MATH 424

Galois Theory of Linear Differential Equations

Note Title 24.06.2021

METU Math. Department

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Textbook: Galois Dream by
Michio Kuga

Video 1

MATH 424 Galois Theory of Covering and

Note Title 9.03.2021

Linear Differential Equations

Definition of Topological Spaces:

A topology on a set X is a collection $\mathcal{T}$ of subsets of X satisfying the following axioms:

1) $\phi, X \in \mathcal{T}$

2) If $U_1, \dots, U_n \in \mathcal{T}$ then so is $U_1 \cap U_2 \cap \cdots \cap U_n$.

3) If $U_\alpha \in \mathcal{T}$ for all $\alpha \in J$, for some index set J, then the union $\bigcup_{\alpha \in J} U_\alpha \in \mathcal{T}$.

In this case the pair $(X, \mathcal{T})$ is called a topological space and elements of the collection $\mathcal{T}$ are called open subsets of the topology.

A subset C of X will be called closed if its complement $X \setminus C$ is open.

Let $A \subseteq X$ be any subset. The closure of A, denoted $\overline{A}$ is the subset defined by

$$\overline{A} = \bigcap_{\substack{A \subseteq F \\ F \subseteq X \text{ closed}}} F,$$ which is clearly a closed subset.

Similarly, interior of A, denoted $\text{Int} A$ or $\mathring{A}$, is the open subset defined by

$$\text{Int} A = \bigcup_{\substack{U \subseteq A \\ U \subseteq X \text{ open}}} U.$$

Examples 1) Finite topologies

$X=\{a,b,c\}$, $\mathcal{T}=\{\phi, \{a,b,c\}, \{a\}, \{b\}, \{a,b\}\}$

Is a topology on X. So $\{a\}$ and $\{b\}$ are open subsets but $\{c\}$ is not. On the other hand, $\{c\}$ is closed but $\{a\}$ and $\{b\}$ are not.

2) X any set. The smallest topology on X is the topology $\{\phi, X\}$. This topology is also called the weakest or coarsest topology on X.

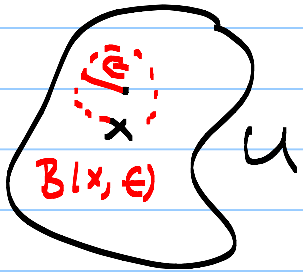
3) X any set. The largest topology on X is the topology given by the power set $\mathcal{P}(X)$.
This topology is also called the strongest or the finest topology on X.

4) $X=\mathbb{R}$, $\mathcal{T}_{std}=\{U \mid U\subseteq\mathbb{R}, \text{ if } x\in U \text{ then } (x-\epsilon,x+\epsilon)\subseteq U, \text{ for some } \epsilon>0\}$.

$\mathbb{R}_{std}$, $\mathbb{R}^2_{std}$, $\mathbb{R}^n_{std}$, $U\subseteq\mathbb{R}^n_{std}$ is open if

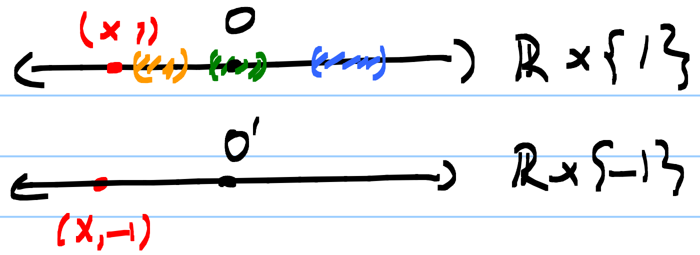
for any $x\in U$ there is some $\epsilon>0$ so that

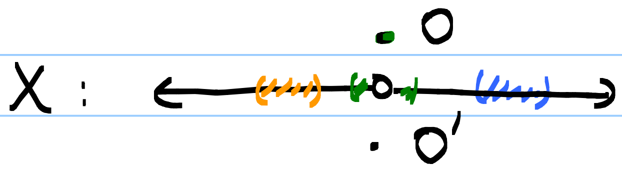
$B(x,\epsilon)=\{y\in\mathbb{R}^n \mid \|x-y\|<\epsilon\}\subseteq U$.



5) Real line with double origin.

$X=\mathbb{R}\times\{-1,1\}/(x,-1)\sim(x,1)$ if $x\neq 0$





Definition: A topological space $(X, \mathcal{T})$ is called T_0 if for any $x, y \in X$ with $x \neq y$ there is an open subset $U \in \mathcal{T}$ so that either ($x \in U$ and $y \notin U$) or ($x \notin U$ and $y \in U$).

Similarly, X is called T_1 if for any $x, y \in X$ with $x \neq y$ there is an open set $U \in \mathcal{T}$ with $x \in U$ and $y \notin U$.

Finally, X is called T_2 (or Hausdorff) if for any $x, y \in X$ with $x \neq y$ there are open subsets U, V of $\mathcal{T}$ so that

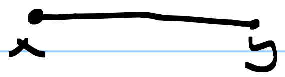
$x \in U$, $y \in V$ and $U \cap V = \phi$.

Clearly, $T_2 \Rightarrow T_1 \Rightarrow T_0$.

Example: The real line with double origin is T_1 but not T_2.

Example: (X, d) metric space. $x, y \in X$, $x \neq y$

$\varepsilon > 0$, $\varepsilon = d(x,y)$



$r = \frac{\varepsilon}{3} > 0$

$U = B(x,r)$, $V = B(y,r)$, $x \in U$, $y \in V$ and $U \cap V = \phi$.

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Hence, any metric space is T_2.

Equivalence of Topologies:

A function $f:(X,\mathcal{T}_X)\to(Y,\mathcal{T}_Y)$ is called continuous if $f^{-1}(U)$ is open in X, whenever U is open in Y.

A continuous bijection $f:(X,\mathcal{T}_X)\to(Y,\mathcal{T}_Y)$, whose inverse $f^{-1}:(Y,\mathcal{T}_Y)\to(X,\mathcal{T}_X)$ is also continuous, is called a homeomorphism. In this case, we say that the topological spaces are homeomorphic.

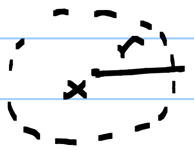
In topology homeomorphic spaces are regarded the same.

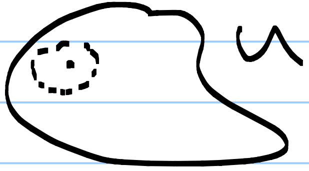
Basis and Subbasis:

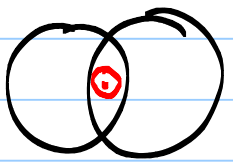
Let $(X,\mathcal{T})$ be a topological space. A subcollection $\mathcal{B}$ of open subsets of $(X,\mathcal{T})$ is called a basis of the topology if the following is satisfied: For any open subset U of X and point $x\in U$, there is some $B\in\mathcal{B}$ so that $x\in B\subseteq U$.

Remark: Let X be any set. A collection $\mathcal{B}$ of subsets of X is called a basis for a topology on X if i) For any $x\in X$ there is some $B\in\mathcal{B}$ s.t. $x\in B$, and ii) For any $x\in B_1\cap B_2$ for any $B_1,B_2\in\mathcal{B}$ there is some $B_3\in\mathcal{B}$ s.t. $x\in B_3\subseteq B_1\cap B_2$. In this case, arbitrary unions of finitely many intersections of B_i's form a topology on X.

Example $X=\mathbb{R}^2_{std}$, $\mathcal{B}=\{B(x,r)\mid x\in\mathbb{R}^2, r>0\}$

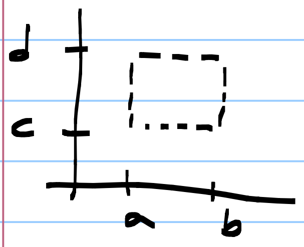






2) Another basis for the same space $\mathbb{R}^2_{std}$.

$$\mathcal{C} = \{(a,b)\times(c,d) \mid a<b,\ c<d,\ a,b,c,d\in\mathbb{R}\}$$

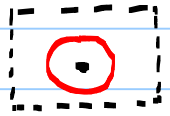


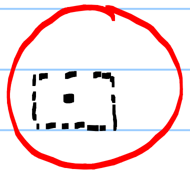
Comparison of Topologies: Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on the same X. We say that $\mathcal{T}_1$ is stronger or finer than $\mathcal{T}_2$ if $\mathcal{T}_2 \subseteq \mathcal{T}_1$.

Remark: 1) If $\mathcal{B}$ is a basis for a topology $\mathcal{T}$ on X then for any open subset U of $(X,\mathcal{T})$ we have

$$U = \bigcup_{\substack{B\in\mathcal{B} \\ B\subseteq U}} B$$

2) (Exercise) Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bases for two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on a set X. Then $\mathcal{T}_1$ is stronger than $\mathcal{T}_2$ if and only if for any $B\in\mathcal{B}_2$ and $x\in B$ there is some $B'\in\mathcal{B}_1$ so that $x\in B'\subseteq B$.





New Topologies from old Ones:

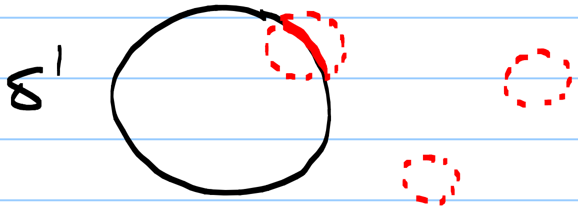
Subspace Topology: $(X,\mathcal{T})$ topological space.

