in H_n(X)$.

This finishes the proof of the theorem.

Exact Sequences and Excision:

An exact sequence of groups, rings or modules is a sequence of the form

$$(A_*) \quad \cdots \to A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \to \cdots,$$

where A_n's are algebraic objects and α_n's are morphisms satisfying $\ker \alpha_n = \operatorname{Im} \alpha_{n+1}$, for all n.

If $\operatorname{Im} \alpha_{n+1} \subseteq \ker \alpha_n$ for all n, then we call A_* a chain complex.

If $A_* = (A_n, \alpha_n)$ is a chain complex then the n^{th} homology is defined as

$$H_n(A_*) = \frac{\ker \alpha_n}{\operatorname{Im} \alpha_{n+1}}.$$

<u>Remark:</u> i) $0 \to A \xrightarrow{\alpha} B$ is exact iff $\ker \alpha = 0$, i.e, α is injective.

ii) $A \xrightarrow{\alpha} B \to 0$ is exact iff $\operatorname{Im} \alpha = B$, i.e, α is surjective.

iii) $0 \to A \xrightarrow{\alpha} B \to 0$ is exact iff α is an isomorphism.

iv) $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is exact iff

α is injective, β is surjective and $\ker \beta = \operatorname{Im} \alpha$.

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Example: 1) $0 \to \underset{\substack{\| \\ A}}{\mathbb{Z}} \xrightarrow{\times 2} \underset{\substack{\| \\ B}}{\mathbb{Z}} \to \underset{\substack{\| \\ C}}{\mathbb{Z}/2\mathbb{Z}} \to 0$

is exact.

2) $0 \to \overset{\substack{A \\ \|}}{\mathbb{Z}} \to \overset{\substack{B \\ \|}}{\mathbb{Z} \oplus \mathbb{Z}} \to \overset{\substack{C \\ \|}}{\mathbb{Z}} \to 0$ is exact

$n \longmapsto (n, 0)$

$(n, m) \longmapsto m$

In the second example $B \cong A \oplus C$ but not in the first example.

X topological space, $A \subseteq X$ subspace.

X/A quotient space

Aim: To obtain a relation among $H_n(X)$, $H_n(A)$ and $H_n(X/A)$.

Theorem: If X is a space and A is nonempty subspace of X that is a deformation retract of some neighborhood in X, then there is an exact sequence

$$\cdots \to \tilde{H}_n(A) \xrightarrow{i_\#} \tilde{H}_n(X) \xrightarrow{j_\#} \tilde{H}_n(X/A) \xrightarrow{\partial} \tilde{H}_{n-1}(A) \xrightarrow{i_\#} \tilde{H}_{n-1}(X) \to \cdots$$

$$\to \tilde{H}_0(X/A) \to 0,$$

where $i_\#$ and $j_\#$ are induced by the inclusion $i: A \to X$ and the quotient map $j: X \to X/A$.

Pairs of spaces (X,A) satisfying the hypothesis of the above theorem are called good pairs.

Remark: We'll see in the proof that an element $x \in \tilde{H}(X/A)$ is represented by a chain α in X with $\partial\alpha$ a cycle in A and the map

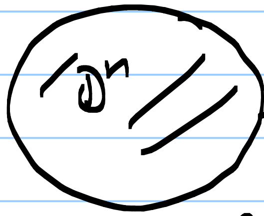
$$H_n(X/A) \xrightarrow{\partial} H_{n-1}(A)$$
$$x = [\alpha] \longmapsto \partial x = [\partial\alpha]$$

Remark: If X is CW-complex and $A \subseteq X$ a subcomplex then (X,A) is a good pair (Appendix 5)

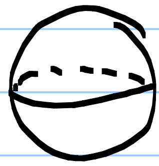
Corollary $\tilde{H}_n(S^n) \simeq \mathbb{Z}$ and $\tilde{H}_i(S^n) = 0$ if $i \neq n$.

Proof: Let $n > 0$. Take $(X,A) = (D^n, \partial D^n = S^{n-1})$

so that $X/A = S^n$.



$\tilde{H}_i(D^n) = 0$ since D^n is contractible.

$D^n/\partial D^n \simeq$ 

$$\cdots \to \tilde{H}_{i+1}(S^{n-1}) \to \underset{\overset{\parallel}{0}}{\tilde{H}_{i+1}(D^n)} \to \tilde{H}_{i+1}(D^n/S^{n-1}) \xrightarrow{\partial} \tilde{H}_i(S^{n-1}) \to \underset{\overset{\parallel}{0}}{\tilde{H}_i(D^n)}$$

$$0 \to \tilde{H}_{i+1}(D^n/\underset{S^n}{\overset{\wr}{S^{n-1}}}) \xrightarrow[\simeq]{\partial} \tilde{H}_i(S^{n-1}) \to 0.$$

Hence, $\partial : \widetilde{H}_{i+1}(S^n) \to \widetilde{H}_i(S^{n-1})$ is an isomorphism for all $i \geq 0$, $n \geq 1$.

$i = n \Rightarrow \widetilde{H}_n(S^n) \simeq \widetilde{H}_{n-1}(S^{n-1}) \simeq \cdots \simeq \widetilde{H}_0(S^0) \simeq \mathbb{Z}$

$$\left(S^0 = \{\pm 1\},\ H_0(S^0) = \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \quad \begin{matrix} 1 \mapsto 1 \\ 1 \mapsto 1 \end{matrix} \right)$$

$i < n$, $\widetilde{H}_i(S^n) \simeq \cdots \simeq \widetilde{H}_0(S^{n-i}) = 0$ because S^{n-i} is connected.

$i > n$, $\widetilde{H}_i(S^n) \simeq \cdots \simeq \widetilde{H}_{n-i}(S^0) = \underset{0}{H_{n-i}(\{1\})} \oplus \underset{0}{H_{n-i}(\{-1\})} = 0.$

This finishes the proof of the corollary. ■

<u>Corollary</u> $\partial D^n = S^{n-1}$ is <u>not</u> a retract of D^n.

Hence, every map $f : D^n \to D^n$ has a fixed point.

<u>Proof</u>: If $r : D^n \to \partial D^n = S^{n-1}$ is a retraction, then $r \circ i = Id_{S^{n-1}}$, where $i : \partial D^n = S^{n-1} \to D^n$, $x \mapsto x$

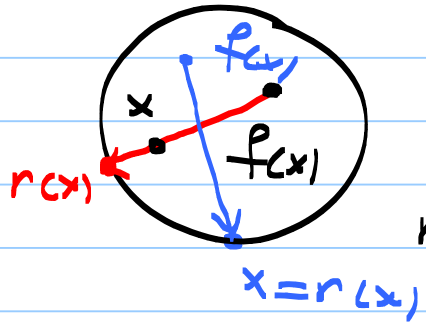
$$S^{n-1} \xrightarrow{i} D^n \xrightarrow{r} S^{n-1}$$

$$\Rightarrow \begin{array}{ccccc} \widetilde{H}_{n-1}(S^{n-1}) & \xrightarrow{i_\#} & \widetilde{H}_{n-1}(D^n) & \xrightarrow{r_\#} & \widetilde{H}_{n-1}(S^{n-1}) \\ \wr\mid & & \| & & \wr\mid \\ \mathbb{Z} & \longrightarrow & (0) & \longrightarrow & \mathbb{Z} \end{array}$$

so that $r_\# \circ i_\# = (r \circ i)_\# = (\mathrm{id}_{S^{n-1}})_\# = \mathrm{id}_{\tilde{H}_{n-1}(S^{n-1})}$.

This is a contradiction since $\tilde{H}_{n-1}(D^n) = \{0\}$.

For the second statement note that if $f: D^n \to D^n$ is a map without any fixed points then we obtain a retraction $r: D^n \to S^{n-1}$ as follows:



$\forall x,\ f(x) \neq x$

$D^n \subseteq \mathbb{R}^n$

$r: D^n \to \partial D^n = S^{n-1}$

$x \longmapsto r(x)$

(Exercise: Show that r is continuous)

Clearly $r(x) = x$ if $x \in \partial D^n = S^{n-1}$, so that r is a retraction. This proves that any continuous $f: D^n \to D^n$ must have a fixed point. ∎

To prove the above theorem we need to introduce so called relative homology groups:

Let X be a space and $A \subseteq X$ any subspace. Define $C_n(X,A)$ as the quotient abelian group

$$C_n(X,A) \doteq C_n(X) / C_n(A)$$

$$\left(\sigma: \Delta^n \to A \subseteq X \Rightarrow C_n(A) \subseteq C_n(X) \right)$$

$$C_n(X) \xrightarrow{\partial} C_{n-1}(X) \longrightarrow C_{n-1}(X)/C_{n-1}(A) = C_{n-1}(X,A)$$

If $\sum n_i \sigma_i \in C_n(A) \subseteq C_n(X)$, then

$$\partial\left(\sum n_i \sigma_i\right) = \sum n_i \partial\sigma_i \in C_{n-1}(A).$$

Hence, the above composition maps $C_n(A)$ to zero in $C_{n-1}(X,A)$. Thus it descends to a homomorphism

$$\begin{tikzcd}
C_n(X) \arrow[r, "\partial"] \arrow[d] & C_{n-1}(X) \arrow[r] & C_{n-1}(X)/C_{n-1}(A) \doteq C_{n-1}(X,A) \\
C_n(X)/C_n(A) \text{ is } C_n(X,A) \arrow[urr, "\partial"] & &
\end{tikzcd}$$

making the diagram commutative.

$$\partial : C_n(X,A) \longrightarrow C_{n-1}(X,A).$$

Since $\partial^2 = 0$ on the complex $C_*(X)$ we see that $\partial^2 = 0$ on $C_*(X,A)$.

$$C_{n+1}(X,A) \xrightarrow{\partial_{n+1}} C_n(X,A) \xrightarrow{\partial_n} C_{n-1}(X,A)$$

$\operatorname{Im}\partial_{n+1} \subseteq \ker\partial_n$ so that we may define

$H_n(X,A) \doteq \dfrac{\ker\partial_n}{\operatorname{Im}\partial_{n+1}}$, the n^{th} relative homology

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group of the pair (X,A).

$$H_n(X,A) = \frac{Z_n(X,A)}{B_n(X,A)}$$

$$Z_n(X,A) = \ker \partial_n : \big(\underbrace{C_n(X,A)}_{= C_n(X)/C_n(A)} \longrightarrow \underbrace{C_{n-1}(X,A)}_{= C_{n-1}(X)/C_{n-1}(A)} \big)$$

$\alpha \in C_n(X)$, $\alpha \in C_n(X)/C_n(A)$

$\partial\alpha = 0$ in $C_{n-1}(X,A) = C_{n-1}(X)/C_{n-1}(A)$

$\Longleftrightarrow \partial\alpha \in C_{n-1}(A)$.

So a relative cycle in $C_n(X,A)$ is a chain in $C_n(X)$ s.t. $\partial\alpha \in C_{n-1}(A)$.

Note that $\alpha = \alpha + \beta$ for any $\beta \in C_n(A)$ as regarded elements of $C_n(X,A)$

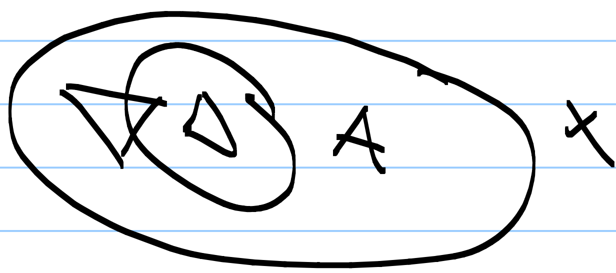
Also a relative cycle α is trivial in $H_n(X,A)$ iff $\alpha = \partial\beta + \gamma$ for some $\beta \in C_{n+1}(X)$ and $\gamma \in C_n(A)$.

Remark: $C_n(A)$ is a direct summand of $C_n(X)$

because $C_n(X)$ has a basis which is an extension of a basis of $C_n(A)$.

$\sigma : \Delta_n \longrightarrow A$

$\sigma : \Delta_n \longrightarrow X$



$B_A = \{\sigma : \Delta_n \to A \mid \sigma \text{ cont.}\}$ is a basis for $C_n(A)$.

$B_A^{\perp} = \{\sigma : \Delta_n \to X \mid \sigma(\Delta_n) \not\subseteq A\}$

$B_A \cap B_A^{\perp} = \phi$

$B_A \cup B_A^{\perp} = \{\sigma : \Delta_n \to X \mid \sigma \text{ cont}\}$ is a basis for $C_n(X)$.

Hence, $C_n(X) = \langle B_A \rangle \oplus \langle B_A^{\perp} \rangle$

$= C_n(A) \oplus \langle B_A^{\perp} \rangle$, which

implies $C_n(X)/C_n(A) \simeq \langle B_A^{\perp} \rangle \subseteq C_n(X)$

so that we may regard $C_n(X)/C_n(A)$ as a subgroup of $C_n(X)$, spanned by singular simplices $\sigma : \Delta \to X$ whose image not lying in A.

<u>Aim</u>: Show that $H_n(A)$, $H_n(X)$ and $H_n(X, A)$ fit into the long exact sequence

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X) \to \cdots$$

$$\to H_0(A) \to H_0(X) \to H_0(X,A) \to 0.$$

<u>Proof</u>: For any integer $n \geq 0$ we have

$$\begin{tikzcd}
0 \arrow[r] & C_n(A) \arrow[r, "i"] \arrow[d, "\partial"] & C_n(X) \arrow[r, "j"] \arrow[d, "\partial"] & \boxed{C_n(X,A)} \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & \boxed{C_{n-1}(A)} \arrow[r, "i"] & C_{n-1}(X) \arrow[r, "j"] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}$$

Instead consider the following diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A_{n+1} \arrow[r, "\partial"] \arrow[d, "i"] & A_n \arrow[r, "\partial"] \arrow[d, "i"] & \boxed{A_{n-1}} \arrow[r] \arrow[d, "i"] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r, "\partial"] \arrow[d, "j"] & B_n \arrow[r, "\partial"] \arrow[d, "j"] & B_{n-1} \arrow[r] \arrow[d, "j"] & \cdots \\
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial"] \arrow[d] & \boxed{C_n} \arrow[r, "\partial"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

All squares are assumed to be commutative.

$\gamma \longmapsto 0$, $\beta \longmapsto \partial\beta$, α

(A_*, ∂), (B_*, ∂), (C_*, ∂) are chain complexes (of abelian groups) and all vertical sequences are short exact:

i: injective, j: surjective and $\operatorname{Im} i = \ker j$.

The i and j induce homomorphism on homology level:

$i_*: H_n(A) \to H_n(B)$ and $j_*: H_n(B) \to H_n(C)$.

$i: Z_n(A) \longrightarrow Z_n(B)$ $\quad \alpha \in Z_n(A) = \ker\partial$

then $\partial(i(\alpha)) = i(\partial\alpha) = i(0) = 0$ so that $i(\alpha) \in Z_n(A)$.

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$$\begin{tikzcd} B_n(A) \subseteq Z_n(A) \arrow[r, "i"] \arrow[d] & Z_n(B) \arrow[d] \\ H_n(A) = \frac{Z_n(A)}{B_n(A)} \arrow[r, "i_*"] & \frac{Z_n(B)}{B_n(B)} = H_n(B) \end{tikzcd}$$

$\alpha \in Z_n(A)$, if $\alpha = \partial \beta$ for some $\beta \in A_{n+1}$ then

$i(\alpha) = i(\partial \beta) = \partial(i\beta) \Rightarrow i(\alpha) \in B_n(B)$.