Any subset A of X inherits a topology from X, called $\mathcal{T}_A$.

$\mathcal{T}_A = \{ A \cap U \,/\, U \in \mathcal{T} \}$.

Exercise: $\mathcal{T}_A$ is a topology on A.

Example: $S^1 = \{(x,y) \in \mathbb{R}^2 \,/\, x^2+y^2=1\} \subseteq \mathbb{R}^2_{std}$.



Definition: Let $f:(X,\mathcal{T}_X) \longrightarrow (Y,\mathcal{T}_Y)$ be a one to one and continuous map. Then the image subset $f(X)$ of Y has the subspace topology.
If the map $f:(X,\mathcal{T}_X) \longrightarrow (f(X), \mathcal{T}_Y|_{f(X)})$ is a homeomorphism (i.e., $f^{-1}: f(X) \to X$ is continuous) then the map $f: X \longrightarrow Y$ is called a topological embedding.

Example: $X = [0,\infty)$, the the collection

$\mathcal{B} = \{(a,b) / 0 < a < b\} \cup \{[0,a) \cup (b,\infty) / a,b > 0\}$

is a basis for a topology on X.

Claim: The topological space $(X,\mathcal{T})$ generated by $\mathcal{B}$ is homeomorphic to S^1 equipped with the subspace topology inherited from

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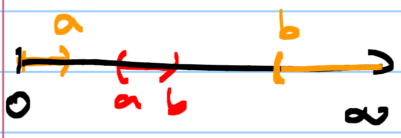
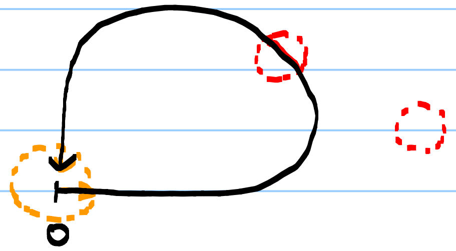
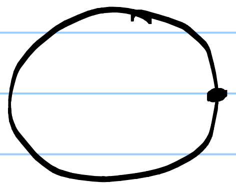
$\mathbb{R}^2$:





Proof: $f:[0,\infty)\to S^1$, $f(t)=e^{2\pi i t/(1+t)}$

$f(t)=\left(\cos\frac{2\pi t}{1+t}, \sin\frac{2\pi t}{1+t}\right)$



$f(0)=(1,0)$

$[0,\infty)\longmapsto[0,2\pi)$

$\frac{t}{1+t}\to 1$

■

Product Spaces: $(X_\alpha,\mathcal{T}_\alpha)_{\alpha\in\Lambda}$ topological spaces

$\prod_{\alpha\in\Lambda}X_\alpha=\{(a_\alpha)\mid a_\alpha\in X_\alpha,\ \alpha\in\Lambda\}$.

$X_1\times X_2\times\dots\times X_n=\{(a_1,a_2,\dots,a_n)\mid a_k\in X_k,\ k=1,\dots,n\}$.

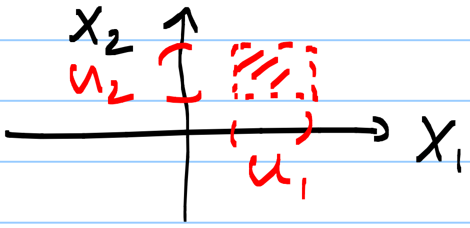
Base for the product topology is given by

$\mathcal{B}=\{\prod_\alpha U_\alpha \mid U_\alpha\subseteq X_\alpha$ open and $U_\alpha=X_\alpha$ for all but finitely many $\alpha\in\Lambda\}$.

Finite case: $X_1\times\dots\times X_n$

$\mathcal{B}=\{U_1\times U_2\times\dots\times U_n\mid U_k\subseteq X_k \text{ open}\}$.

n=2



Quotient Topology: Let (X,τ) be a topological space and $f: X \to Y$ a surjection, when Y is a set. The collection

$$\tau' = \{U \subseteq Y \mid f^{-1}(U) \text{ is open in } X\}$$

defines a topology on Y called the quotient topology on Y induced by $f: X \to Y$.

Claim: τ' is a topology on Y.

Proof: 1) $\phi \in \tau$, $\phi = f^{-1}(\phi) \in \tau'$
$X \in \tau$ and $X = f^{-1}(Y) \in \tau$ and thus $Y \in \tau'$.

2) Let $U_1, \ldots, U_n \in \tau'$. Then $f^{-1}(U_i)$ is open in X for all $i = 1, \ldots, n$. So
$f^{-1}(U_1) \cap \cdots \cap f^{-1}(U_n)$ is open in X.

Thus, $f^{-1}(U_1 \cap \cdots \cap U_n) = f^{-1}(U_1) \cap \cdots \cap f^{-1}(U_n)$ is open in X. So, $U_1 \cap \cdots \cap U_n \in \tau'$.

3) $U_\lambda \in \tau'$, $\lambda \in \Lambda$. Then $f^{-1}(U_\lambda) \in \tau$ for all $\lambda \in \Lambda$.
So, $\bigcup_{\lambda \in \Lambda} f^{-1}(U_\lambda) = f^{-1}(\bigcup_{\lambda} U_\lambda)$ is open and thus $\bigcup_{\lambda} U_\lambda \in \tau'$.

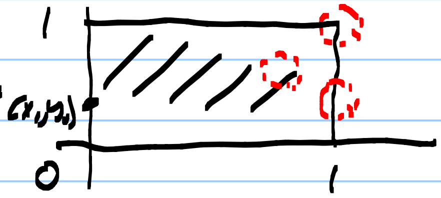
Remark: $f: X \to Y$ onto map of sets. This induces a partition of X as follows via an equivalence relation:
$x_1, x_2 \in X$, $x_1 \sim x_2$ if and only if $f(x_1) = f(x_2)$.

The equivalence classes of this relation

$[x]=\{x'\in X \mid x\sim x'\}$ gives a partition of X.

$$X=\dot{\bigcup_{x\in X}}[x]$$

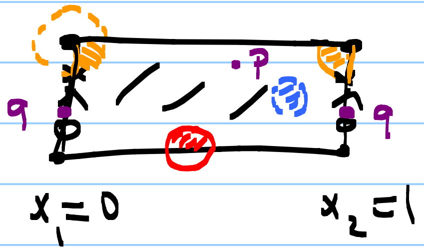
Example:

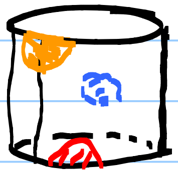


$X=[0,1]\times[0,1]$

Define an equivalence relation on X as follows:
$(x_1,y_1)\sim(x_2,y_2)$ if and only if

1) $(x_1=0,\ x_2=1$ and $y_1=y_2)$ or
2) $(x_1,y_1)=(x_2,y_2)$.



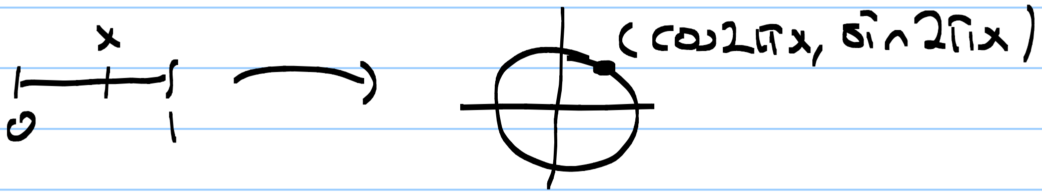


$$Y=X/\!\sim\ =\{[p] \mid p\in X\}$$

An embedding of Y into $\mathbb{R}^3$ is given by

$$f: X=[0,1]\times[0,1]\longrightarrow Y\subseteq\mathbb{R}^3$$

$$f(x,y)=(\cos 2\pi x, \sin 2\pi x, y)$$



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Example:

$X=[0,1]\times[0,1]$

$(x_1,y_1)\sim(x_2,y_2)$ if and only if.

1) $(x_1=0,\ x_2=1,\ y_1+y_2=1)$ or

2) $(x_1,y_1)=(x_2,y_2)$.

$X/\sim\ = MB$

Mōbius Band.

Proposition: Let $\pi: X\to Y$ be a quotient map. Let $f: Y\to Z$ be any map. Then f is continuous if and only if $f\circ\pi: X\to Z$ is continuous.

Proof:

$$\begin{tikzcd} X \arrow[r, "f\circ\pi"] \arrow[d, "\pi"'] & Z \\ Y=X/\sim \arrow[ur, "f"'] & \end{tikzcd}$$

First assume f is continuous.
must show: $f\circ\pi$ is continuous.

Let $U\subseteq Z$ be an open subset. Since f is continuous $f^{-1}(U)$ is open in Y. However, π is a quotient map and thus $\pi^{-1}(f^{-1}(U))=(f\circ\pi)^{-1}(U)$ is open in X. Hence, $f\circ\pi$ is continuous.

Conversely, assume $f\circ\pi$ is continuous.
must show: f is continuous.

Let $U \subseteq Z$ be open in Z. So, $(f\circ\pi)^{-1}(U)$ is open in X. However,
$(f\circ\pi)^{-1}(U) = \pi^{-1}(f^{-1}(U))$ and π is a quotient map. Hence, $f^{-1}(U)$ is open in Y. Thus, $f: Y \to Z$ is continuous. ■

Corollary Let $\pi: X \to Y$ be a quotient map and $g: X \to Z$ be any map.

$$\begin{tikzcd} x \arrow[d, mapsto] & X \arrow[r, "g"] \arrow[d, "\pi"] & Z \\ {[x]} \in & Y = X/\sim \arrow[ur, dotted, "f"'] & \end{tikzcd}$$

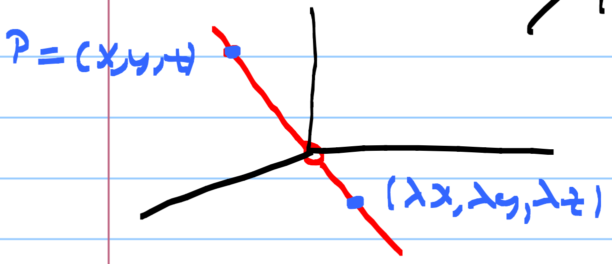
Then there is a map $f: Y \to Z$ so that $g = f\circ\pi$ if and only if g is constant on each equivalence class (equivalently, g is constant on the fibers of π).

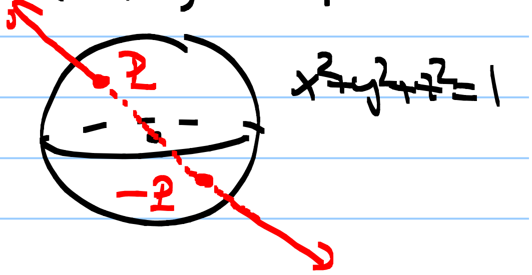
$\left([x] = \pi^{-1}(x) = \{x' \in Y \mid \pi(x) = \pi(x')\}\right)$

Moreover, in this case, f is continuous if and only if g is continuous.

Example: $\mathbb{RP}^2$: the real projective space.

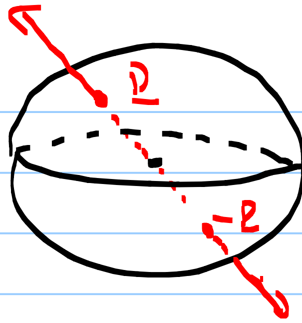
$$\mathbb{RP}^2 = \mathbb{R}^3 \setminus \{(0,0,0)\} \Big/ (x,y,z) \sim \lambda(x,y,z), \quad \lambda \neq 0$$

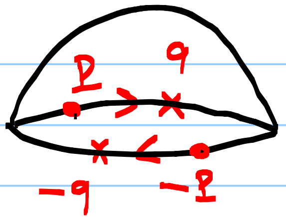


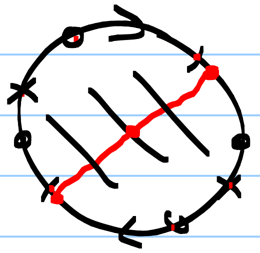


$$\mathbb{RP}^2 = S^2 \Big/ (x,y,z) \sim (-x,-y,-z)$$

$\pi: S^2 \to \mathbb{RP}^2$ quotient map. Since S^2 is compact and connected so is its image $\mathbb{RP}^2$ under π.







Fact: $\mathbb{RP}^2$ does not admit an embedding into $\mathbb{R}^3$.

Proposition: $\mathbb{RP}^2$ embeds into $\mathbb{R}^4$.

Proof: $\mathbb{RP}^2 = S^2 /_{(x,y,z)\sim(-x,-y,-z)}$

$$\begin{tikzcd}
(x,y,z) \arrow[d] & S^2 \arrow[r, "f"] \arrow[d, "\pi"'] & \mathbb{R}^5 \\
{[x:y:z]} \in & \mathbb{RP}^2 \arrow[ur, "g"'] &
\end{tikzcd}$$

$f = g \circ \pi$

$f(x,y,z) = f(-x,-y,-z)$

$[x:y:z] = \{(x,y,z), (-x,-y,-z)\}$.

Let $f : S^2 \to \mathbb{R}^5$, $f(x,y,z) = (x^2, y^2, xy, xz, yz)$.

f is clearly constant on the fibers of π. Hence, f induces a map $g : \mathbb{RP}^2 \to \mathbb{R}^5$ given by

$$g([x:y:z]) \doteq f(x,y,z).$$

Since f is continuous g is also continuous.

Exercise: Show that g is one to one.

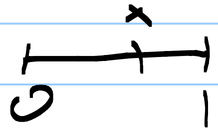
$g : \mathbb{RP}^2 \to \mathbb{R}^5$ is continuous, one to one and $\mathbb{RP}^2$ is compact. Thus $g : \mathbb{RP}^2 \to g(\mathbb{RP}^2)$ is a homeomorphism.

Exercise: Modify g so that it gives an embedding of $\mathbb{RP}^2$ into $\mathbb{R}^4$.

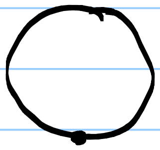
Video 5

More Examples of Quotient spaces:

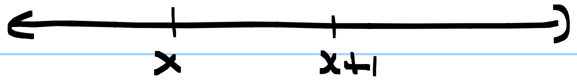
1) Circle: $S^1 = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2=1\}$



$/0\sim 1 \quad = S^1$



OR: $\mathbb{R}/x\sim x+1,\ x\in\mathbb{R}$

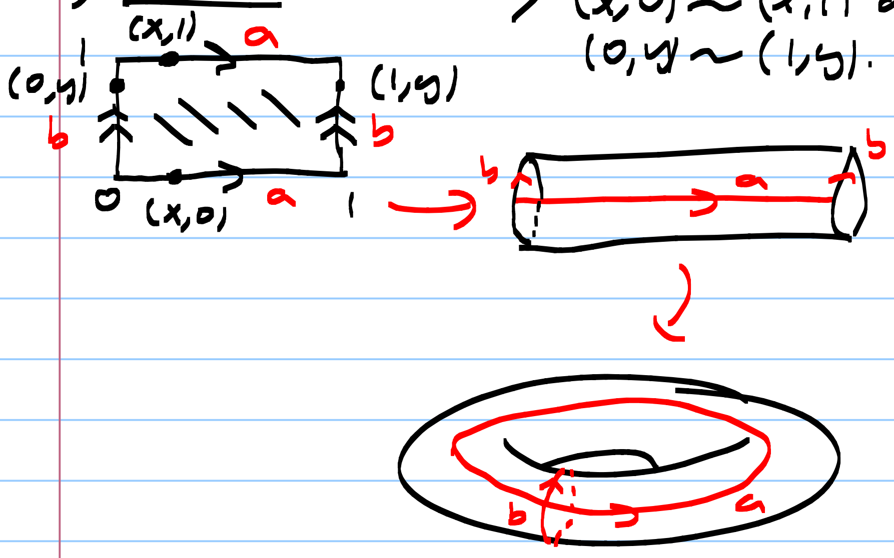


$$\begin{tikzcd}
{[0,1]} \arrow[r, "f"] \arrow[d, "\pi"'] & S^1 \\
{[0,1]/0\sim 1} \arrow[ur, "g"'] &
\end{tikzcd}$$

$f(t) = (\cos 2\pi t, \sin 2\pi t)$

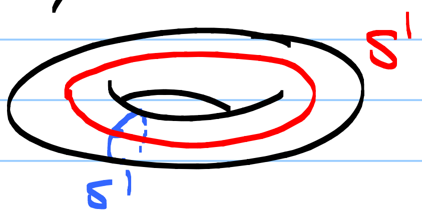
g is 1-1, onto and continuous. Since $[0,1]/\sim$ is compact we see that g is a homeomorphism.

2) Torus: $I\times I/(x,0)\sim(x,1)$ and $(0,y)\sim(1,y)$.

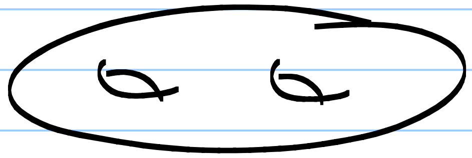


$T^2 = I\times I/\sim \;= I/_{0\sim1} \times I/_{0\sim 1}$

$= S^1 \times S^1$

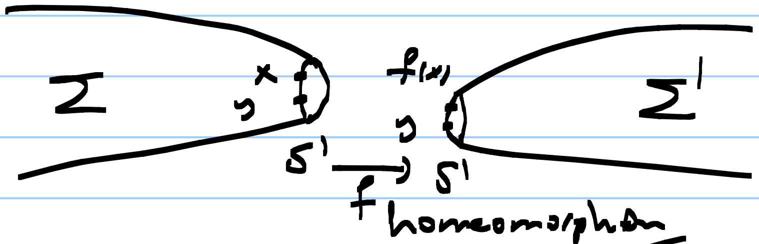


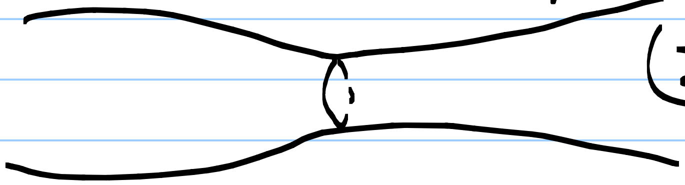
3) Σ_2: genus 2 orientable surface



How to obtain Σ_2 from the previous spaces?