Similarly, $j : B_n \to C_n$ induces a homomorphism

$j_* : H_n(B) \to H_n(C)$.

Now we define the boundary map

$\partial : H_n(C) \to H_{n-1}(A)$ defined as

$\partial([\gamma]) \doteq [\alpha]$, where α is as in the

above diagram.

<u>must show:</u> α is a cycle and the map is

independent of the several choices made in the definition.

$\partial\alpha = 0$ is seen by diagram chasing.

Other claims can be proven again by diagram chasing.

So it gives a long exact sequence

$$\cdots \to H_n(A) \to H_n(B) \to H_n(C) \to H_{n-1}(A) \to \cdots \cdots \to H_0(C).$$

This algebraic fact taking $A_n = C_n(A)$, $B_n = C_n(X)$ and $C_n = C_n(X, A)$ we obtain the long exact sequence of the pair (X, A):

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Example: $(X, A) = (D^n, \partial D^n = S^{n-1})$

$$H_k(D^n) \to H_k(D^n, S^{n-1}) \to H_{k-1}(S^{n-1}) \to H_{k-1}(D^n) \to \cdots$$

$$\underset{0}{\underset{\parallel}{H_{n+1}(D^n)}} \to \underset{0}{\underset{\parallel}{H_{n+1}(D^n, S^{n-1})}} \to \underset{0}{\underset{\parallel}{H_n(S^{n-1})}} \to \underset{0}{\underset{\parallel}{H_n(D^n)}}$$

$\Rightarrow$

$$\to \underset{\mathbb{Z}}{\underset{\parallel}{H_n(D^n, S^{n-1})}} \xrightarrow{\cong} \underset{\mathbb{Z}}{\underset{\text{is}}{H_{n-1}(S^{n-1})}} \to \underset{0}{\underset{\parallel}{H_{n-1}(D^n)}} \to \cdots$$

$\Rightarrow$

It follows that $H_k(D^n, S^{n-1}) = \begin{cases} \mathbb{Z} & \text{if } k = n \\ 0 & \text{if } k \neq n \end{cases}$

$$\underset{\mathbb{Z} \text{ or } \mathbb{Z} \oplus \mathbb{Z}}{\underset{\text{is}}{H_0(S^{n-1})}} \xrightarrow[\text{onto}]{\downarrow} \underset{\mathbb{Z}}{\underset{\text{is}}{H_0(D^n)}} \xrightarrow{\downarrow} \underset{0}{\underset{\parallel}{H_0(D^n, S^{n-1})}} \to 0$$

$\Rightarrow = 0$

Example: (Exercise) For any space X and $x_0 \in X$ we have $H_n(X, x_0) \simeq \tilde{H}_n(X)$.

$A = \{x_0\}$

$$\longrightarrow \underset{0}{\underset{\shortparallel}{H_n(A)}} \longrightarrow H_n(X) \xrightarrow{\cong} H_n(X, A) \longrightarrow \underset{0}{\underset{\shortparallel}{H_{n-1}(A)}} \longrightarrow \cdots$$

$n \geq 2$

$n=2$ $$\underset{0}{\underset{\shortparallel}{H_2(A)}} \to H_2(X) \xrightarrow{\cong} H_2(X,A) \to \underset{0}{\underset{\shortparallel}{H_1(A)}} \to H_1(X) \xrightarrow{\cong} H_1(X,A)$$

$$0 \to \underset{\mathbb{Z}}{\underset{\text{is}}{H_0(A)}} \xrightarrow{\text{inj.}} H_0(X) \longrightarrow H_0(X,A) \longrightarrow 0$$

$\Rightarrow H_k(X, x_0) = H_k(X)$ $\forall$ $k \geq 1$.

$$0 \to \underset{\mathbb{Z}}{\underset{\text{is}}{H_0(x_0)}} \longrightarrow \underset{\mathbb{Z} \oplus \cdots}{\underset{\shortparallel}{H_0(X)}} \longrightarrow \underset{H_0(X)/H_0(x_0) \simeq \tilde{H}_0(X)}{\underset{\shortparallel}{H_0(X,A)}} \longrightarrow 0$$

$\mathbb{Z} \longmapsto \mathbb{Z} \oplus \cdots$ ■

Proposition: If two maps $f, g : (X,A) \to (Y,B)$ are homotopic through maps of pairs $(X,A) \to (Y,B)$, then $f_* = g_* : H_n(X,A) \longrightarrow H_n(Y,B)$.

Proof The prism operator $P: C_n(X) \to C_{n+1}(Y)$ takes $C_n(A)$ to $C_{n+1}(B)$ and this induces a relative Prism operator

$$P: C_n(X)/C_n(A) \longrightarrow C_{n+1}(Y)/C_{n+1}(B).$$

One can finish the proof using this prism operator.

Remark: Assume that we have $B \subseteq A \subseteq X$. Let

$$A_n = C_n(A, B), \quad B_n = C_n(X, B), \quad C_n = C_n(X, A)$$

Then we have exact sequence:

$$\begin{array}{ccccccccc} 0 \to & A_n & \longrightarrow & B_n & \longrightarrow & C_n & \to 0 & \text{for all } n. \\ & \| & & \| & & \| & & \\ 0 \to & C_n(A)/C_n(B) & \longrightarrow & C_n(X)/C_n(B) & \longrightarrow & C_n(X)/C_n(A) & \to 0 & \end{array}$$

Now the algebraic result gives the long exact sequence of the triple (B, A, X)

$$\cdots \to H_n(A,B) \to H_n(X,B) \to H_n(X,A) \to H_{n-1}(A,B) \to \cdots$$

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Excision:

Theorem: Given subspaces $Z \subset A \subset X$ such that $\overline{Z} \subseteq \text{Int}(A)$, the inclusion map
$$i : (X \setminus Z, A \setminus Z) \longrightarrow (X, A)$$
induces isomorphisms
$$H_n(X \setminus Z, A \setminus Z) \longrightarrow H_n(X, A), \text{ for all } n.$$

Equivalently, for subspaces $A, B \subset X$ so that $X = \text{Int}(A) \cup \text{Int}(B)$, the inclusion
$$i\,(B, A \cap B) \hookrightarrow (X, A) \text{ induces}$$
isomorphism
$$H_n(B, A \cap B) \longrightarrow H_n(X, A).$$

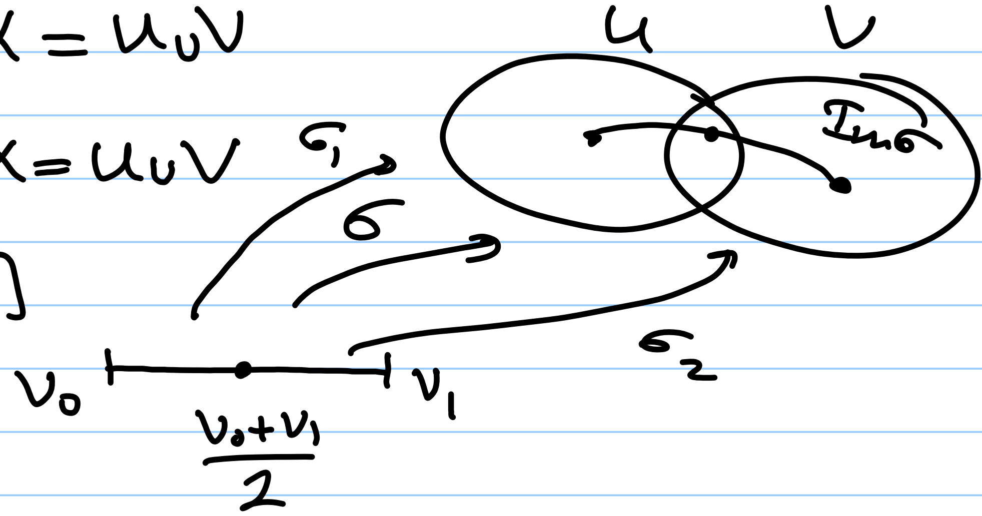
Remark: The equivalence of the two statement can be seen taking $B = X \setminus Z$ and thus $Z = X \setminus B$. Then $A \cap B = A \setminus Z$ and
$\overline{Z} \subset \text{Int}(A)$ is equivalent to $X = \text{Int}(A) \cup \text{Int}(B)$.

Main Idea:

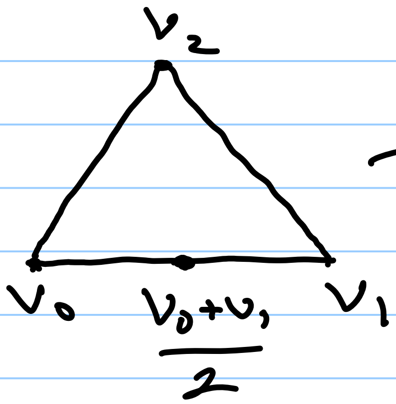
$\sigma : \Delta^n \to X = U \cup V$

$\sigma : \Delta^1 \to X = U \cup V$

$\Delta^1 = [v_0, v_1]$



$\sigma - (\sigma_1 + \sigma_2) = \partial \tau$, for some two simplex τ.



$\xrightarrow{\tilde{\tau}} X$

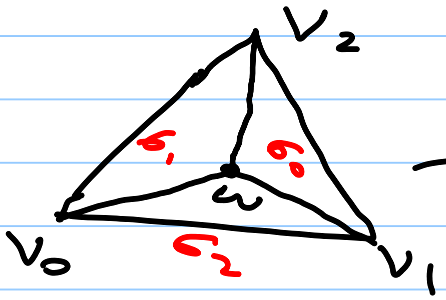
$$\begin{tikzcd}
{[v_0, v_1, v_2]} \arrow[r] \arrow[rr, bend right, "\tilde{\tau}"'] & {[v_0, v_1, \frac{v_0+v_1}{2}]} \arrow[r, "\sigma"] & X
\end{tikzcd}$$

$$\partial\tilde{\tau} = \tilde{\tau}|_{[v_1,v_2]} - \tilde{\tau}|_{[v_0,v_2]} + \tilde{\tau}|_{[v_0,v_1]}$$

$$= \sigma|_{[v_1, \frac{v_0+v_1}{2}]} - \sigma|_{[v_0, \frac{v_0+v_1}{2}]} + \sigma$$

$$= -\sigma_2 - \sigma_1 + \sigma. \text{ So, } \sigma - (\sigma_1 + \sigma_2) = \partial\tilde{\tau}.$$

Similarly, for a two simplex $\sigma : [v_0, v_1, v_2] \to X$ we decompose σ to three "smallest" simplices.



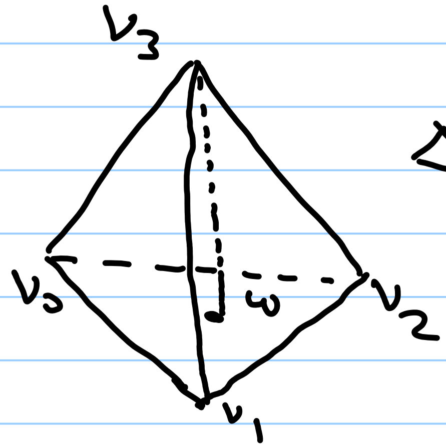
$\xrightarrow{\sigma} X$

$$w = \frac{v_0+v_1+v_2}{3}$$

$$\sigma - \overbrace{(\sigma_1 + \sigma_2 + \sigma_3)}^{S(\sigma)} = \partial\tilde{\tau}$$

for some 3-simplex $\tilde{\tau}$.

$$T(\sigma) = \tilde{\tau}$$



$$\begin{tikzcd} \Delta^3 \arrow[r] \arrow[rr, bend left, "\tilde{\sigma}"] & \Delta^2 \arrow[r, "\sigma"] & X \end{tikzcd}$$

$$[v_0, v_1, v_2, v_3] \longrightarrow [v_0, v_1, v_2] \xrightarrow{\sigma} X$$

$$v_i \longmapsto v_i \quad i = 0, 1, 2.$$
$$v_3 \longmapsto \frac{1}{3}(v_0 + v_1 + v_2)$$

For a space X, let $\mathcal{U} = \{U_\delta\}$ be a collection of subspaces of X whose interiors form an open cover of X, and let $C_n^{\mathcal{U}}(X)$ be the subgroup of chains $\sum n_i \sigma_i$ in $C_n(X)$ such that each σ_i has image contained in some set in the cover $\mathcal{U}$.

$\sum n_i \sigma_i \in C_n^{\mathcal{U}}(X)$, $\quad \operatorname{Im} \sigma_i \subseteq U_i$, for some $U_i \in \mathcal{U}$. Then $\operatorname{Im}(\partial \sigma_i) \subseteq U_i$ for each i so that ∂ takes the subgroup $C_n^{\mathcal{U}}(X)$ to $C_{n-1}^{\mathcal{U}}(X)$.

$$\begin{array}{rccc} \partial : & C_n(X) & \longrightarrow & C_{n-1}(X) \\ & \cup\!\shortmid & & \cup\!\shortmid \\ \partial| : & C_n^{\mathcal{U}}(X) & \longrightarrow & C_{n-1}^{\mathcal{U}}(X) \end{array}$$

Since $\partial^2 = 0$ the restriction $\partial|$ also satisfies $\partial|^2 = 0$. Hence, we can define the homology group

$$H_n^{\mathcal{U}}(X) = \frac{\ker \partial : C_n^{\mathcal{U}}(X) \to C_{n-1}^{\mathcal{U}}(X)}{\operatorname{Im} \partial : C_{n+1}^{\mathcal{U}}(X) \to C_n^{\mathcal{U}}(X)}.$$

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Proposition: The inclusion $i: C_n^{\mathcal{U}}(X) \to C_n(X)$ is a chain homotopy equivalence, that is, there is a chain map

$$\rho: C_n(X) \to C_n^{\mathcal{U}}(X) \text{ such that}$$

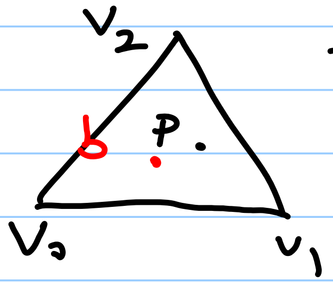
$i \circ \rho$ and $\rho \circ i$ are chain homotopic to the identity. Hence, i induces isomorphisms

$$H_n^{\mathcal{U}}(X) \cong H_n(X), \text{ for all } n.$$

Proof: Proof has four steps.

Step 1 Barycentric Subdivision of Simplices:

$$[v_0, v_1, \dots, v_n] = \left\{ \sum t_i v_i \mid t_i \geq 0, \sum t_i = 1 \right\}.$$

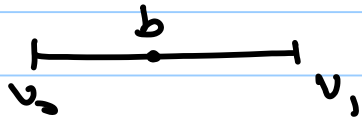


The barycenter (center of gravity) is defined to be the point

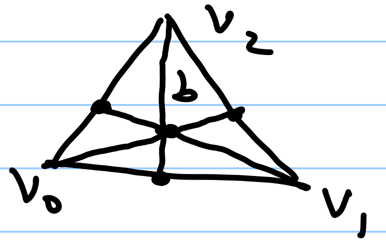
$$b = \frac{1}{n+1}(v_0 + \dots + v_n).$$

The barycentric subdivion of $[v_0, \dots, v_n]$ is the decomposition of $[v_0, \dots, v_n]$ into n-simplicis $[b, w_0, \dots, w_{n-1}]$, where inductively $[w_0, \dots, w_{n-1}]$ is an $(n-1)$-simplex in the barycentric subdivion of a face $[v_0, \dots, \hat{v}_i, \dots, v_n]$.

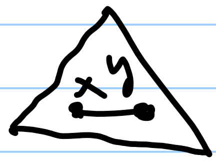
$n=0$: Barycentric subdivision of $[v_0]$ is just $[v_0]$.
$n=1$: Barycentric subdivision of $[v_0, v_1]$ is $[b, v_0]$ and $[b, v_1]$.



$n=2$: The barycentric subdivision of $[v_0, v_1, v_2]$ is given as in the diagram below:



Diameter of a the simplex $[v_0, v_1, \dots, v_n]$ is defined to be the maximum of the distance between any two points in $[v_0, v_1, \dots, v_n]$.

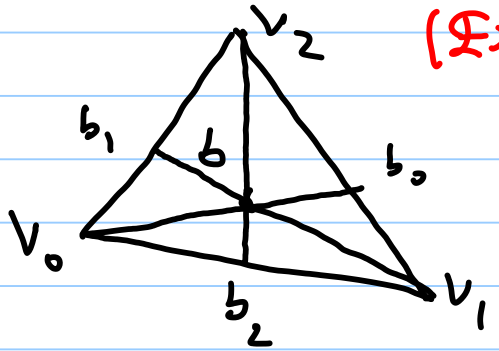


$$\text{Diameter}([v_0, v_1, \dots, v_n]) = \max\{|v_i - v_j| \mid i,j = 0,\dots,n\}.$$

<u>Fact:</u> If $[b, w_0, \dots, w_{n-1}]$ is a simplex in the barycentric subdivision of $[v_0, v_1, \dots, v_n]$ then

$$\operatorname{diam}([b, w_0, \dots, w_{n-1}]) \leq \frac{n}{n+1} \operatorname{diam}([v_0, v_1, \dots, v_n]).$$

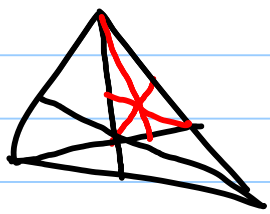
(Exercise!)



$[b, b_0, v_2]$, $[b, v_1, b_0]$

$$\operatorname{diam}([b, w_0, w_1]) \leq \frac{2}{3} \operatorname{diam}([v_0, v_1, v_2]).$$

Note that for fixed n, then the limit $\left(\frac{n}{n+1}\right)^r \to 0$ as $r \to \infty$.



Step 2: Barycentric Subdivision of linear Chains

Aim: Construct a subdivision operator

$$S : C_n(X) \to C_n(X).$$

Let Y be a convex subset in some $\mathbb{R}^n$. Let $LC_n(Y)$ denote the subgroup of $C_n(Y)$ generated by linear n-chains in Y.

$\sigma : \Delta^n = [v_0, \dots, v_n] \longrightarrow Y$

$\sigma(v_i) = u_i \quad i = 0, \dots, n$

$p = \sum_i t_i v_i$, then $\sigma(p) = \sum_i t_i u_i$

Convention: $LC_{-1}(Y) = \mathbb{Z} = \langle [\phi] \rangle$ where

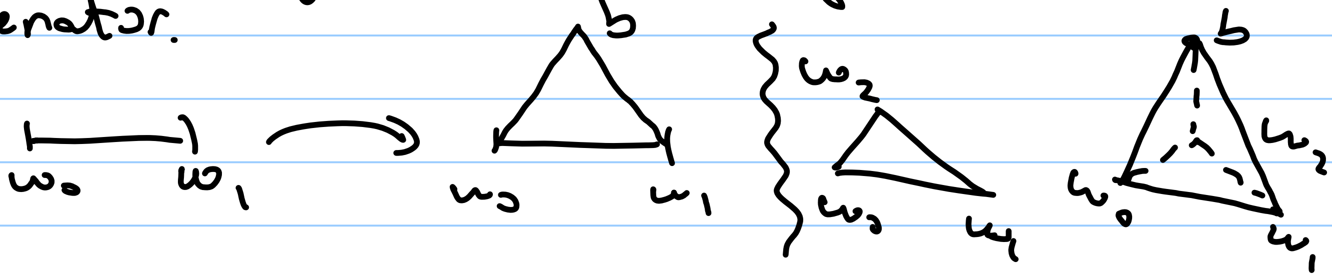
$\partial[w_0] = [\phi]$.

Note that each point $b \in Y$ determines a homomorphism

$$b : LC_n(Y) \longrightarrow LC_{n+1}(Y) \text{ defined on}$$

basis elements by $b([w_0, \dots, w_n]) = [b, w_0, \dots, w_n]$.

Geometrically this can be regarded as a cone operator.



Note that $\partial b([w_0,\dots,w_n]) = \partial[b, w_0, \dots, w_n]$

$$\Rightarrow \partial b([w_0,\dots,w_n]) = [w_0,\dots,w_n] - [b, w_1,\dots,w_n] + [b, w_0, w_2, w_3,\dots,w_n] + (-1)^n [b, w_0, w_1, \dots, w_{n-1}]$$

$$\Rightarrow \partial b([w_0,\dots,w_n]) = [w_0,\dots,w_n] - b\,\partial([w_0,\dots,w_n])$$

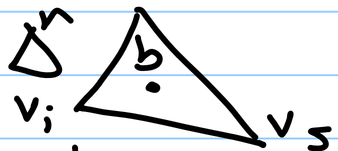
Hence, $\partial b(\alpha) = \alpha - b\partial(\alpha)$, for any $\alpha \in LC_n(Y)$.

In other words, $\partial b + b\partial = \mathrm{Id}_{LC_n(Y)}$.

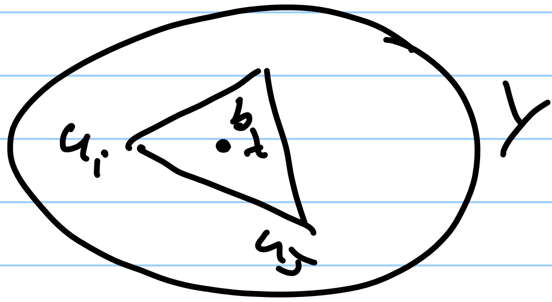
Next we define the subdivision homomorphism

$$S : LC_n(Y) \to LC_n(Y).$$

Let $\lambda : \Delta^n \to Y$ be a generator of $LC_n(Y)$ and let b_λ be the image of the barycenter of Δ^n under λ.



$b = \frac{1}{n+1}(v_0 + \dots + v_n)$



$$u_i = \lambda(v_i), \quad b_\lambda = \lambda(b)$$

The inductive formula for S is

$$S(\lambda) = b_\lambda(S\partial\lambda)$$

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$\lambda(v_0)$

$\lambda : \Delta^0 = [v_0] \longrightarrow Y$, $S(\lambda) = ?$

$\partial\lambda = [\phi]$, $S\partial Y = [\phi]$, $b_\lambda(S\partial\lambda) = [b_\lambda] = [\lambda(v_0)]$.

Convention: $S([\phi]) \doteq [\phi]$

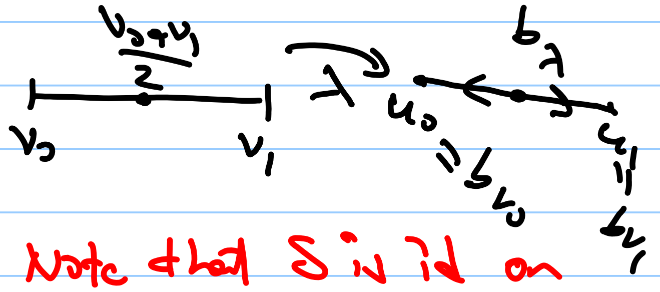
$\lambda : \Delta^1 = [v_0, v_1] \longrightarrow Y$, $S(\lambda) = ?$

$S(\lambda) = b_\lambda(S\partial\lambda) = b_\lambda S([v_1] - [v_0])$

$= b_\lambda(b_{v_1} - b_{v_0})$

$= [b_\lambda, b_{v_1}] - [b_\lambda, b_{v_0}]$

$= [b_\lambda, u_1] - [b_\lambda, u_0]$



Note that S is id on $LC_{-1}(Y)$ and $LC_0(Y)$.

Claim: $\partial S = S\partial$ so that S is a chain map from $LC_n(Y)$ to itself.

Proof: Now for any linear simplex $\lambda : \Delta^n \to Y$

$\partial S\lambda = \partial(b_\lambda S\partial\lambda)$ $\qquad \partial b_\lambda + b_\lambda\partial = \mathrm{id}$

$= \partial b_\lambda(S\partial\lambda)$

$= (\mathrm{id} - b_\lambda\partial)(S\partial\lambda)$

$= S\partial\lambda - b_\lambda\partial S\underbrace{\partial\lambda}_{n-1 \text{ chain}}$ $\qquad$ by induction on n we may assume $\partial S = S\partial$ on $n-1$-chains

$= S\partial\lambda - b_\lambda S\underbrace{\partial\partial}_{=0}\lambda$

$= S\partial\lambda$.

Next we need to construct a chain homotopy $T: LC_n(Y) \to LC_{n+1}(Y)$ between S and identity, fitting into a diagram

$$\begin{tikzcd}
\cdots \arrow[r] & LC_2(Y) \arrow[r] \arrow[d, shift right, "S"'] \arrow[d, shift left, "\mathrm{id}"] & LC_1(Y) \arrow[r] \arrow[d, shift right, "S"'] \arrow[d, shift left, "\mathrm{id}"] \arrow[dl, "T"] & LC_0(Y) \arrow[r] \arrow[d, shift right, "S"'] \arrow[d, shift left, "\mathrm{id}"] \arrow[dl, "T"] & LC_{-1}(Y) \arrow[r] \arrow[d, shift right, "S"'] \arrow[d, shift left, "\mathrm{id}"] \arrow[dl, "T"] & 0 \\
\cdots \arrow[r] & LC_2(Y) \arrow[r] & LC_1(Y) \arrow[r] & LC_0(Y) \arrow[r] & LC_{-1}(Y) \arrow[r] & 0
\end{tikzcd}$$

Define T inductively: $T=0$ on $LC_{-1}(Y)$

and $T\lambda = b_\lambda(\lambda - T\partial\lambda)$ for $n \geq 0$.

Claim: $\partial T + T\partial = \mathrm{id} - S$ on $LC_n(Y)$.

Proof: $n=-1$, $T=0$ and $S = \mathrm{id}$

For $n \geq 0$ we have

$$\partial T\lambda = \partial(b_\lambda(\lambda - T\partial\lambda))$$

$$= \lambda - T\partial\lambda - b_\lambda(\partial(\lambda - T\partial\lambda)) \quad \text{since } \partial b_\lambda = \mathrm{id} - b_\lambda\partial$$

$$= \lambda - T\partial\lambda - b_\lambda(\partial\lambda - \underbrace{\partial T}_{\substack{\shortparallel \\ \mathrm{id} - S - T\partial \text{ (by Induction)}}}\underbrace{\partial\lambda}_{n-1-\text{chain}})$$

$$= \lambda - T\partial\lambda - b_\lambda(\underbrace{\partial\lambda - \partial\lambda}_{=0} + S\partial\lambda + T\underbrace{\partial\partial}_{=0}\lambda)$$

$$= \lambda - T\partial\lambda - b_\lambda S\partial\lambda$$

$$= \lambda - T\partial\lambda - S\lambda$$

$$\Rightarrow (\partial T + T\partial)\lambda = \lambda - S\lambda, \text{ for any } \lambda$$

$$\partial T + T\partial = \mathrm{id} - S.$$

This finishes the proof of the claim.

3) Barycentric Subdivision of General Chains

Define $S : C_n(X) \to C_n(X)$ by setting

$$S\sigma \doteq \sigma_{\#} S\Delta^n \qquad \sigma : \Delta^n \to X \text{ cont.}$$

$$S\Delta^n \to \underset{\substack{\shortparallel \\ Y}}{\Delta^n} \xrightarrow{\sigma} X$$

Claim: The operator S is a chain map.

$$\partial S = S\partial.$$

Proof: $\sigma : \Delta^n \to X$

$$\partial S\sigma = \partial \sigma_{\#} S\Delta^n = \sigma_{\#}\partial S\Delta^n = \sigma_{\#} S\partial\Delta^n$$

$$\Rightarrow \partial S\sigma = \sigma_{\#} S\Big(\sum_i (-1)^i \Delta^n_i\Big), \quad \Delta^n_i = i\text{th face of } \Delta^n$$

$$= \sum_i (-1)^i \sigma_{\#} S\Delta^n_i$$

$$= \sum_i (-1)^i S(\sigma|_{\Delta^n_i})$$

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$$= S\left(\sum_i (-1)^i \sigma|_{\Delta_i^n}\right)$$

$$= S\,\partial\sigma$$

This finishes the proof.

Similarly, define $T: C_n(X) \to C_{n+1}(X)$ by

$T\sigma = \sigma_\# T\Delta^n \qquad (\sigma : \Delta^n \to X)$.

One can see that $\partial T\sigma = \sigma - S\sigma - T(\partial\sigma)$

so that $\partial T + T\partial = id - S$ on $C_n(X)$.

4) <u>Iterated Barycentric Subdivision</u>

The operator $D_m = \sum_{i=0}^{m-1} T S^i$ gives the chain homotopy between id and S^m:

$$\partial D_m + D_m \partial = \sum_{i=0}^{m-1} (\partial T S^i + T S^i \partial)$$

$$= \sum_{i=0}^{m-1} (\partial T S^i + T \partial S^i) \quad \text{since } S\partial = \partial S.$$

$$= \sum_{i=0}^{m-1} (\partial T + T\partial) S^i$$

$$= \sum_{i=0}^{m-1} (id - S) S^i$$

$$= \sum_{i=0}^{m-1} S^i - S^{i+1} = id - S^m.$$

For each singular simplex $\sigma: \Delta^n \to X$ there is some m s.t. $S^m(\sigma)$ lies in $C_n^{\mathcal{U}}(X)$.

[$\sigma(\Delta^n) \subset \bigcup_{U \in \mathcal{U}} U$ and $\sigma(\Delta^n)$ is compact. This open cover has a Lebesgue number say $\rho > 0$. Hence, for any $p \in \sigma(\Delta^n)$ the ball $B(p, \rho) \subseteq U$ for some $U \in \mathcal{U}$. Now choose m so that each piece of $S^m \sigma$ has diameter less than ρ. Now $S^m \sigma$ lies in $C_n^{\mathcal{U}}(X)$.]

Note that for each $\sigma: \Delta^n \to X$ we may need a different m.

Define $m(\sigma)$ to be smallest m so that $S^m \sigma \in C_n^{\mathcal{U}}(X)$.

Define $D: C_n(X) \to C_{n+1}(X)$ by $D\sigma = D_{m(\sigma)} \sigma$.

Let's see if D defines a chain homotopy?