Connected sum of surfaces:

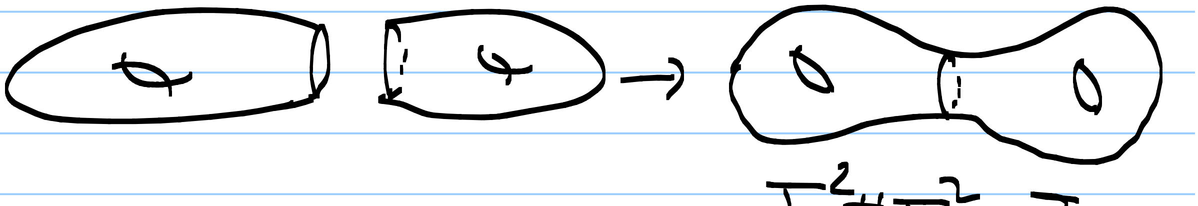




$$(\Sigma \setminus D^2) \sqcup (\Sigma' \setminus D^2) \Big/ \begin{matrix} x \sim f(x) \\ f: \partial D^2 \to \partial D^2 \\ \text{homeo.} \end{matrix}$$

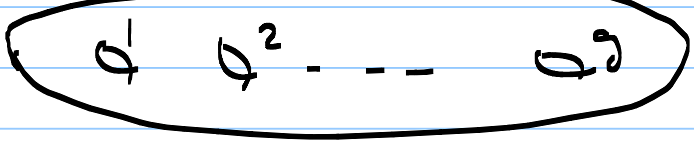
The resulting quotient space is called the connected sum of the surfaces Σ and Σ' and will be denoted as $\Sigma \# \Sigma'$.

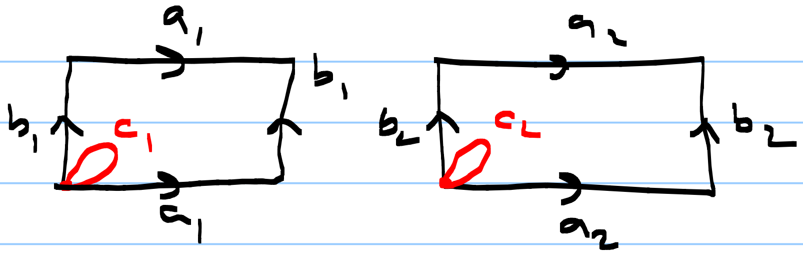
Some Examples: 1) $T^2 \# T^2$

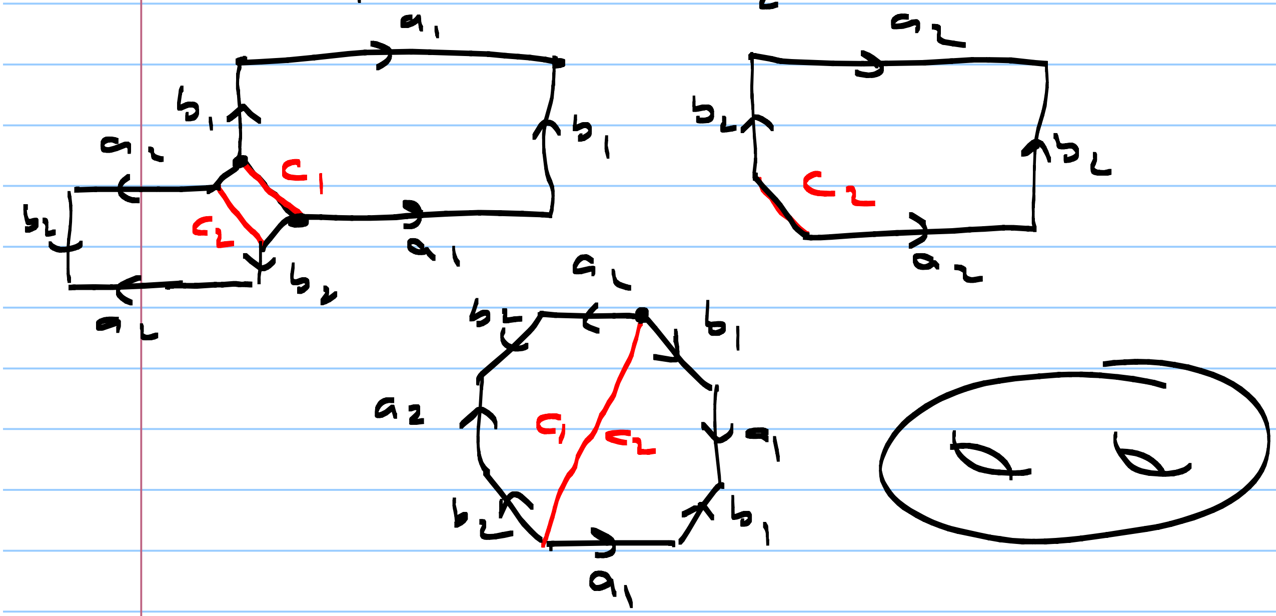


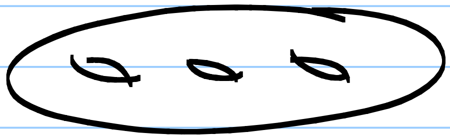
$T^2 \# T^2 = \Sigma_2$

Similarly, $\underbrace{T^2 \# T^2 \# \cdots \# T^2}_{g\text{-copies}} = \Sigma_g$



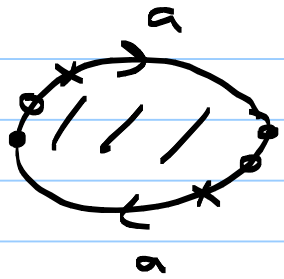






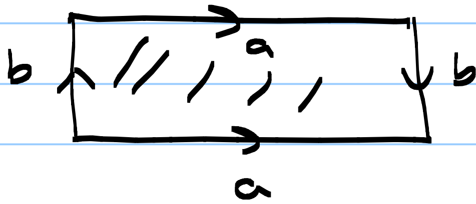
12-gon
$a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} a_3 b_3 a_3^{-1} b_3^{-1}$

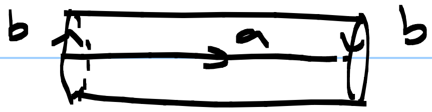
2)

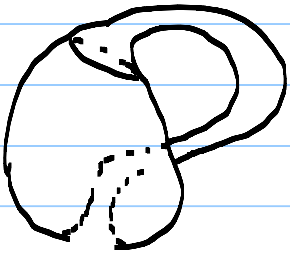


$\mathbb{RP}^2$

$\mathbb{RP}^2 \# \mathbb{RP}^2 = KB$, Klein Bottle

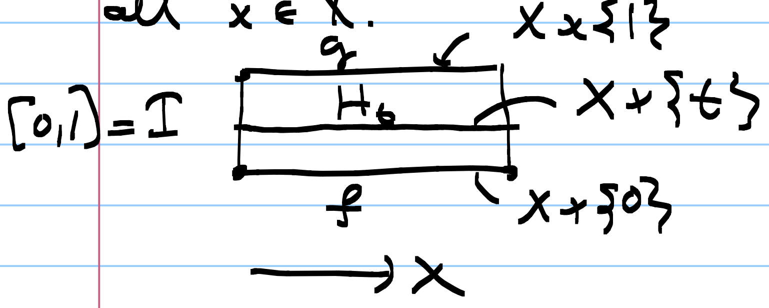






Definition: (Homotopy)

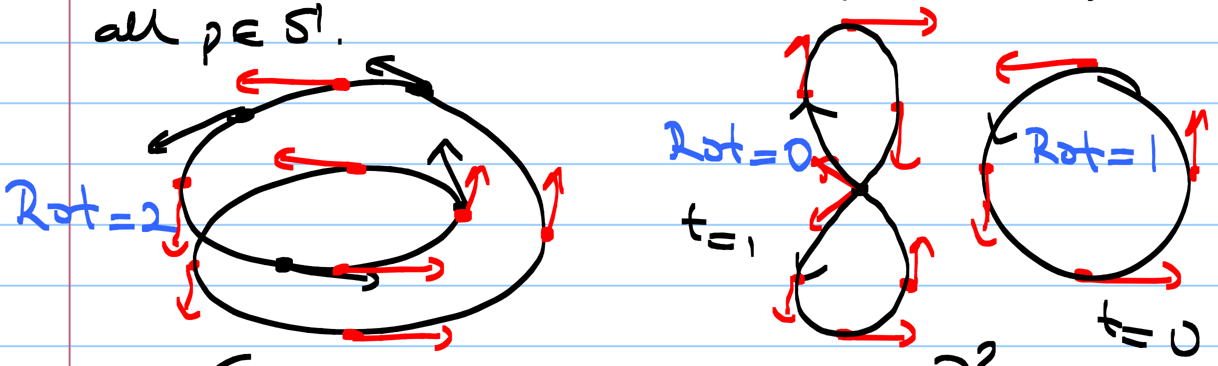
Let $f: X \to Y$ and $g: X \to Y$ be two continuous maps of topological spaces. We say that f and g are homotopic maps if there is a continuous map
$$H: X \times I \to Y, \quad I = [0,1],$$
so that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$.



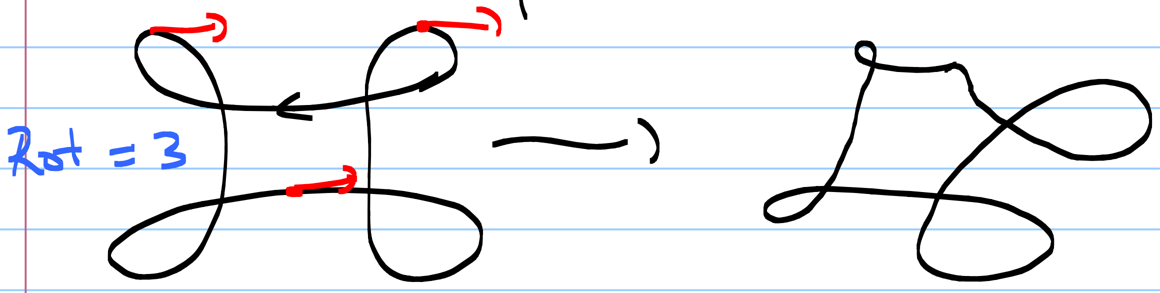
$$H_t(x) = H(x,t) \qquad H_0 = f, \; H_1 = g$$

Example (Rotation number)

Immersion: $f: S^1 \to \mathbb{R}^2$, $df(p) \neq 0$ for all $p \in S^1$.



Immersion of circle in $\mathbb{R}^2$.



Video 6

Question: Are two above immersions homotopic through immersions?

Back to the Homeomorphism Theorem:

$f: X \to Y$ one to one continuous map, X compact Y Hausdorff.
Then $f: X \to Z \doteq f(X) \subseteq Y$ is a homeomorphism.

$f: X \to Z$, 1-1, onto and continuous.

Let $g: Z \to X$ be the inverse of $f: X \to Z$.

must show: g is continuous.

Let $A \subseteq X$ be a closed subset of X. It is enough to show that $g^{-1}(A)$ is closed in Z.

$g^{-1}(A) = \{y \in Z \,|\, g(y) \in A\}$

Any $y \in Z$, $y = f(x)$ for a unique $x \in X$.

$g(y) = f^{-1}(y) = x$. So, $g^{-1}(A) = f(A)$.

Since $A \subseteq X$ is closed and X is compact A is a compact subset of X. Hence $f(A)$ is a compact subset of Y.

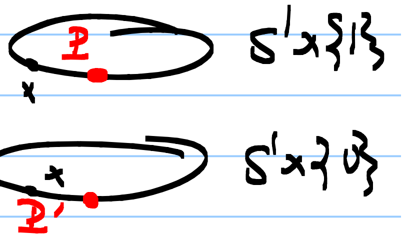
Fact: Since Y is Hausdorff any compact subset of Y is closed.
Proof: Almost the same for metric spaces. ∎

Since Y is Hausdorff and $f(A)$ is a compact subset of Y, $f(A)$ is closed in Y.

This finishes the proof of the homeomorphism theorem. ■

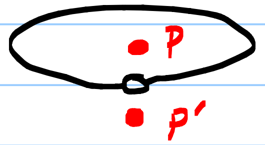
Example: Let X be the quotient space

$$X = S^1 \times \{0,1\} / \sim$$



$(x,0) \sim (x,1)$ if and only if $x \neq p$.

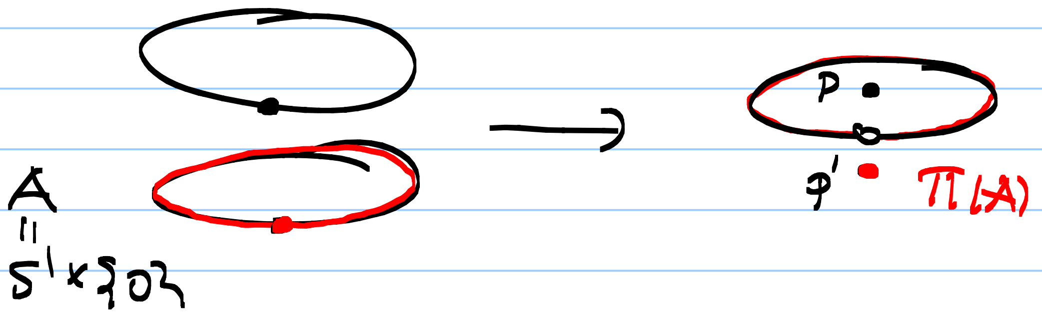
$X =$



Let $\pi : S^1\times\{0,1\} \longrightarrow X = S^1\times\{0,1\}/\sim$ be the quotient map.

Clearly, X is not Hausdorff.

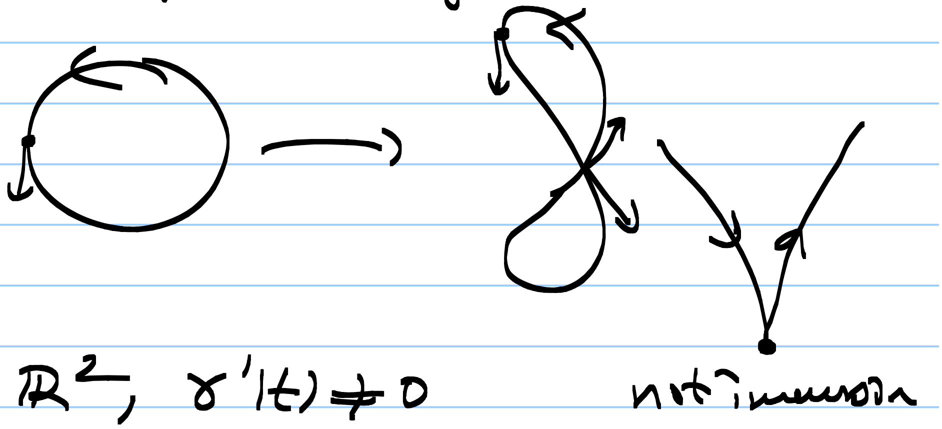
$A = S^1\times\{0\} \subseteq S^1\times\{0,1\}$ is a compact subset.
Hence, $\pi(A)$ is a compact subset of X.



Note that $\pi(A)$ is <u>not</u> a closed subset.
Because, $X \setminus \pi(A) = \{p\}$ is not an open subset.

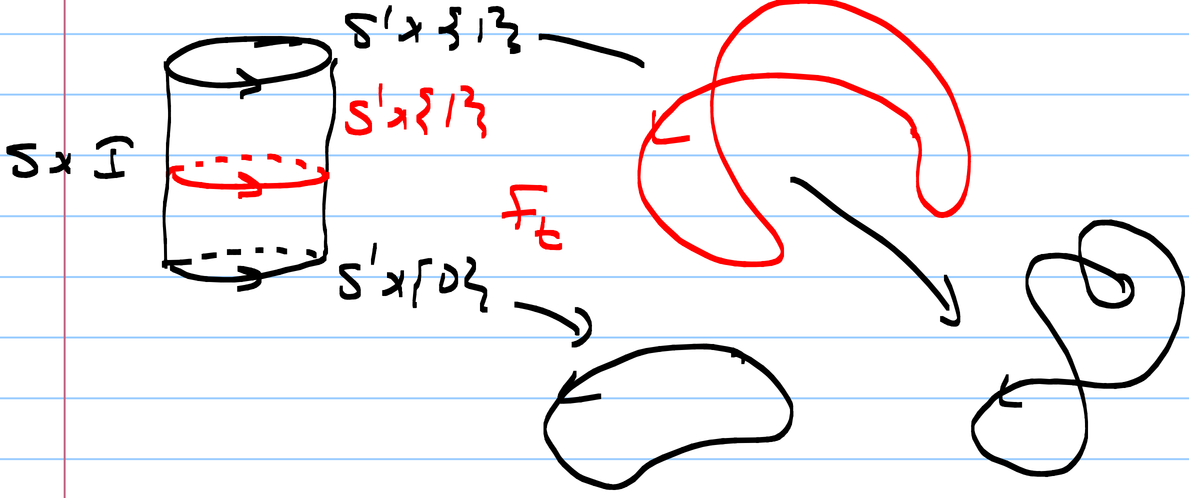
Proposition: Rotation number is invariant under homotopies through immersions.

Proof:



$\gamma : S^1 \to \mathbb{R}^2$, $\gamma'(t) \neq 0$

Let $F : S^1 \times I \to \mathbb{R}^2$ be a homotopy so that $F_t : S^1 \times \{t\} \to \mathbb{R}^2$ is an immersion.

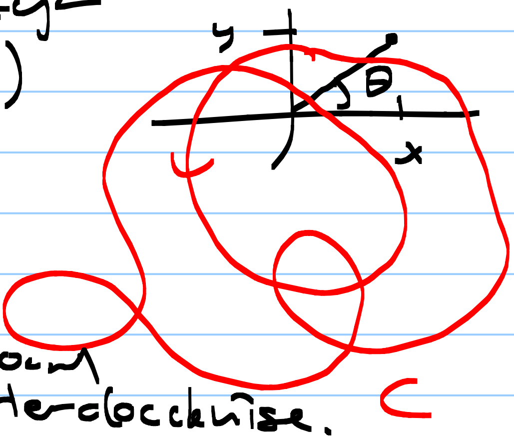


The Rotation numbers of the immersions of $S^1 \times \{0\}$ and $S^1 \times \{1\}$ are the same.

Proof: $\omega = \dfrac{x\,dy - y\,dx}{x^2+y^2} \in \Omega^1(\mathbb{R}^2 \setminus \{(0,0)\})$.

$= \pm d\left(\tan \frac{y}{x}\right)$

$= d\theta$



$\int_C \omega =$ the number times C goes around the origin counter-clockwise.