We know that for any $\sigma: \Delta^n \to X$

$$\partial D_{m(\sigma)} \sigma + D_{m(\sigma)} \partial \sigma = \sigma - S^{m(\sigma)} \sigma.$$

Using this we write

$$\partial D \sigma + D \partial \sigma = \sigma - \left[S^{m(\sigma)} \sigma + D_{m(\sigma)}(\partial \sigma) - D(\partial \sigma) \right]$$

Let $\rho(\sigma) \doteq S^{m(\sigma)} \sigma + D_{m(\sigma)}(\partial \sigma) - D(\partial \sigma)$

[$\partial D \sigma + D \partial \sigma = \sigma - \rho(\sigma)$]

Claim: $\rho(\sigma) \in C_n^{\mathcal{U}}(X)$.

Proof: This is trivial for the term $\delta_\sigma^{m(\sigma)}$.

For the term $D_{m(\sigma)}(\partial\sigma) - D(\partial\sigma)$ note that if σ_j denote the restriction of σ to the j^{th} face of Δ^n, then $m(\sigma_j) \leq m(\sigma)$, so every term $TS^i(\sigma_j)$ in $D(\partial\sigma)$ will be a term in $D_{m(\sigma)}(\partial\sigma)$.

Thus $D_{m(\sigma)}(\partial\sigma) - D(\partial\sigma)$ is a sum of terms $TS^i(\sigma_j)$ with $i \geq m(\sigma_j)$ and these terms lie in $C_n^{\mathcal{U}}(X)$ since T takes $C_{n-1}^{\mathcal{U}}(X)$ to $C_n^{\mathcal{U}}(X)$.

So we may define $\rho : C_n(X) \to C_n^{\mathcal{U}}(X)$ as above. Moreover, by the definition

$$\partial\rho(\sigma) = \partial\sigma - \partial D\partial(\sigma) = \rho(\partial\sigma)$$

Finally, we have $\partial D + D\partial = \mathrm{Id} - i\rho$, where

$i : C_n^{\mathcal{U}}(X) \to C_n(X)$ the inclusion map.

Note that $\rho i = \mathrm{id}$ since D is identically zero on $C_n^{\mathcal{U}}(X)$ as $m(\sigma) = 0$ if σ is in $C_n^{\mathcal{U}}(X)$.

Thus ρ is a chain homotopy inverse for $i : C_n^{\mathcal{U}}(X) \to C_n(X)$. ∎

<u>Proof of the Excision Theorem</u>: Let's prove the second version: $X = \text{Int}(A) \cup \text{Int}(B)$

$$C_n(A+B) = C_n^{\mathcal{U}}(X) \qquad (\mathcal{U} = \{A, B\})$$

$$\partial D + D\partial = id - ip \quad \text{and} \quad pi = id.$$

All the maps in the above equation takes chains in A to chains in A, so they induce quotient maps when we factor out chain in A.

$$C_n(A+B)/C_n(A) \hookrightarrow C_n(X)/C_n(A)$$

induces isomorphism in homology

On the other hand, the map

$$C_n(B)/C_n(A \cap B) \longrightarrow C_n(A+B)/C_n(A)$$ is an isomorphism. So we get

$$H_n(B, A \cap B) \simeq H_n(X, A).$$

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Proposition: For good pairs (X,A), the quotient map $q:(X,A) \to (X/A, A/A)$ induces isomorphisms

$$q_*: H_n(X,A) \to H_n(X/A, A/A) \simeq \tilde{H}_n(X/A), \text{ for all } n.$$

Proof: (X,A) is a good pair means there is some open subset V so that $A \subseteq V \subseteq X$, which deformation retracts onto A.

$A \subseteq V \subseteq X \Rightarrow (V,A) \to (X,A) \to (X,V)$ induces an exact sequence of pairs

$$\to \underset{\substack{\text{IS} \\ 0}}{H_n(V,A)} \to H_n(X,A) \xrightarrow[\simeq]{j} H_n(X,V) \to \underset{\substack{\text{IS} \\ 0}}{H_{n-1}(V,A)} \to \cdots$$

$$\left(\to H_n(A) \xrightarrow{\simeq} H_n(V) \xrightarrow[0]{} \underset{\substack{\text{IS} \\ 0}}{H_n(V,A)} \xrightarrow[0]{} H_{n-1}(A) \xrightarrow{\simeq} H_{n-1}(V) \to \right)$$

Now consider the commutative diagram below:

$$\begin{array}{ccccc}
H_n(X,A) & \xrightarrow{\simeq} & H_n(X,V) & \xleftarrow{\simeq} & H_n(X \setminus A, V \setminus A) \\
\downarrow q_* & & \downarrow q_* & \text{Excision} & \simeq \downarrow q_* \\
H_n(X/A, A/A) & \xrightarrow{\simeq} & H_n(X/A, V/A) & \xleftarrow{\simeq} & H_n(X/A \setminus A/A, V/A - A/A)
\end{array}$$

The right hand side q_* is an isomorphism since $q:(X \setminus A, V \setminus A) \to (X/A \setminus A/A, V/A - A/A)$ is a homeomorphism on the complement of A.

Hence, the first two q_*'s are also isomorphisms.

Finally, the isomorphism $H_n(X/A, A/A) \cong \tilde{H}_n(X/A)$ follows from the exact sequence of the pair $(X/A, A/A)$:

$$\underset{H_n(pt)}{\underset{\|}{H_n(A/A)}} \to H_n(X/A) \to H_n(X/A, A/A) \to \underset{H_{n-1}(pt)}{\underset{\|}{H_{n-1}(A/A)}}$$

Note that $H_n(pt) = \begin{cases} 0 & \text{if } n > 0 \\ \mathbb{Z} & \text{if } n = 0. \end{cases}$

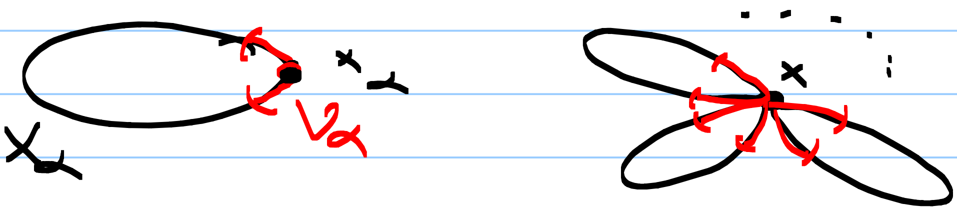
<u>Corollary</u> For a wedge sum $\bigvee_\alpha X_\alpha$, the inclusions

$i_\alpha : X_\alpha \to \bigvee_\alpha X_\alpha$ induce isomorphisms

$$\bigoplus_\alpha i_{\alpha *} : \bigoplus \tilde{H}_n(X_\alpha) \to \tilde{H}_n\left(\bigvee_\alpha X_\alpha\right), \text{ provided}$$

that the wedge sum is formed at basepoints $x_\alpha \in X_\alpha$ such that the pairs (X_α, x_α) are good pairs.

<u>Proof</u>: Exercise. (X_α, x_α) is good means there is some open subset $x_\alpha \in V_\alpha \subset X_\alpha$ so that V_α deformation retracts onto x_α.



$\bigvee_\alpha V_\alpha \to x$

$x = \bigvee_\alpha \{x_\alpha\}$

Just write down the homology exact sequence of the pair

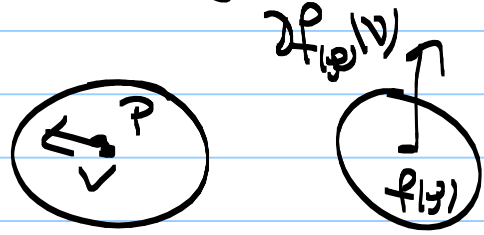
$$(X,A) = \left(\coprod_\alpha X_\alpha, \coprod_\alpha \{x_\alpha\}\right) \qquad X/A = \bigvee_\alpha X_\alpha$$

$$H_n(X,A) \simeq \tilde{H}_n(X/A)$$

__Invariance of Domain:__

Let $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ be open subsets. If there is a diffeomorphism $f: U \to V$ then the derivative map at any point $p \in U$ gives an isomorphism

$$\begin{array}{ccc} Df_{(p)}: T_pU & \xrightarrow{\simeq} & T_{f(p)}V \\ \wr\wr & & \wr\wr \\ \mathbb{R}^n & & \mathbb{R}^m \end{array}$$



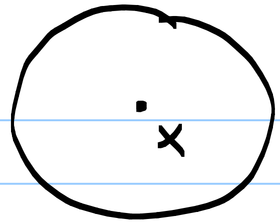
(as vector spaces) $\Longrightarrow m=n$.

__Theorem__ (1910 Brouwer – Invariance of Domain)

If nonempty open subsets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ are homeomorphic then $m=n$.

__Proof:__ Let $x \in U$, then by excision

$H_k(U, U \setminus \{x\}) \simeq H_k(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\})$, when we excise

A 

$A = \overline{X \setminus U}$

$\mathbb{R}^m \setminus \{x\}$ deformation retracts onto the sphere with center x and radius, say 1.

$$\mathbb{R}^n \setminus \{x\} \longrightarrow S^{m-1}$$
$$y \longmapsto \frac{y-x}{\|y-x\|} + x$$

Hence, $H_k(\mathbb{R}^m - \{x\}) \simeq H_k(S^{m-1})$

$$H_k(U, U \setminus \{x\}) \simeq H_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \simeq \tilde{H}_{k-1}(S^{m-1}) = \begin{cases} \mathbb{Z} & k=m \\ 0 & k \neq m \end{cases}$$

$$\begin{array}{ccccccc} \tilde{H}_k(\mathbb{R}^m) & \to & H_k(\mathbb{R}^n, \mathbb{R}^m \setminus \{x\}) & \to & \tilde{H}_{k-1}(\mathbb{R}^m \setminus \{x\}) & \to & \tilde{H}_{k-1}(\mathbb{R}^n) \\ \| & & & & \wr & & \| \\ 0 & & & & \tilde{H}_{k-1}(S^{m-1}) & & 0 \end{array}$$

Finally, if $f: U \to V$ is a homeomorphism then for any $x \in U$ the map of pairs

$f: (U, U \setminus \{x\}) \longrightarrow (V, V \setminus \{x\})$ is also a homeomorphism

and thus $H_k(U, U \setminus \{x\}) \underset{f_*}{\simeq} H_k(V, V \setminus \{f(x)\})$, $\forall k$.

$U \subseteq \mathbb{R}^m$, $V \subseteq \mathbb{R}^n \implies m = n$. ∎

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The Mayer—Vietoris Exact Sequence:

$X = \mathrm{Int}(A) \cup \mathrm{Int}(B)$ for some subsets $A, B \subseteq X$.

Then we have an exact sequence of chain groups, for each n,

$$0 \to C_n(A\cap B) \xrightarrow{\varphi} C_n(A) \oplus C_n(B) \xrightarrow{\psi} C_n(A+B) \to 0$$

$$x \longmapsto (x, -x)$$

$$(x, y) \longmapsto x+y$$

φ is clearly injective and ψ is surjective.

Also clearly $\mathrm{Im}\,\varphi \subseteq \ker\psi$. One can see also that $\ker\psi \subseteq \mathrm{Im}\,\varphi$.

$$\begin{tikzcd}
 & \arrow[d, "\partial"] & \arrow[d, "\partial"] & \arrow[d, "\partial"] & \\
0 \arrow[r] & C_{n+1}(A\cap B) \arrow[r] \arrow[d, "\partial"] & C_{n+1}(A)\oplus C_{n+1}(B) \arrow[r] \arrow[d, "\partial"] & C_{n+1}(A+B) \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & C_n(A\cap B) \arrow[r] \arrow[d, "\partial"] & C_n(A)\oplus C_n(B) \arrow[r] \arrow[d, "\partial"] & C_n(A+B) \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & C_{n-1}(A\cap B) \arrow[r] \arrow[d, "\partial"] & C_{n-1}(A)\oplus C_{n-1}(B) \arrow[r] \arrow[d, "\partial"] & C_{n-1}(A+B) \arrow[r] \arrow[d, "\partial"] & 0 \\
 & {} & {} & {} &
\end{tikzcd}$$

The above diagram induces a long exact sequence of homology groups

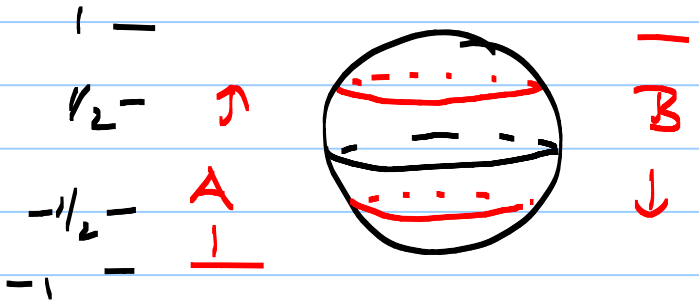
$$\to H_n(A\cap B) \to H_n(A) \oplus H_n(B) \to H_n(A+B) \to H_{n-1}(A\cap B) \to$$

where $H_n(A+B) \simeq H_n(A \cup B) = H_n(X)$.

So, we've obtained

$$\cdots \to H_n(A\cap B) \to H_n(A)\oplus H_n(B) \to H_n(X) \to H_{n-1}(A\cap B) \to \cdots$$

__Example:__ $S^n = A \cup B$



$A \simeq D^n \simeq B$

$A \cap B = S^{n-1} \times [-1/2, 1/2] \simeq S^{n-1}$

By induction one can compute $H_k(S^n)$ using Mayer–Vietoris sequence.

$(n=1,\ S^{n-1} = S^0 = \{\pm 1\}$ two points$)$

__Definition:__ For any space X and a point $x \in X$ the homology of the pair $(X, X\setminus\{x\})$, $H_n(X, X\setminus\{x\})$, is called the local homology of X at the point x.

__Example__ $X = U \subseteq \mathbb{R}^m$ open. $x \in U$

$$H_k(U, U\setminus\{x\}) \simeq \begin{cases} \mathbb{Z} & k=m \\ 0 & k\neq m \end{cases}$$

If $f: X \to Y$ is a map, which is a homeomorphism on an open set U containing $x \in X$ then

$H_n(X, X\setminus\{x\}) \simeq H_n(Y, Y\setminus\{f(x)\})$.

<u>Proof</u>- $f: X \to Y$, $f|_U : U \to V \subseteq Y$

is a homeomorphism.

$$\begin{tikzcd}
H_n(U, U\setminus\{x\}) \arrow[r, "\simeq", "f_*"'] & H_n(V, V\setminus\{f(x)\}) \\
H_n(X, X\setminus\{x\}) \arrow[u, "\text{Excision}\ \simeq"] & H_n(Y, Y\setminus\{f(x)\}) \arrow[u, "\simeq\ \text{Excision}"']
\end{tikzcd}$$

$(A = X\setminus U)$ $\qquad$ $(B = Y\setminus V)$

<u>Naturality</u>: An example of naturality.

Suppose that we have a map of pairs $f: (X,A) \to (Y,B)$. Then we have a commutative diagram of exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & \boxed{H_n(A)} \arrow[r] \arrow[d, "f_*"] & H_n(X) \arrow[r] \arrow[d, "f_*"] & H_n(X,A) \arrow[r] \arrow[d, "f_*"] & H_{n-1}(A) \arrow[r] \arrow[d, "f_*"] & \cdots \\
\cdots \arrow[r] & H_n(B) \arrow[r] & \boxed{H_n(Y)} \arrow[r] & H(Y,B) \arrow[r] & H_{n-1}(B) \arrow[r] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, "i_A"] \arrow[d, "f|_A"'] & X \arrow[d, "f"] \\
B \arrow[r, "i_B"'] & Y
\end{tikzcd}
\text{ commutative} \implies
\begin{tikzcd}
H_n(A) \arrow[r, "i_{A*}"] \arrow[d, "f_*"] & H_n(X) \arrow[d, "f_*"] \\
H_n(B) \arrow[r, "i_{B*}"'] & H_n(Y)
\end{tikzcd}$$

is commutative.

In this case we say that the exact sequence of pairs of spaces (X,A) is natural with respect to morphism (continuous maps of pairs).

Similarly, we have a commutative diagram for homology of quotient spaces as below, which is also an example of naturality:

$$f:(X,A)\longrightarrow(Y,B)\Longrightarrow(X/A,A/A)\xrightarrow[\bar{f}]{}(Y/B,B/B)$$

$$\begin{array}{ccccccccc}
\cdots\to\tilde{H}_n(A) & \to & \tilde{H}_n(X) & \to & \tilde{H}_n(X/A) & \xrightarrow{\partial} & \tilde{H}_{n-1}(A) & \to\cdots \\
\downarrow f_* & & \downarrow f_* & & \downarrow \bar{f}_* & & \downarrow f_* & \\
\cdots\to\tilde{H}_n(B) & \to & \tilde{H}_n(Y) & \to & \tilde{H}_n(Y/B) & \xrightarrow{\partial} & H_{n-1}(B) & \to
\end{array}$$

The above diagram is also commutative.

We say that the sequence

$$\to\tilde{H}_n(A)\to\tilde{H}_n(X)\to\tilde{H}_n(X/A)\xrightarrow{\partial}\tilde{H}_{n-1}(A)\to\cdots$$

is natural with respect to morphisms $f:(X,A)\to(Y,B)$.

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The Equivalence of Simplicial and Singular Homology

(X, A) simplicial complex, A subcomplex of X (or Δ-complex).

$\Delta_n(X)$ and $\Delta_n(A)$ are already defined.

We may define $\Delta_n(X,A) = \Delta_n(X)/\Delta_n(A)$, which is a free abelian group with generators $\sigma : \Delta^n \to X$, whose image is <u>not</u> contained in A.

Similarly, we define $H_n^{\Delta}(X,A)$ as the homology of the chain complex

$$\cdots \xrightarrow{\partial} \Delta_{n+1}(X,A) \xrightarrow{\partial} \Delta_n(X,A) \xrightarrow{\partial} \Delta_{n-1}(X,A) \to \cdots$$

We have a natural map $\Delta_n(X,A) \to C_n(X,A)$ sending each n-simplex of X to its characteristic map $\sigma : \Delta^n \to X$. This gives us a homomorphism on homology level

$$H_n^{\Delta}(X,A) \longrightarrow H_n(X,A), \text{ for each } n.$$

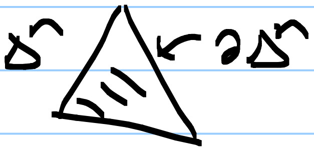
Theorem: The homomorphism $H_n^{\Delta}(X,A) \to H_n(X,A)$ are isomorphisms for all n and all Δ complex pairs (X,A).

Before we start proving the theorem we study the following example.

Example: (Example 2.3 in the book)

Let's find explicit generators for the infinite cyclic groups $H_n(D^n, \partial D^n)$ and $\tilde{H}_n(S^n)$.

Note that the pair $(D^n, \partial D^n)$ is homeomorphic to $(\Delta^n, \partial\Delta^n)$.



Let $i_n : \Delta^n \to \Delta^n$ be the identity map representing a singular simplex generating $H_n(\Delta^n, \partial\Delta^n)$ as we'll show below:

We use induction: For $n=0$, $\Delta^0 = pt$, $\partial\Delta^0 = \emptyset$
$H_0(\Delta^0, \partial\Delta^0) = H_0(pt, \emptyset) \simeq \mathbb{Z}$ with generator $i_0 : \Delta^0 \to \Delta^0$.

Now assume the result for $0, 1, \dots, n-1$. Let $\Lambda \subset \Delta^n$ be the union of all but one of the $(n-1)$-dimensional faces of Δ^n.

<u>Claim</u>: We have isomorphisms

$$H_n(\Delta^n, \partial\Delta^n) \xrightarrow{\cong} H_{n-1}(\partial\Delta^n, \Lambda) \xleftarrow{\cong} H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}).$$

$$i_n \longmapsto \partial i_n = \pm i_{n-1}$$

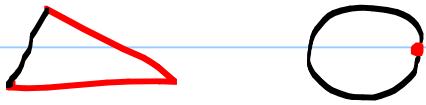
The first isomorphism is the boundary map of the exact sequence of the triple $(\Delta^n, \partial\Delta^n, \Lambda)$, where its terms $H_i(\Delta^n, \Lambda)$ are all trivial since Δ^n deformation retract onto Λ.

$$\begin{tikzcd}
\cdots \arrow[r] & H_i(\partial\Delta^n, \Lambda) \arrow[r] & H_i(\Delta^n, \Lambda) \arrow[r] \arrow[d, phantom, "\parallel"] & H_i(\Delta^n, \partial\Delta^n) \arrow[r, "\partial", "\cong"'] & H_{i-1}(\partial\Delta^n, \Lambda) \arrow[d] \\
 & & 0 & & 0 = H_{i-1}(\Delta^n, \Lambda)
\end{tikzcd}$$

For the second map $H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}) \to H_{n-1}(\partial\Delta^n, \Lambda)$ note that we have a homeomorphism of the quotient spaces $\Delta^{n-1}/\partial\Delta^{n-1}$ and $\partial\Delta^n/\Lambda$

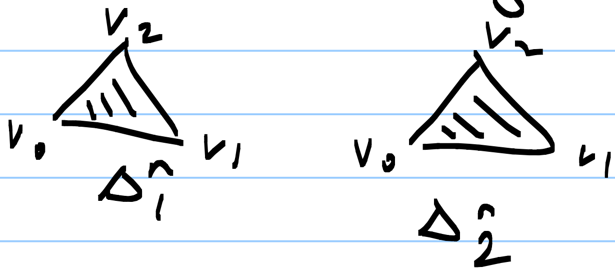
when both pairs are good.

$\Delta^{n-1}/\partial\Delta^{n-1}$ 

$\partial\Delta^n/\Lambda$ 

The induction step then follows since the map ι_n is sent to $\partial\iota_n = \pm\iota_{n-1}$ in $C_{n-1}(\partial\Delta^n, \Lambda)$.

For the generator of $\tilde{H}_n(S^n)$ regard S^n as the union of two copies of Δ^n glued along their boundaries in the obvious way:



$\partial(\Delta_1^n - \Delta_2^n) = 0$ so that $\Delta_1^n - \Delta_2^n$ is a cycle.

<u>Claim:</u> $\tilde{H}_n(S^n)$ is generated by $\Delta_1^n - \Delta_2^n$.

<u>Proof:</u> Consider the isomorphisms

$$\tilde{H}_n(S^n) \xrightarrow{\cong} H_n(S^n, \Delta_2^n) \xleftarrow{\cong} H_n(\Delta_1^n, \partial\Delta_1^n).$$

The first isomorphism comes from the exact sequence of the pair (S^n, Δ_2^n)

$$\to \underset{\substack{\wr\wr \\ 0}}{\tilde{H}_n(\Delta_2^n)} \to \tilde{H}_n(S^n) \xrightarrow[\cong]{} H_n(S^n, \Delta_2^n) \to \underset{\substack{\wr\wr \\ 0}}{\tilde{H}_{n-1}(\Delta_2^n)}$$

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For the second isomorphism note that both pairs $(S^n, \partial\Delta_2^n)$ and $(\Delta_1^n, \partial\Delta_1^n)$ are good pairs and thus

$$H_n(S^n, \partial\Delta_2^n) \simeq H_n(S^n/\partial\Delta_2^n) \quad \text{and}$$

$$H_n(\Delta_1^n, \partial\Delta_1^n) \simeq H_n(\Delta_1^n/\partial\Delta_1^n).$$

Finally, $H_n(S^n/\partial\Delta_2^n) \simeq H_n(\Delta_1^n/\partial\Delta_1^n)$ since we have a homeomorphism

$$S^n/\partial\Delta_2^n = \Delta_1^n \cup \Delta_2^n/\partial\Delta_2^n = \Delta_1^n/\partial\Delta_1^n.$$

Note that under these isomorphisms the cycle $\Delta_1^n - \Delta_2^n$ in the first groups maps to the cycle Δ_1^n in the third group so that $\Delta_1^n - \Delta_2^n$ represent a generator of $\tilde{H}_n(S^n)$.

Proof of the theorem:

First assume that X is finite dimensional and $A = \phi$.

X^k – k^{th} skeleton of X, union of all simplices of dimension 0 to k. Then we have a commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
H^\Delta_{n+1}(X^k, X^{k-1}) & \to & H^\Delta_n(X^{k-1}) & \to & H^\Delta_n(X^k) & \to & H^\Delta_n(X^k, X^{k-1}) & \to & H^\Delta_{n-1}(X^{k-1}) \\
① \downarrow & & ② \downarrow & & ③ \downarrow & & ④ \downarrow & & ⑤ \downarrow \\
H_{n+1}(X^k, X^{k-1}) & \to & H_n(X^{k-1}) & \to & H_n(X^k) & \to & H_n(X^k, X^{k-1}) & \to & H_{n-1}(X^{k-1})
\end{array}$$

Aim: Show that all vertical maps are isomorphism.

Claim: ① and ④ are isomorphisms for all n.

Proof: The chain groups $\Delta_n(X^k, X^{k-1}) = 0$ for $n \neq k$ and is free abelian with basis the k-simplices of X when $n = k$. It follows that the homology of the chain complex

$$\to \Delta_{n+1}(X^k, X^{k-1}) \to \Delta_n(X^k, X^{k-1}) \to \Delta_{n-1}(X^k, X^{k-1}) \to$$ is

isomorphic to $\Delta_n(X^k, X^{k-1})$.

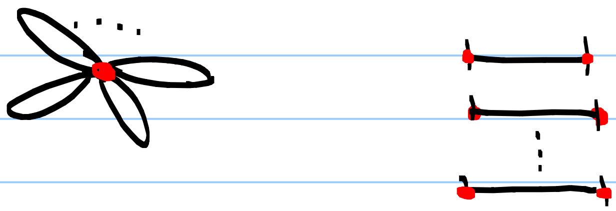
$$H_n^{\Delta}(X^k, X^{k-1}) = \Delta_n(X^k, X^{k-1}).$$

On the other hand, $H_n(X^k, X^{k-1})$ can be computed using the map

$\Phi : \coprod_\alpha (\Delta_\alpha^k, \partial \Delta_\alpha^k) \to (X^k, X^{k-1})$, formed by the

characteristic maps $\Delta^k \to X$, for all the k-simplices of X. Φ induces a homeomorphism

$$\coprod_\alpha \Delta_\alpha / \coprod_\alpha \partial \Delta_\alpha \simeq X^k / X^{k-1}.$$

Γ $k=1$ $\quad X^k / X^{k-1} =$  $\quad \Delta^1$ $\quad \coprod_\alpha \partial \Delta^1$ $\lrcorner$

and thus an isomorphism on singular homology:

$$H_n(X^k, X^{k-1}) \simeq H_n(X^k / X^{k-1}) \simeq H_n(\coprod_\alpha \Delta_\alpha^k / \coprod_\alpha \partial \Delta_\alpha^k)$$

$$= H_n(\coprod_\alpha \Delta_\alpha^k, \coprod_\alpha \partial \Delta_\alpha^k)$$

$$= \bigoplus_\alpha H_n(\Delta^k_\alpha, \partial\Delta^k_\alpha)$$

$$= \bigoplus H^\Delta_n(\Delta^k_\alpha, \partial\Delta^k_\alpha)$$

$$= H^\Delta_n(X^k, X^{k-1}).$$

By induction on k we may assume that ② and ⑤ are also isomorphisms.
The map ③ is an isomorphism follows from a purely algebraic fact known as the Five Lemma

<u>The Five Lemma</u>: In a commutative diagram of abelian groups as below, if the rows are exact and the homomorphism α, β, δ and ε are isomorphisms then so is γ.

$$\begin{array}{ccccccccc} A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{l} & E \\ \downarrow{\alpha} & & \downarrow{\beta} & & \downarrow{\gamma} & & \downarrow{\delta} & & \downarrow{\varepsilon} \\ A' & \xrightarrow[i']{} & B' & \xrightarrow[j']{} & C' & \xrightarrow[k']{} & D' & \xrightarrow[l']{} & E' \end{array}$$

<u>Proof</u> is done by diagram chasing.
Indeed one can prove:
a) γ is surjective if β and δ are surjective and ε is injective

b) γ is injective if β and δ are injective and α is surjective.

Now by the 5-lemma ③ is an isomorphism.
This finishes the proof in the

special case that X is finite dimensional and $A = \emptyset$.

Now assume that X is infinite dimensional. Since any chain in X has compact image any chain can meet only finitely many open simplices of X. More precisely, if C is a compact set in X intersecting infinitely many open simplicies we would get an infinite sequence of points x_i each lying in a different open simplex. In this case the open sets

$$u_i = X \setminus \left(\bigcup_{j \neq i} \{x_j\} \right)$$ form an open cover for

C having no finite subcover (note that the set $\{x_i\}$ is discrete and thus closed).

Take any element of $H_n(X)$ represented by a singular n-cycle z. Since z has compact image it lies in some X^k.

However, $H_n^\Delta(X^k) \to H_n(X^k)$ is an isomorphism and $[z] \in H_n(X^k)$ and thus the class $[z]$ is represented by a delta cycle. Hence the map $H_n^\Delta(X) \to H_n(X)$ is onto.

A similar argument proves that $H_n^\Delta(X) \to H_n(X)$ is also injective: $H_n^\Delta(X) \to H_n(X)$

$$[\alpha] \longmapsto [\alpha] = 0, \quad \alpha = \partial \beta.$$

For relative homology groups $H_n^\Delta(X,A) \to H_n(X,A)$ consider the below diagram together with 5-lemma:

$$\begin{array}{ccccccccccc}
\cdots \to & H_n^\Delta(A) & \to & H_n^\Delta(X) & \to & H_n^\Delta(X,A) & \to & H_{n-1}^\Delta(A) & \to & H_{n-1}^\Delta(X) & \to \cdots \\
& \simeq \downarrow 1 & & \simeq \downarrow 2 & & \simeq \downarrow 3 & & \simeq \downarrow 4 & & \simeq \downarrow 5 & \\
\cdots \to & H_n(A) & \to & H_n(X) & \to & H_n(X,A) & \to & H_{n-1}(A) & \to & H_{n-1}(X) & \to \cdots
\end{array}$$

1, 2, 4, 5 are isom. by the above arguments. 3 is $\simeq$ by 5-Lemma

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Degree of a map of spheres.