If $\gamma : S^1 \longrightarrow \mathbb{R}^2$ is an immersion then $\gamma' : S^1 \longrightarrow \mathbb{R}^2 \setminus \{(0,0)\}$ is smooth map.

$$\int_{S^1} (\gamma')^*(\omega) = \text{Rotation number.}$$

$F : S^1 \times [0,1] \longrightarrow \mathbb{R}^2,$

$dF : S^1 \times [0,1] \longrightarrow \mathbb{R}^2 \setminus \{(0,0)\},$

$$\int_{S^1 \times [0,1]} dF^*\omega = \int_{\partial(S^1 \times [0,1])} F^*\omega = \int_{S^1 \times \{1\}} F^*\omega - \int_{S^1 \times \{0\}} F^*\omega$$

$$\overset{\shortparallel}{\int_{S^1 \times [0,1]} F^* \underbrace{d\omega}_{0} = 0} \Rightarrow \int_{S^1 \times \{1\}} F^*\omega = \int_{S^1 \times \{0\}} F^*\omega.$$

Free Groups: (Michio Kuga: Galois' Dream)

Group: $G \neq \emptyset$, $G \times G \to G$, $(x,y) \mapsto x \cdot y$

1) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$
2) $\exists\, e \in G$, $e \cdot x = x = x \cdot e$
3) $x \in G, \exists\, x^{-1} \in G$ s.t. $x \cdot x^{-1} = e = x^{-1} \cdot x$.

Definition: The free group on a set A consists of words with letters, elements of A.

Example $A = \{x, y, z\}$

$x, xy, yx, x \cdot xyz, x^{-1}y, x^{-1}y^{-1}z, x^3 = xxx$

Identity element e = Empty word.

Product of two words: $(xyz) \cdot (x^2 y) = xyzx^2 y$

$x \cdot x^{-1}$ = empty word, usually denoted as e.

Cancellation property: $(xyz)(z^{-1}x^2) = xy\underbrace{zz^{-1}}_{e}x^2$

$= xyx^2$

Remark: Multiplication is not abelian.

$xy \neq yx$

Video 7

Proposition: Let F be a free group on an alphabeth $A=\{x_\alpha \mid \alpha\in\Delta\}$ and G be any group. For each $\alpha\in\Delta$ choose some $g_\alpha \in G$. Then there is a unique group homomorphism $\varphi: F\longrightarrow G$ so that $\varphi(\alpha)=g_\alpha$, $\alpha\in\Delta$.

Group Presentation:

Let G be any group with generating set $B=\{g_\alpha \mid \alpha\in\Delta\}$. In this case, we write $G=\langle B\rangle$

Ex: $G=(\mathbb{Z},+)$, $B=\{1\}$, $B=\{-1\}$ or $B=\{2,3\}$

Let A be the alphabeth $A=\{x_\alpha \mid \alpha\in\Delta\}$ and F be the free group on A. Then there is a unique homomorphism

$$\varphi: F\longrightarrow G,\ \varphi(x_\alpha)=g_\alpha,\ \alpha\in\Delta.$$

φ is clearly onto. Hence, the first isomorphism theorem implies that

$$G=\operatorname{Im}\varphi\simeq F/\ker\varphi.$$

In this case, we write

$$G\simeq\langle x_\alpha \mid r_\lambda\in\ker\varphi\rangle.$$

Ex: $G=\mathbb{Z}$, $\mathbb{Z}=\langle x\mid -\rangle$

$B=\{1\}$, $A=\{x\}$, F free group on A.

$F=\{x, x^2, x^3, x^{-2}, x^{-1}, x^{-3}, \dots\}$

$\varphi: F \longrightarrow \mathbb{Z}$, $\varphi(x)=1$, $\varphi(x^2)=\varphi(x)\,\varphi(x)$

$$=\varphi(x)+\varphi(x)$$

$$=1+1=2$$

$\varphi(x^n)=n$, φ is onto and 1-1.

φ is an isomorphism and $\ker\varphi=\langle e\rangle$.

$\mathbb{Z}=\langle x \mid - \rangle$

Ex $\mathbb{Z}_5=(\{\bar{0},\bar{1},\dots,\bar{4}\},\oplus)$

$B=\{\bar{1}\}$, $A=\{x\}$, $\varphi: F \longrightarrow G=\mathbb{Z}_5$

$\varphi(x)=\bar{1}$. $\varphi(x^n)=\bar{n}$ and thus $\varphi(x^n)=\bar{0}$ if and only if $n\equiv 0 \pmod 5$.

$\ker\varphi=\langle x^5\rangle=\{e, x^5, x^{-5}, x^{10}, x^{-10}, \dots\}$

$\mathbb{Z}_5 \simeq \langle x \mid x^5\rangle=\{e, x, x^2, x^3, x^4\}$

$x^5=e \Rightarrow x\cdot x^4=e \Rightarrow x^{-1}=x^4$

Ex: $G=\mathbb{Z}\times\mathbb{Z}=\{(m,n)\mid m,n\in\mathbb{Z}\}$ abelian group under pointwise addition.

$(2,3)+(5,-7)=(7,-4)$

$B=\{(1,0),(0,1)\}$ generating set for G.

Choose $A=\{x,y\}$ and let F be the group on A.
Let $\varphi : F \to G = \mathbb{Z}\times\mathbb{Z}$ be given by

$\varphi(x)=(1,0)$ and $\varphi(y)=(0,1)$.

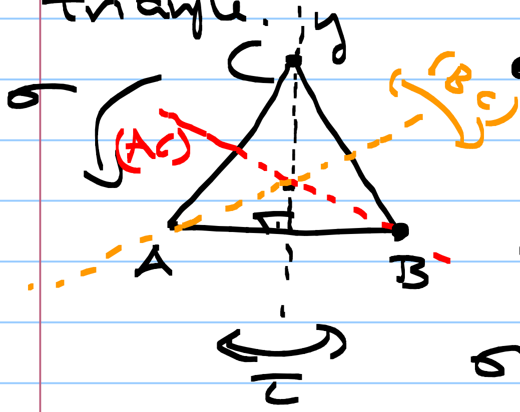
$$\begin{aligned}\varphi(xyx^{-1}y^{-1}) &= \varphi(x)+\varphi(y)+\varphi(x^{-1})+\varphi(y^{-1})\\ &= (1,0)+(0,1)+(\varphi(x))^{-1}+(\varphi(y))^{-1}\\ &= (1,0)+(0,1)+(-1,0)+(0,-1)\\ &= (0,0).\end{aligned}$$

So, $xyx^{-1}y^{-1}\in\ker\varphi$. Indeed, $\ker\varphi=\langle xyx^{-1}y^{-1}\rangle$.

$\mathbb{Z}\times\mathbb{Z}\simeq\langle x,y \mid xy=yx\rangle=\langle x,y\mid xyx^{-1}y^{-1}\rangle$

$(xyx^{-1}y^{-1}=e \Rightarrow xy=yx)$

Example: Symmetry group on equilateral triangle.



σ: $2\pi/3$ radian counterclockwise rotation

$\sigma \longleftrightarrow (ABC)$

σ^2: $4\pi/3$ radian c.c.w. rotation

$\sigma^2 \longleftrightarrow (ACB)$

σ^3: 2π radian rotation = Identity.

Also let τ be the reflection w.r.t. the y-axis.

$\tau \longleftrightarrow (AB)$

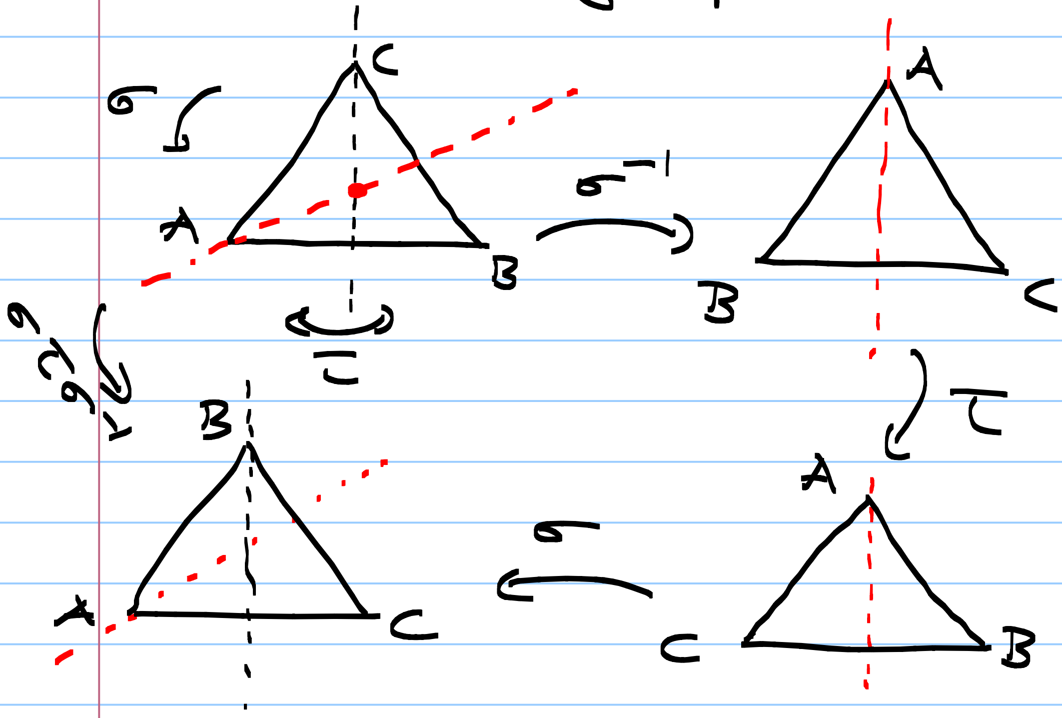
There are two more reflections corresponding to the permutations (AC) and (BC).