$f: S^n \to S^n$ continuous map

$$\Rightarrow f_*: \underset{\cong}{H_n(S^n)} \to \underset{\cong}{H_n(S^n)} \quad \text{homomorphism}$$

$$\mathbb{Z} \xrightarrow{\times k} \mathbb{Z}$$

$$1 \longmapsto k$$

$f_*([\alpha]) = \deg(f)\,[\alpha]$, for any $[\alpha] \in H_n(S^n)$.

Some properties of degree:

1) If $f = id_{S^n}: S^n \to S^n$ is the identity map then $f_* = id_{H_n(S^n)} : H_n(S^n) \to H_n(S^n)$ is the identity map and hence $\deg(f) = 1$.

2) If $f: S^n \to S^n$ is not surjective then f is homotopic to a constant map and thus $f_* = 0$ is the trivial map.

Hence, $\deg(f) = 0$.

3) If $f \simeq g$ are homotopic maps then $f_* = g_*$ and thus $\deg(f) = \deg(g)$.

4) Let $f: S^n \to S^n$, $g: S^n \to S^n$ be two maps

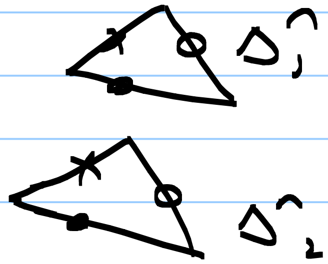
then $(f \circ g)_* = f_* \circ g_*$

$$\underset{\underset{\mathbb{Z}}{\cong}}{H_n(S^n)} \xrightarrow[\times \deg(g)]{g_*} \underset{\underset{\mathbb{Z}}{\cong}}{H_n(S^n)} \xrightarrow[\times \deg(f)]{f_*} \underset{\underset{\mathbb{Z}}{\cong}}{H_n(S^n)}$$

Hence, $\deg(f\circ g)=\deg(f)\cdot\deg(g)$.

5) If $f: S^n \to S^n$ is the reflection map with respect to S^{n-1} then $\deg(f)=-1$.

$$H_n(S^n)\cong H_n^{\Delta}(S^n)\cong \mathbb{Z} = \langle \Delta_1^n-\Delta_2^n\rangle$$



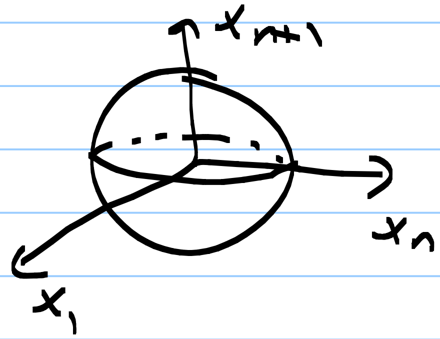
$$f(\Delta_1^n)=\Delta_2^n$$

$$f_*([\Delta_1^n-\Delta_2^n]) = [\Delta_2^n-\Delta_1^n]$$
$$= -[\Delta_1^n-\Delta_2^n].$$

$\Rightarrow \deg(f_*)=-1.$

$$S^n=\{(x_1,\dots,x_{n+1})\in\mathbb{R}^{n+1} \mid x_1^2+\dots+x_{n+1}^2=1\}$$

$$f(x_1,\dots,x_{n+1}) = (x_1,\dots,x_n,-x_{n+1})$$



6) If $f: S^n\to S^n$ is the antipodal map

$f(x_1,\dots,x_{n+1}) = (-x_1,\dots,-x_{n+1})$ then

$\deg(f)=(-1)^{n+1}$.

$$(x_1,\dots,x_{n+1}) \xmapsto[\deg=-1]{} (-x_1,\dots,x_n) \xmapsto[\deg=-1]{} (-x_1,-x_2,\dots,x_n) \mapsto \dots \xmapsto[\deg=-1]{} \xmapsto[\deg=-1]{} (-x_1,-x_2,\dots,-x_{n+1})$$

7) If $f: S^n \to S^n$ has no fixed points then $\deg(f) = (-1)^{n+1}$.

Proof: If $f(x) \neq x$ for all $x \in S^n$ then consider the homotopy

$$f_t(x) = \frac{(1-t)f(x) - tx}{\|(1-t)f(x) - tx\|}$$

$S^n \subseteq \mathbb{R}^{n+1}$

$x, f(x) \in S^n$

Since $f(x) \neq x$ for all $x \in S^n$ $(1-t)f(x) \neq tx$. This is because if $(1-t)f(x) = tx$ then

$$|1-t|\underbrace{|f(x)|}_{=1} = t\underbrace{|x|}_{=1} \Rightarrow (1-t) = t \Leftrightarrow t = 1/2$$

$$\Rightarrow \quad \frac{1}{2}f(x) = \frac{1}{2}x \Rightarrow f(x) = x, \text{ a contradiction.}$$

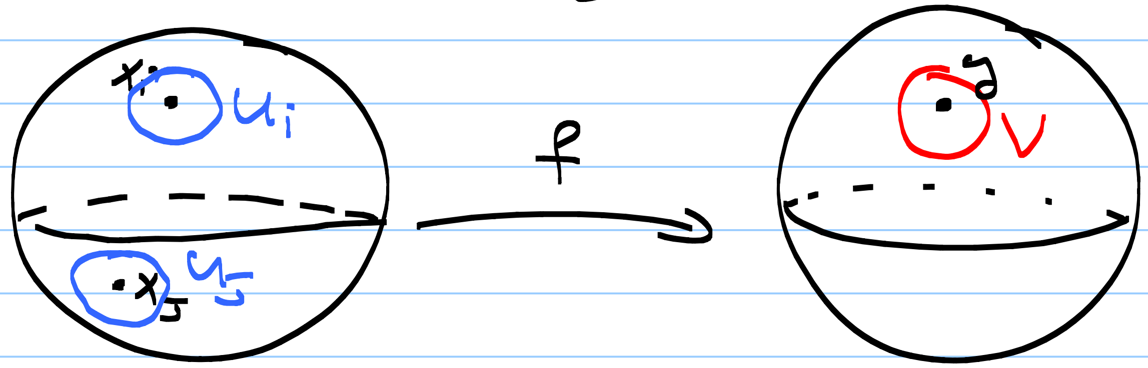
Hence, f_t is well defined. Now $f_0(x) = \frac{f(x)}{\|f(x)\|} = f(x)$

and $f_1(x) = \frac{-x}{\underbrace{|-x|}_{=1}} = -x$.

Hence, f is homotopic to $-id: S^n \to S^n$ and thus $\deg(f) = \deg(-id) = (-1)^{n+1}$.

Local degree of f: Let $f: S^n \to S^n$ be a map, $n>0$. Let $y \in S^n$ so that $f^{-1}(y)$ consists of finitely many points. Let V be a neighborhood of y and U_i be a neighborhood of x_i, where $\{x_1, \dots, x_m\} = f^{-1}(y)$ so that

$$f(U_i \setminus \{x_i\}) \subseteq V \setminus \{y\}. \quad (\text{Assume } U_i \cap U_j = \emptyset \text{ if } i \neq j)$$



Now consider the following commutative diagram

$$\begin{tikzcd}
 & H_n(U_i, U_i \setminus \{x_i\}) \arrow[dl, "\approx"', "Ex"] \arrow[d, "k_i"] \arrow[r, "f_*", "\times \deg f|_{x_i}"'] & H_n(V, V - \{y\}) \arrow[d, "\approx", "Ex"'] \\
H_n(S^n, S^n \setminus \{x_i\}) & H_n(S^n, S^n \setminus \{x_1, \dots, x_m\}) \arrow[l, "p_i"'] \arrow[r, "f_*"] & H_n(S^n, S^n \setminus \{y\}) \\
 & H_n(S^n) \arrow[ul, "\approx"] \arrow[u, "j"] \arrow[r, "f_*", "\times \deg(f)"'] & H_n(S^n) \arrow[u, "\approx"']
\end{tikzcd}$$

$H_n(U_i, U_i \setminus \{x_i\}) \cong \mathbb{Z}$, $H_n(V, V - \{y\}) \cong \mathbb{Z}$, $H_n(S^n, S^n \setminus \{x_i\}) \cong \mathbb{Z}$, $H_n(S^n, S^n \setminus \{y\}) \cong \mathbb{Z}$

$(S^n \setminus U_i)$ $\qquad$ $(S^n \setminus V)$

$(0, \dots, 1, \dots, 0) \longmapsto \deg f|_{x_i}$

$1 \longmapsto ?$

$$f_*: H_n(U_i, U_i \setminus \{x_i\}) \longrightarrow H_n(V, V \setminus \{y\})$$

$$\begin{tikzcd}
H_n(U_i, U_i \setminus \{x_i\}) \arrow[r, "f_*"] \arrow[d, phantom, "\wr"] & H_n(V, V \setminus \{y\}) \arrow[d, phantom, "\wr"] \\
\mathbb{Z} \arrow[r, "\times \deg f|_{x_i}"'] & \mathbb{Z}
\end{tikzcd}$$

The integer $\deg f|_{x_i}$ is called the local degree of f at x_i.

$H_n(S^n, S^n \setminus \{x_1, \dots, x_m\})$

$A = S^n \setminus (U_1 \cup \dots \cup U_m)$ closed set in $S^n \setminus \{x_1, \dots, x_m\}$. Excise A from S^n and $S^n \setminus \{x_1, \dots, x_m\}$ to get

$$H_n(S^n, S^n \setminus \{x_1, \dots, x_m\}) \simeq H_n\left(\dot\bigcup U_i, \dot\bigcup (U_i \setminus \{x_i\})\right)$$
$$\simeq \bigoplus_{i=1}^{m} H_n(U_i, U_i \setminus \{x_i\}) \simeq \mathbb{Z}^m$$

$$\begin{array}{ccccc}
H_n(U_i, U_i \setminus \{x_i\}) & \xrightarrow{k_i} & H_n(S^n, S^n \setminus \{x_1, \dots, x_m\}) & \xrightarrow{p_i} & H_n(S^n, S^n \setminus f^{-1}(b)) \\
\wr\| & & \wr\| & & \wr\| \\
\mathbb{Z} & & \mathbb{Z}^m & & \mathbb{Z} \\
e & \longmapsto & (0, \dots, \underset{i\text{th place}}{e}, \dots 0) & \longmapsto & e \\
 & & (e_1, \dots, e_i, \dots, e_m) & \longmapsto & e_i
\end{array}$$

Clearly the diagram is commutative. It follows that
$$J(1) = (1, 1, \dots, 1) \in \mathbb{Z}^m \simeq H_n(S^n, S^n \setminus \{x_1, \dots, x_m\}).$$

Since $J(1) = (1, 1, \dots, 1) = (1, 0, \dots, 0) + (0, 1, 0 \dots 0) + \dots + (0, \dots, 1)$ and $(0, \dots, 1, \dots 0)$ is mapped to $\deg f|_{x_i}$ in $H_n(S^n)$

$f_* : H_n(S^n) \to H_n(S^n)$ has degree

$$\deg f = \sum_{i=1}^{m} \deg f|_{x_i}$$

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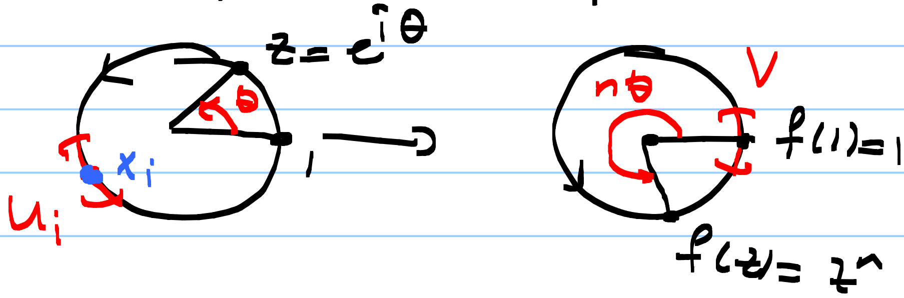
Proposition: Assume the above set up. Then

$$\deg f = \sum_{i=1}^{m} \deg f|_{x_i}.$$

Example: $f: S^1 \to S^1$, $f(z) = z^n$, $z \in S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$



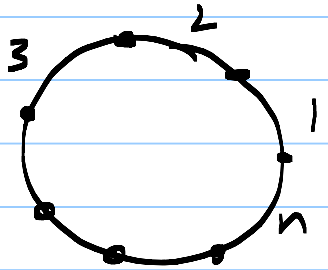
$y = 1$

$f^{-1}(y) = \{z_0, \dots, z_{n-1}\}$

$z_k = e^{2\pi i k/n}$ $\quad k = 0, \dots, n-1.$

$f: (U_i, U_i \setminus \{x_i\}) \to (V, V \setminus \{1\})$ are all homeomorphism

and hence $\deg f|_{x_i} = 1$.



Cellular Homology

Lemma: If X is a CW-complex, then

a) $H_k(X^n, X^{n-1})$ is zero for $k \neq n$ and is free abelian for $k = n$, with a basis in one-to-one correspondence with the n-cells of X.

b) $H_k(X^n)=0$ for $k>n$. In particular, if X is finite dimensional then $H_k(X)=0$ if $k>\dim X$.

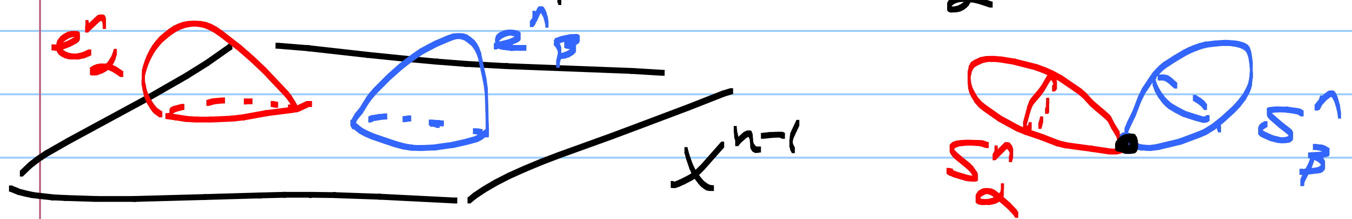
c) The inclusion $i: X^n \hookrightarrow X$ induces an isomorphism

$i_*: H_k(X^n) \to H_k(X)$ if $k<n$.

Proof: $X^n = X^{n-1} \cup \coprod_\alpha e^n_\alpha \qquad f_\alpha: \partial e^n_\alpha = S^{n-1}_\alpha \to X^{n-1}$

Since (X^n, X^{n-1}) is a good pair and thus

$$H_k(X^n, X^{n-1}) \simeq \tilde{H}_k(X^n/X^{n-1}) \simeq \tilde{H}_k(\bigvee_\alpha S^n_\alpha)$$



So, $H_k(X^n, X^{n-1}) = 0$ if $k \neq n$ and

$$H_n(X^n, X^{n-1}) \simeq \tilde{H}_n(\bigvee_\alpha S^n_\alpha) \simeq \bigoplus_\alpha \mathbb{Z}$$

b) For (b) consider the exact sequence of the pair

$$\cdots \to \underset{\substack{\text{by (a)}\\ \text{is } 0}}{H_{k+1}(X^n, X^{n-1})} \to H_k(X^{n-1}) \xrightarrow[\simeq]{i} H_k(X^n) \to \underset{\substack{\text{is } 0\\ \text{by (a)}}}{H_k(X^n, X^{n-1})} \to H_{k-1}(X^{n-1})$$

if $k>n$

So $H_k(X^{n-1}) \simeq H_k(X^n)$ if $k>n$. $(\Rightarrow k>n\geq 0)$

Hence, $H_k(X^n) \simeq H_k(X^{n-1}) \simeq H_k(X^{n-2}) \simeq \dots \simeq H_k(X^0) = 0$.
This proves (b).

Now (c) assume first X is finite dimensional.
If $k<n$, then by a similar argument we have

$$H_k(X^n) \approx H_k(X^{n+1}) \approx \dots \approx H_k(X^{n+m}) = H_k(X)$$

if $\dim X = n+m$.

If X is infinite dimensional then more care is needed. (Exercise, Read from the Book p. 138).

Cellular Homology X CW complex.

$H_k(X^n) = 0$ if $k>n$ and $i_*: H_k(X^n) \to H_k(X)$ an isomorphism if $k<n$.

$$H_k(X^n, X^{n-1}) = \begin{cases} 0 & \text{if } k \neq n \\ \bigoplus_{e^n_\alpha} \mathbb{Z} & \text{if } k=n \end{cases}$$

The exact sequences of the pairs (X^{n+1}, X^n), (X^n, X^{n-1}) and (X^{n-1}, X^{n-2}) fit into a diagram as below

$$
\begin{tikzcd}
 & & & 0 & \\
0 \arrow[dr] & & H_n(X^{n+1}) \simeq H_n(X) \arrow[ur] & & \\
 & H_n(X^n) \arrow[ur, "\text{surj}"] \arrow[dr, "j_n\ \text{-inj}"] & & & \\
\cdots \to H_{n+1}(X^{n+1}, X^n) \arrow[ur, "\partial_{n+1}"] \arrow[rr, "d_{n+1}"] & & H_n(X^n, X^{n-1}) \arrow[rr, "d_n"] \arrow[dr, "\partial_n"'] & & H_n(X^{n-1}, X^{n-2}) \to \cdots \\
 & & & H_{n-1}(X^{n-1}) \arrow[ur, "j_{n-1}\ \text{-inj}"'] & \\
 & & 0 \arrow[ur] & &
\end{tikzcd}
$$

$d_{n+1} \doteq j_n \circ \partial_{n+1}$

$d_n = j_{n-1} \circ \partial_n$

$$
\begin{array}{ccccccc}
\to H_{n+1}(X^{n+1}, X^n) & \xrightarrow{d_{n+1}} & H_n(X^n, X^{n-1}) & \xrightarrow{d_n} & H_{n-1}(X^{n-1}, X^{n-2}) & \to & . \\
\| & & \| & & \| & & \\
A_{n+1} & \xrightarrow{d_{n+1}} & A_n & \xrightarrow{d_n} & A_{n-1} & &
\end{array}
$$

<u>Claim</u> $d_n \circ d_{n+1} = 0$

<u>Proof</u> $d_n \circ d_{n+1} = j_{n-1} \circ \underbrace{\partial_n \circ j_n}_{\text{is } 0} \circ \partial_{n+1} = 0.$ ∎

Hence, $(A_*, d_*) = (H_n(X^n, X^{n-1}), d_n)$ is a chain complex.

The homology of the chain complex is called the cellular homology of the CW-complex X and will be denoted as $H_n^{CW}(X)$.

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Theorem: $H_n^{CW}(X) \simeq H_n(X)$.

Proof: $H_n(X) \simeq H_n(X^n)/\mathrm{Im}\,\partial_{n+1}$. Moreover, since j_n is injective it maps $\mathrm{Im}\,\partial_{n+1}$ isomorphically onto $\mathrm{Im}\,\partial_{n+1} = \mathrm{Im}\, d_{n+1}$ and $H_n(X^n)$ isomorphically onto $\mathrm{Im}\,\partial_n = \ker \partial_n$. Also j_{n-1} is injective and thus $\ker \partial_n = \ker d_n$. Thus j_n induces an isomorphism

$$\underset{\substack{\text{\rotatebox{90}{$\simeq$}}\\ H_n(X)}}{j_n : H_n(X^n)/\mathrm{Im}\,\partial_{n+1}} \xrightarrow{\sim} \frac{\ker d_n}{\mathrm{Im}\, d_{n+1}} = H_n^{CW}(X)$$

Remarks 1) If X has no n-cells then $H_n(X^n, X^{n-1}) = 0 \Rightarrow H_n^{CW}(X) = 0$

(X is a CW complex) $\Rightarrow H_n(X) = 0$.

2) If X has k n-cells then $H_n(X^n, X^{n-1})$ has rank n. So $\ker d_n$ has rank at most k.

$$\underset{\substack{\text{\rotatebox{90}{$\simeq$}}\\ \mathbb{Z}^k}}{H_n(X^n, X^{n-1})} \xrightarrow{d_n} H_{n-1}(X^{n-1}, X^{n-2})$$

So, $H_n^{CW}(X) = \dfrac{\ker d_n}{\mathrm{Im}\, d_{n+1}}$ can be generated by k elements.

3) If X is a CW-complex having no two of its cells in adjacent dimensions, then $H_n(X)$ is free abelian with basis in one-to-one correspondence with the n-cells of X.

$$\begin{array}{ccccc} \to H_{n+1}(X^{n+1},X^n) & \longrightarrow & H_n(X^n,X^{n-1}) & \longrightarrow & H_{n-1}(X^{n-1},X^{n-2}) \to \\ \| & & & & \| \\ 0 & \longrightarrow & ? & \longrightarrow & 0 \end{array}$$

$$H_n^{CW}(X) = \frac{H_n(X^n,X^{n-1})}{(0)} \cong H_n(X^n,X^{n-1})$$

<u>Ex</u> $\mathbb{CP}^n = e^0 \cup e^2 \cup e^4 \cup \cdots \cup e^{2n-2} \cup e^{2n}$

$$H_k(\mathbb{CP}^n) = \begin{cases} \mathbb{Z}, & k = 0, 2, \ldots, 2n \\ 0, & \text{otherwise} \end{cases}$$

Cellular Boundary Formula

$$\begin{array}{ccc} H_n(X^n,X^{n-1}) & \xrightarrow{d_n} & H_{n-1}(X^{n-1},X^{n-2}) \\ \wr\| & & \wr\| \\ H_n(X^n/X^{n-1}) & & H_{n-1}(X^{n-1}/X^{n-2}) \\ \wr\| & & \wr\| \\ \bigvee_\alpha S^n_\alpha & & \bigvee_\beta S^{n-1}_\beta \end{array}$$

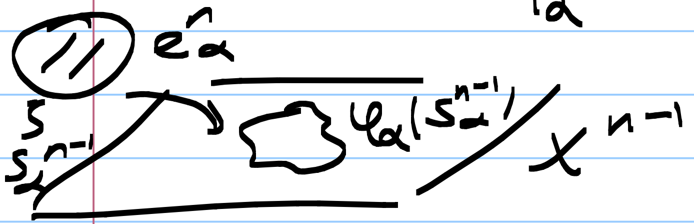
e^n_α

$$H_n(X^n,X^{n-1}) \simeq \langle e^n_\alpha \rangle$$
$$H_{n-1}(X^{n-1},X^{n-2}) = \langle e^{n-1}_\beta \rangle$$

Claim: $d_n(e^n_\alpha) = \sum_\beta \underline{d_{\alpha\beta}}\, e^{n-1}_\beta$, where

$d_{\alpha\beta}$ is the degree of the map below

$$\begin{tikzcd}
\partial e^n_\alpha = S^{n-1}_\alpha \arrow[r, "\varphi_\alpha"] & X^{n-1} \arrow[r] & X^{n-1}/X^{n-2} = \bigvee_\beta S^{n-1}_\beta \arrow[d, "\text{collapsing all spheres except } S^{n-1}_\beta"] \\
 & & S^{n-1}_\beta
\end{tikzcd}$$



collapsing all spheres except S^{n-1}_β.

$$\partial D^n_\alpha = S^{n-1}_\alpha \longrightarrow S^{n-1}_\beta$$

More precise argument can be given using the diagram below:

$$\begin{tikzcd}
H_n(D^n_\alpha, \partial D^n_\alpha) \arrow[r, "\partial"] \arrow[d, "\Phi_{\alpha *}"] & \tilde{H}_{n-1}(\partial D^n_\alpha) \arrow[r, "D_{\alpha\beta *}"] \arrow[d, "\varphi_{\alpha *}"] & \tilde{H}_{n-1}(S^{n-1}_\beta) \\
H_n(X^n, X^{n-1}) \arrow[r, "\partial_n"] \arrow[dr, "d_n"'] & \tilde{H}_{n-1}(X^{n-1}) \arrow[r, "q_*"] \arrow[d, "j_{n-1}"] & \tilde{H}_{n-1}(X^{n-1}/X^{n-2}) \arrow[u, "q_{\beta *}"'] \arrow[d, "\cong"] \\
 & \underline{H_{n-1}(X^{n-1}, X^{n-2})} \arrow[r, "\cong"] & H_{n-1}(X^{n-1}/X^{n-2}, X^{n-2}/X^{n-2})
\end{tikzcd}$$

Here

1) Φ_α the characteristic map of e^n_α and φ_α is the attaching map.

2) $q : X^{n-1} \longrightarrow X^{n-1}/X^{n-2}$ is the quotient map

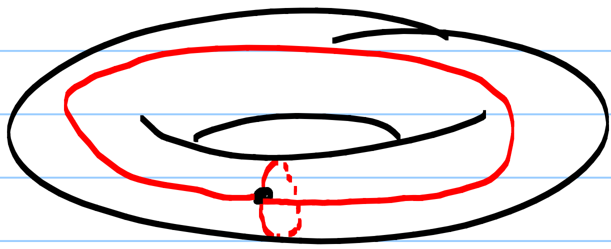
3) $q_\beta : X^{n-1}/X^{n-2} \longrightarrow S^{n-1}_\beta$ is the collapsing map of all the spheres except S^{n-1}_β.

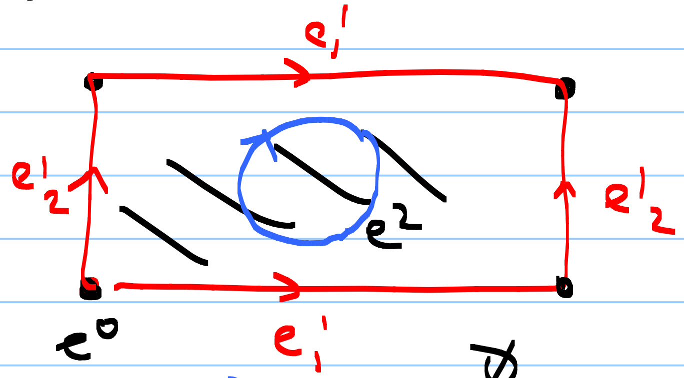
4) $\Delta_{\alpha\beta} : \underset{\substack{\| \\ S^{n-1}_\alpha}}{\partial D^n_\alpha} \longrightarrow S^{n-1}_\beta$ is described above.

<u>Examples</u> 1) T^2

$T^2 = e^0 \cup e^1_1 \cup e^1_2 \cup e^2_1$

$X^0 = e^0, \quad X^1 = e^0 \cup e^1_1 \cup e^1_2, \quad X^2 = T^2$





$$
\begin{tikzcd}
0 \arrow[r] & H_2(X^2, X^1) \arrow[r, "d_2"] \arrow[d, phantom, "\cong" description] & H_1(X^1, X^0) \arrow[r, "d_1 = 0"] \arrow[d, phantom, "\cong" description] & H_0(X^0, \underset{\substack{\| \\ \emptyset}}{X^{-1}}) \arrow[r] \arrow[d, phantom, "\cong" description] & {} \\
& \mathbb{Z} \arrow[d, phantom, "\|" description] & \mathbb{Z} \oplus \mathbb{Z} \arrow[d, phantom, "\|\quad\|" description] & \mathbb{Z} \arrow[d, phantom, "\|" description] & \\
& \langle e^2 \rangle \arrow[r] & \langle e^1_1 \rangle \quad \langle e^1_2 \rangle \arrow[r, "0"] & \langle e^0 \rangle &
\end{tikzcd}
$$

$d_1 = ?$ $\quad e^1_1$ $\quad \overset{-}{\bullet} \longrightarrow \overset{+}{\bullet}$ $\qquad \varphi^1_1 : \partial e^1_1 \longrightarrow X^0 = e^0$

$d_1(e^1_1) = e^0 - e^0 = 0.$

Similarly, $d_1(e^1_2) = 0.$

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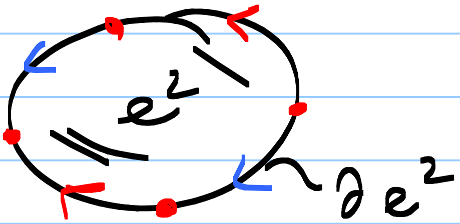
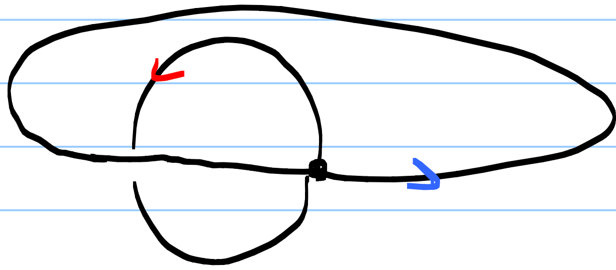
$d_2 = ?$ $\quad d_2(e^2) = e_1^1 - e_2^1 - e_1^1 + e_2^1 = 0$





$$0 \longrightarrow \mathbb{Z} \xrightarrow{d_2 = 0} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{d_1 = 0} \mathbb{Z} \xrightarrow{d_0} 0$$

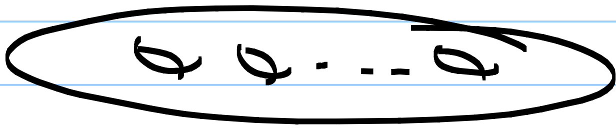
$$H_2(T^2) = H_2^{CW}(T^2) = \frac{\ker d_2}{\operatorname{Im} d_3} = \frac{\mathbb{Z}}{(0)} \simeq \mathbb{Z}$$

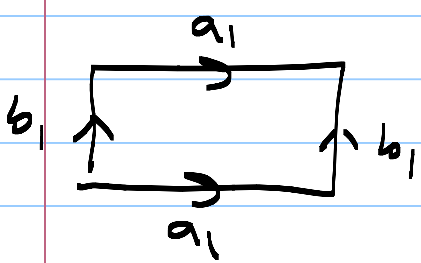
$$H_1(T^2) = H_1^{CW}(T^2) = \frac{\ker d_1}{\operatorname{Im} d_2} = \frac{\mathbb{Z} \oplus \mathbb{Z}}{(0)} \simeq \mathbb{Z} \oplus \mathbb{Z}$$

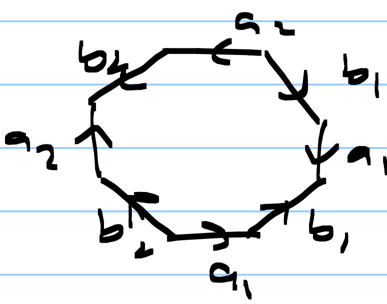
$$H_0(T^2) = H_0^{CW}(T^2) = \frac{\ker d_0}{\operatorname{Im} d_1} = \frac{\mathbb{Z}}{(0)} \simeq \mathbb{Z}.$$