So if G is the group of symmetries of the triangle then G is isomorphic to

$$G = \{e, (ABC), (ACB), (AB), (AC), (BC)\}$$

$$= \{e, (12), (13), (23), (123), (132)\} = S_3$$

the symmetric group on the set $\{1,2,3\}$.



$$G = \{e, \sigma, \sigma^2, \tau, \sigma\tau\sigma^{-1}, \sigma^2\tau\sigma^{-2}\}$$

$$\simeq \langle x, y \mid x^3, y^2, yxy^{-1} = \underset{\substack{\| \\ x^{-1}}}{x^2} \rangle$$

$A = \{x, y\}$, $x \longmapsto \sigma$, $y \longmapsto \tau$

$\sigma^3 = e$, $\tau^2 = e$

$y^2 = e \Rightarrow y = y^{-1}$ and $x^3 = e \Rightarrow x \cdot x^2 = e$
$\Rightarrow x^2 = x^{-1}$

Video 8

$yxy^{-1} = x^2$? $(AB)(ABC)(AB) \overset{?}{=} (ACB)$

$(ACB) = (ACB)$ ✓

So, we may write $S_3 = \langle x, y \mid x^3, y^2, yxy = x^2 \rangle$

$\varphi : F_2 \longrightarrow S_3$, $\varphi(x) = \sigma$, $\varphi(y) = \tau$

$\ker\varphi = \langle x^3, y^2, yxyx \rangle$

Formal Definition of Group Presentation:

$G \simeq \langle x_\alpha, \alpha \in \Delta \mid r_\lambda, \lambda \in \Gamma \rangle$

F: free group on the set $\{x_\alpha \mid \alpha \in \Delta\}$

$r_\lambda \in F$, $\lambda \in \Gamma$ and

$G \simeq F/N$, N is normal closure of the set $\{r_\lambda \mid \lambda \in \Gamma\}$ of relations.

$$N = \bigcap_{\substack{H \triangleleft F \\ \{r_\lambda \mid \lambda \in \Gamma\} \subseteq H}} H$$

Ex $\underset{x}{\underset{|}{\mathbb{Z}_2}} \times \underset{y}{\underset{|}{\mathbb{Z}_2}} = \langle x, y \mid x^2, y^2, \underset{xy = yx}{\underset{\updownarrow}{xyx^{-1}y^{-1}}} \rangle$

$$\underline{\text{Ex}}\quad \mathbb{Z}_2 \underset{\substack{\uparrow\\ \text{free product}}}{*} \mathbb{Z}_3 = \langle x, y \mid x^2, y^3 \rangle \simeq PSL(2,\mathbb{Z})$$

$$PSL(2,\mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \,\middle|\, a,b,c,d \in \mathbb{Z},\ ad-bc=1 \right\} \Big/ \left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle$$

$$x \longleftrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$y \longleftrightarrow \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

$$x^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},\quad y^2 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$$

$$y^3 = y^2 \cdot y = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$z = x^3 Y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^3 \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$\underline{\text{Example:}}$ $A = \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix} \in SL(2,\mathbb{Z})$

$$A = \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix} \xrightarrow{X} \begin{pmatrix} 7 & 5 \\ -3 & -2 \end{pmatrix} \xrightarrow{z} \begin{pmatrix} 4 & 3 \\ -3 & -2 \end{pmatrix} \xrightarrow{z} \begin{pmatrix} 1 & 1 \\ -3 & -2 \end{pmatrix}$$

$$\xrightarrow{X} \begin{pmatrix} -3 & -2 \\ -1 & -1 \end{pmatrix} \xrightarrow{X} \begin{pmatrix} -1 & -1 \\ 3 & 2 \end{pmatrix} \xrightarrow{z} \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \xrightarrow{X} \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix}$$

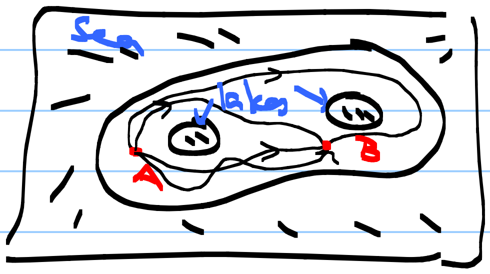
$$\xrightarrow{z} \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \xrightarrow{z} \begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix} \xrightarrow{X} \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix} \xrightarrow{z^2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\xrightarrow{X} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$ Hence, $xz^2xz^2xzx^2z^2xA = I$

so that A is a word in X and Y.

In Kugel's Book the Fourth Week:

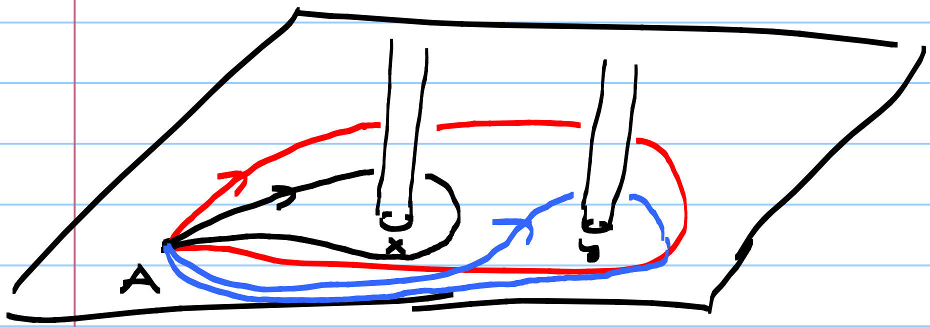
The fundamental Group of a Surface:

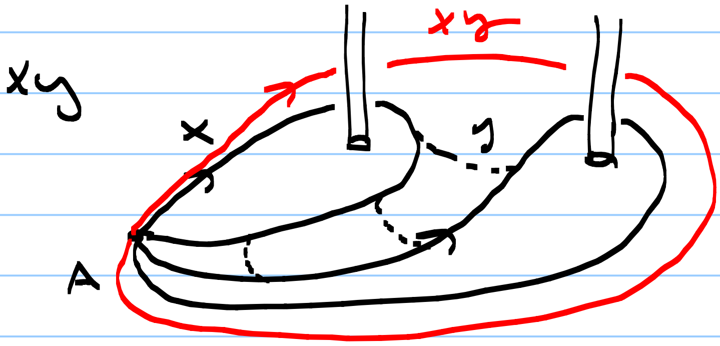


The fundamental group basically counts the ways of going from one point to another, when paths, which can be deformed to each other, regarded the same.

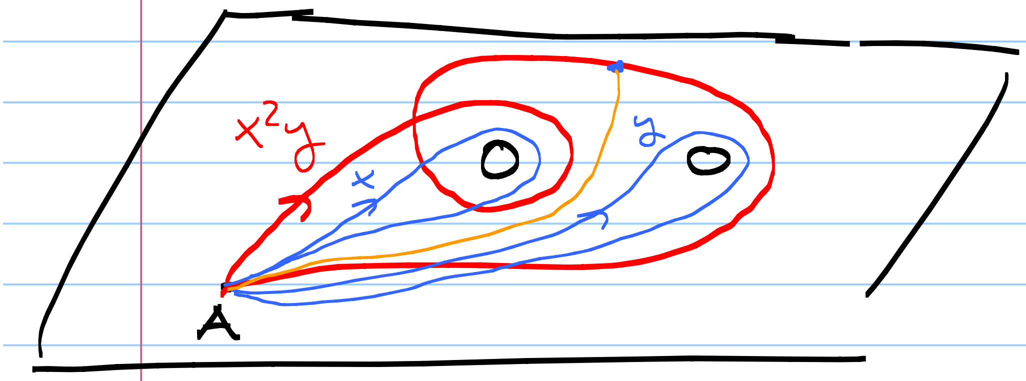
Group operation: We may add two path if the terminal point of one of them is the initial point of the other.

Relation to free groups:





Video 9



x^2y

$\pi_1(\mathbb{R}^2 \setminus \{2\text{points}\}) \simeq F_2$ free group on two generators.

$(\pi_1(\mathbb{R}^2 \setminus \{(0,0)\}) \simeq \mathbb{Z})$

The Fundamental Group:

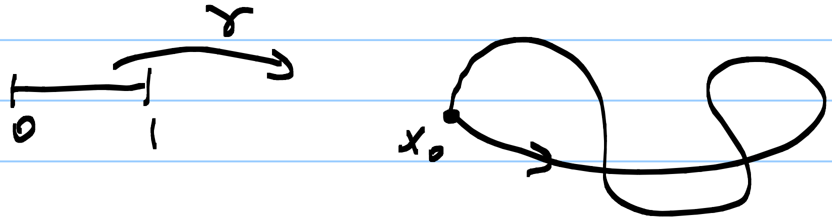
X topological space, $x_0 \in X$. Then the pair (X, x_0) is called a based space.