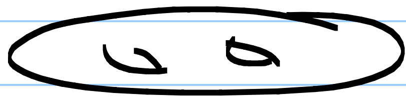
$$H_k(T^2) = \begin{cases} \mathbb{Z} & \text{if } k = 0, 2 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases}$$

2) Σ_g:

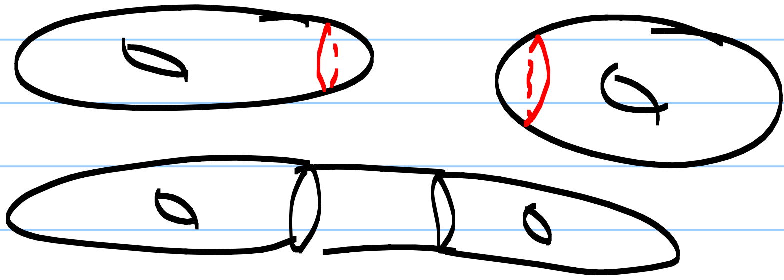


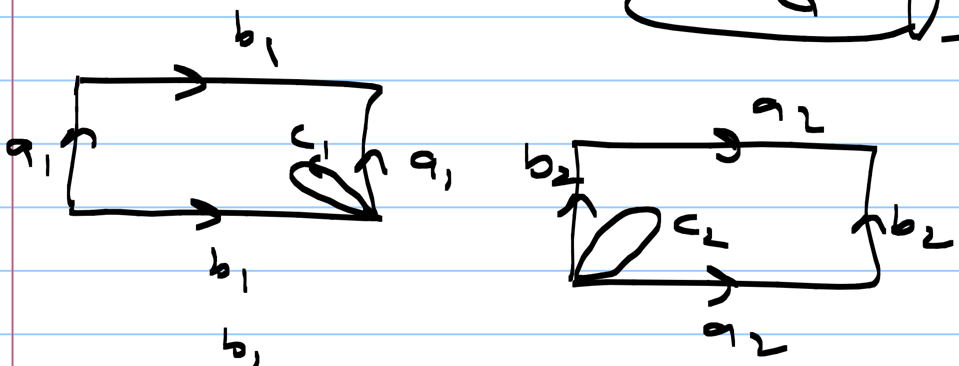


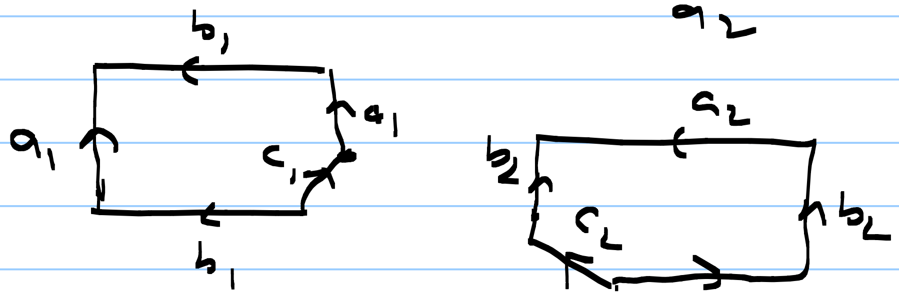


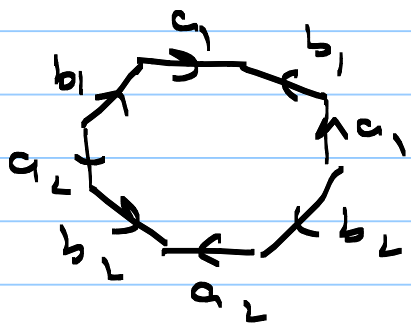


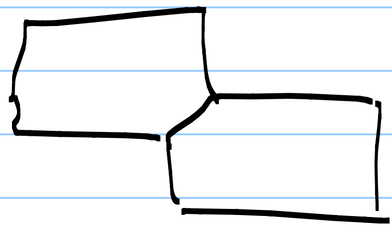
$\Sigma_2 = T^2 \# T^2$



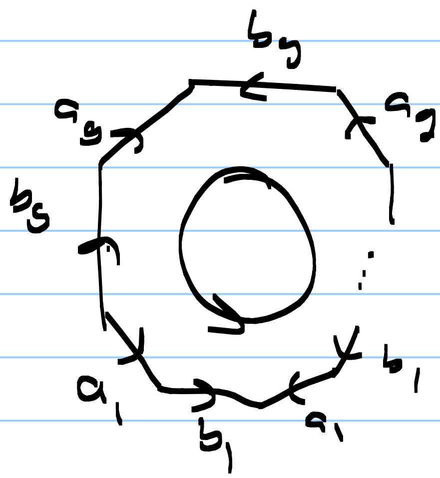








$$\Sigma_g = \underbrace{T^2 \# T^2 \# T^2 \# \quad \# T^2}_{g\text{-copies}}$$



$$\Sigma_g = e^0 \cup e^1_1 \cup e^1_2 \cup \cdots \cup e^1_{2g} \cup e^2$$

$d_1 = 0$

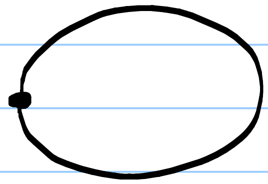
$d_2(e^2) = e^1_1 + e^1_2 - e^1_1 - e^1_2 + \cdots + \cdots = 0$

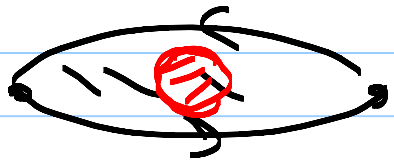
$$0 \to \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^{2g} \xrightarrow{d_1} \mathbb{Z} \xrightarrow{d_0} 0$$

$H_2(\Sigma_g) \simeq \mathbb{Z}, \quad H_1(\Sigma_g) = \mathbb{Z}^{2g}, \quad H_0(\Sigma_g) = \mathbb{Z}.$

Ex Exercise $H_k(\mathbb{RP}^2) = ?$, $H_k(KB) = ?$

$\mathbb{RP}^2 = e^0 \cup e^1 \cup e^2$





$$d_2 : H_2(X^2, X^1) \to H_1(X^1, X^0)$$

$$\begin{array}{ccc} \cong & & \cong \\ \mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z} \end{array}$$

$KB = \mathbb{RP}^2 \# \mathbb{RP}^2$

Homology and Fundamental Group

① X path connected space, $x_0 \in X$ base point

$h: \pi_1(X, x_0) \longrightarrow H_1(X)$

$h([f]) = [f]$

$f: [0,1] \longrightarrow X$ cont. $h(0) = h(1) = x_0$

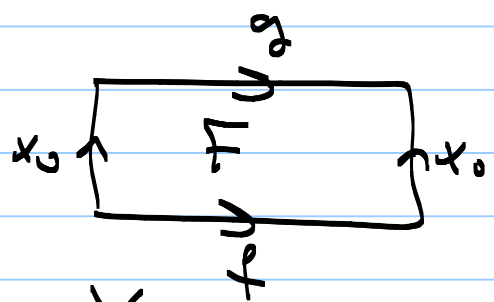
$\partial [f] = [f|_1] - [f|_0] = [x_0] - [x_0] = 0.$

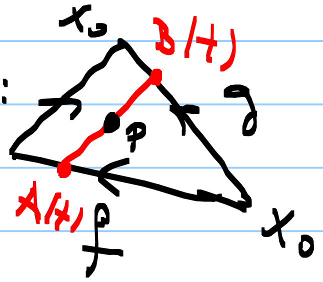
So, $[f]$ is a cycle and hence $h([f]) \in H_1(X)$.

② h is well defined:

$[f] = [g]$ in $\pi_1(X, x_0)$

$[f] = [g]$ in $H_1(X)$?



τ: 

$\tau(p) = F(s,t) \longrightarrow X$

$\partial \tau = [f] - \underset{\underset{0}{\parallel}}{[x_0]} - [g] = 0$ in H_1

$p = (1-t)A(t) + tB(t)$

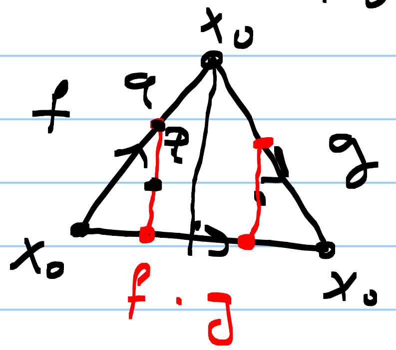
$\Rightarrow [f] = [g]$.

③ $h: \pi_1(X, x_0) \longrightarrow H_1(X)$ is a group homomorphism.

$[f], [g] \in \pi_1(X, x_0)$

must show $h([f \cdot g]) = h([f]) + h([g])$

$$(f\cdot g)(s) = \begin{cases} f(\underline{\underline{2s}}) & 0\le s\le 1/2 \\ g(\underline{\underline{2s-1}}), & 1/2\le s\le 1 \end{cases}$$



$\tau(p) = f(q)$

$\tau : \Delta_2 \longrightarrow X$

$$0 = [\partial \tau] = [f\cdot g - g - f] = 0.$$

$$= [f\cdot g] - [g] - [f]$$

$$= h([f\cdot g]) - h([g]) - h([f])$$

$$\Rightarrow h([f\cdot g]) = h([f]) + h([g]).$$

So, h is a homomorphism

Note that $0 = h([x_0]) = h([f][\bar{f}])$

$0 = h([f]) + h([\bar{f}]) \Rightarrow h([f]^{-1}) = -h([f]).$

$h : \pi_1(X, x_0) \longrightarrow H_1(X)$ homomorphism.

<u>h is onto:</u> Let $[\alpha] \in H_1(X)$, then

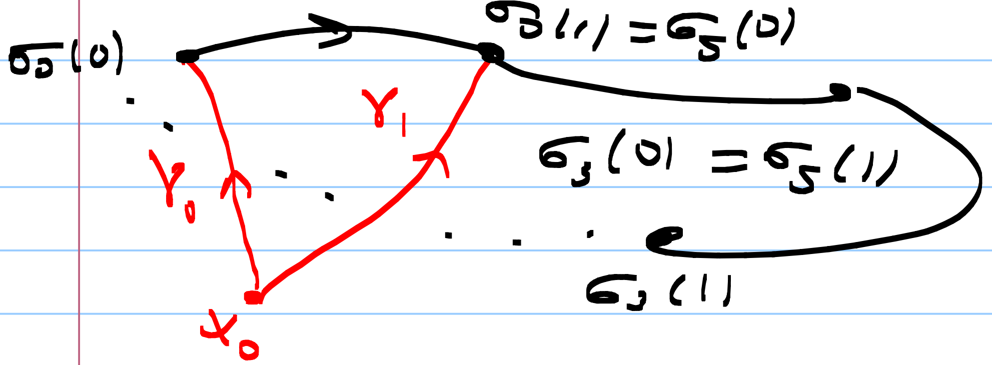
$\alpha = \sum n_i \sigma_i\ ,\ n_i \in \mathbb{Z}$

We can assume each $n_i = \pm 1$ by writing σ_i several times. Writing $-\sigma$ as another one simplex we can assume that each $n_i = 1$.

$\alpha = \sum \sigma_i$, $\sigma_i : [0,1] \to X$ singular simplex.

$0 = \partial\alpha = \sum \partial\sigma_i = \sum_i (\sigma_i(1) - \sigma_i(0))$

So each $\sigma_i(1) = \sigma_j(0)$ for some σ_j.



$\alpha = \sigma_1 + \sigma_2 + \cdots + \sigma_n$

$= \underbrace{(\cancel{\gamma_0} + \sigma_1 - \cancel{\gamma_1})}_{\text{loop}} + \underbrace{(\cancel{\gamma_1} + \sigma_3 - \cancel{\gamma_2})}_{\text{loop}} + \cdots$

$= h([\gamma_0 \sigma_1 \bar{\gamma}_1]) + h([\gamma_1 \sigma_2 \cdot \bar{\gamma}_2]) + \cdots$

$= h([\gamma_0 \sigma_1 \bar{\gamma}_1 \cdot \gamma_1 \sigma_2 \cdot \bar{\gamma}_2 \cdots])$

$\Rightarrow h$ is onto.

So, $h : \pi_1(X, x_0) \to H_1(X)$ is onto.

Hence, $H_1(X) = \mathrm{im}(h) = \pi_1(X, x_0) / \ker h$

$\ker h = ?$

Claim: $\ker h = [\pi_1(X,x_0), \pi_1(X,x_0)]$

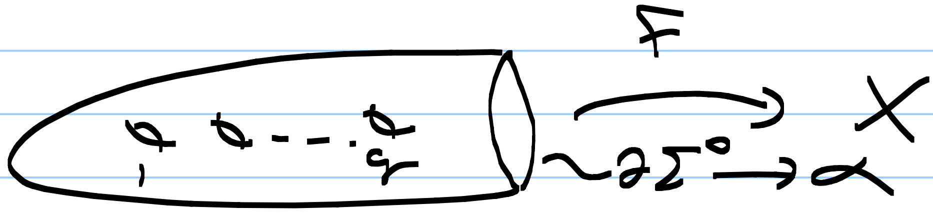
Proof: Let $h([\alpha]) = 0$ for some homotopy class $[\alpha] \in \pi_1(X,x_0)$. Hence, as a 1-cycle α is the boundary of a 2-chain.

$\alpha = \partial\beta$, $\beta = \sum n_i \tau_i$, τ_i singular 2-simplex.

$\tau : [v_0, v_1, v_2] \longrightarrow X$

This gives us a continuous map from an orientable surface Σ^0 with one boundary component into X so that

$F : \Sigma^0 \longrightarrow X$

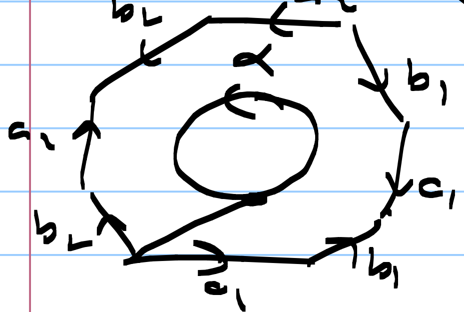


$F(\partial\Sigma^0) = [\alpha]$, $\Sigma^0 \sim \beta = \sum_i \tau_i$

$\pi_1(\Sigma^0) \longrightarrow \pi_1(X) \xrightarrow{F_\#} H_1(X)$

$[\alpha] = [a_1][b_1][a_1]^{-1}[b_1]^{-1} \cdots [a_g][b_g][a_g]^{-1}[b_g]^{-1}$

$\underbrace{x_1 y_1 x_1^{-1} y_1^{-1}}_{\in[\pi_1(X), \pi_1(X)]} \cdots \underbrace{x_g y_g x_g^{-1} y_g^{-1}}_{\in[\pi_1(X), \pi_1(X)]}$



$[\alpha] \in [\pi_1(X), \pi_1(X)]$.

$\ker h \subseteq [\pi_1(X), \pi_1(X)]$.

Finally, since $H_1(X)$ is abelian the kernel of $h : \pi_1(X) \to H_1(X)$ lies in the commutator.

So, $\ker h = [\pi_1(X), \pi_1(X)]$ and hence

$$H_1(X) = \pi_1(X) \big/ [\pi_1(X), \pi_1(X)] \doteq \pi_1(X)_{ab}$$