Fundamental group can be thought as an assignment a group to a given based space:

$$(X, x_0) \rightsquigarrow \pi_1(X, x_0)$$

Definition: Given a based topological space (X, x_0) Let $\mathcal{L}$ be the set of all loops at x_0:

$$\mathcal{L} = \{\gamma : [0,1] \to X \,/\, \gamma \text{ continuous}, \gamma(0) = x_0 = \gamma(1)\}.$$

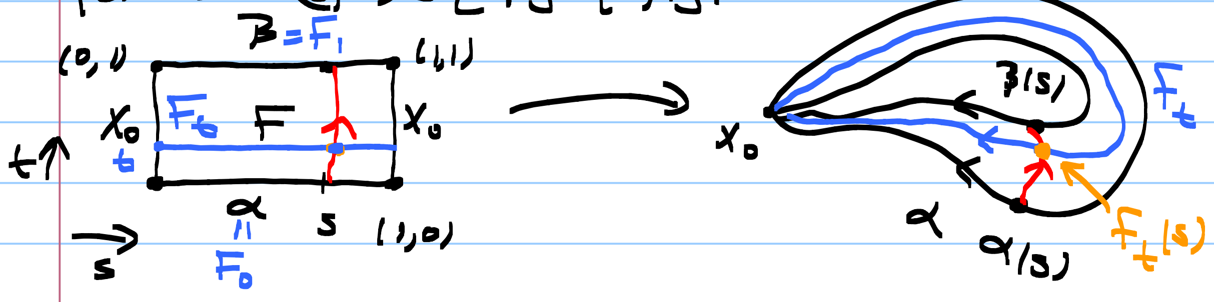


Define a homotopy relation on $\mathcal{L}$ as follows:

If $\alpha, \beta \in \mathcal{L}$, then we say that α is homotopic to β and write $\alpha \sim \beta$, if there is a homotopy

$$F:[0,1]\times[0,1]\longrightarrow X \quad \text{so that}$$

$F(s,0)=\alpha(s)$, $F(s,1)=\beta(s)$, $F(0,t)=x_0=F(1,t)$, for all $(s,t)\in[0,1]\times[0,1]$.



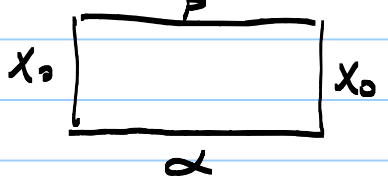
__Proposition:__ Homotopy relation is an equivalence relation on $\mathcal{L}$.

__Proof:__ 1) Reflexive: Given $\alpha \in \mathcal{L}$, let $F: I\times I \to X$ be given by $F(s,t)=\alpha(s)$.

$F(s,0)=\alpha(s)$, $F(s,1)=\alpha(s)$, $F(0,t)=\alpha(0)=x_0$ and $F(1,t)=\alpha(1)=x_0$.

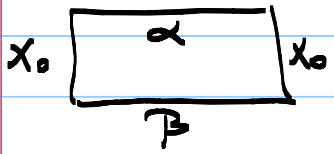
2) __Symmetric:__ If $\alpha, \beta \in \mathcal{L}$ and $\alpha \sim \beta$, the there is some $F: I\times I \to X$ so that

$F(s,0)=\alpha(s)$, $F(s,1)=\beta(s)$, $F(0,t)=x_0=F(1,t)$, for all $(s,t)\in I\times I$.



Let $G: I\times I \to X$ be given by $G(s,t)=F(s,1-t)$.

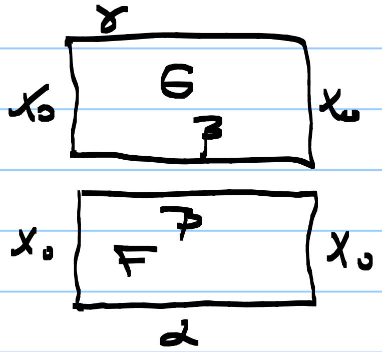
Then $G(s,0) = F(s,1) = \beta(s)$, $G(s,1) = F(s,0) = \alpha(s)$ and $G(0,t) = F(0,1-t) = x_0$, $G(1,t) = F(1,1-t) = x_0$.

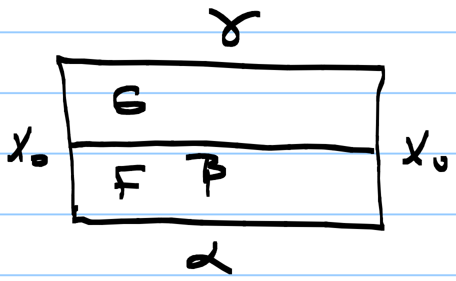


3) Transitive: Assume that $\alpha \sim \beta$ and $\beta \sim \gamma$. Then there are homotopies $F: I \times I \to X$ and $G: I \times I \to X$ so that

$F(s,0) = \alpha(s)$, $F(s,1) = \beta(s)$, $G(s,0) = \beta(s)$, $G(s,1) = \gamma(s)$

$F(0,t) = F(1,t) = G(0,t) = G(1,t) = x_0$, for all $(s,t) \in I \times I$.





H = composition of F and G, defined by

$$H(s,t) = \begin{cases} F(s,2t), & 0 \leq t \leq 1/2 \\ G(s,2t-1), & 1/2 \leq t \leq 1 . \end{cases}$$

When $t = 1/2$, $F(s, 2 \cdot 1/2) = F(s,1) = \beta(s)$ and $G(s, 2 \cdot 1/2 - 1) = G(s,0) = \beta(s)$ so that $F(s, 2 \cdot 1/2) = G(s, 2 \cdot 1/2 - 1)$, which implies that H is continuous (by the Pasting Lemma).

Moreover, $H(s,0) = F(s, 2 \cdot 0) = F(s,0) = \alpha(s)$, $H(s,1) = G(s, 2 \cdot 1 - 1) = G(s,1) = \gamma(s)$, and $H(0,t) = H(1,t) = x_0$, for all $(s,t) \in I \times I$.

Hence, being homotopic is an equivalence relation of $\mathcal{L}$.

The fundamental group of (X, x_0) is defined to be the set of equivalence classes of the homotopy relation on $\mathcal{L}$. It will be denoted as $\pi_1(X, x_0)$.

$$\pi_1(X,x_0) = \mathcal{L}/\sim$$

<u>Notation:</u> The equivalence (homotopy) class of a loop $\alpha \in \mathcal{L}$ will be denoted as $[\alpha]$.

$$[\alpha] = \{\beta \in \mathcal{L} \mid \alpha \sim \beta\}.$$

<u>Group Operation on $\pi_1(X, x_0)$:</u>

Let $[\alpha], [\beta] \in \pi_1(X, x_0)$. The product $[\alpha]\cdot[\beta]$ is defined by the formula

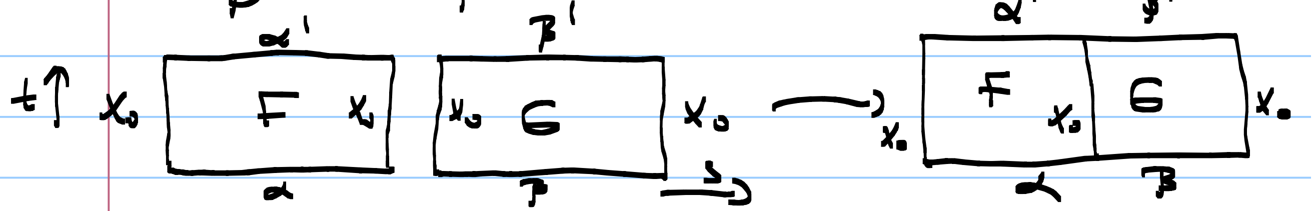
$[\alpha]\cdot[\beta] = [\alpha\cdot\beta]$, where

$$\alpha\cdot\beta : I \to X, \quad (\alpha\cdot\beta)(s) = \begin{cases} \alpha(2s), & 0 \le s \underline{\le 1/2} \\ \beta(2s-1), & \underline{1/2 \le} s \le 1. \end{cases}$$

Note that $\alpha(2\cdot 1/2) = \alpha(1) = x_0 = \beta(0) = \beta(2\cdot 1/2 - 1)$ so that $\alpha\cdot\beta : I \to X$ is continuous.

Note that we need to show that this operation is well defined. In other words, it must be independent of the choice of representatives α and β.

<u>must show:</u> If $\alpha \sim \alpha'$ and $\beta \sim \beta'$ then $\alpha\cdot\beta \sim \alpha'\cdot\beta'$.



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$$F\cdot G : I\times I \longrightarrow X,\ (F\cdot G)(s,t)=\begin{cases} F(2s,t), & 0\leq s\leq 1/2 \\ G(2s-1,t), & 1/2\leq s\leq 1. \end{cases}$$

$$(F\cdot G)(s,0)=\begin{cases} F(2s,0), & 0\leq s\leq 1/2 \\ G(2s-1,0), & 1/2\leq s\leq 1 \end{cases} = \begin{cases} \alpha(2s), & 0\leq s\leq 1/2 \\ \beta(2s-1), & 1/2\leq s\leq 1 \end{cases}$$

$$=(\alpha\cdot\beta)(s),$$

and similarly, $(F\cdot G)(s,1)=(\alpha'\cdot\beta')(s)$

Hence, $\alpha\cdot\beta$ is homotopic to $\alpha'\cdot\beta'$.

Therefore, the group operation on $\pi_1(X,x_0)$ is well defined.

For the operation defined above on $\pi_1(X,x_0)$ to induce a group stucture we need to show the followings:

1) There must be an identity element $e\in\pi_1(X,x_0)$ so that $e\cdot[\alpha]=[\alpha]\cdot e=[\alpha]$ for each $[\alpha]\in\pi_1(X,x_0)$.

2) For any $[\alpha]\in\pi_1(X,x_0)$ there is some $[\beta]\in\pi_1(X,x_0)$ so that $[\alpha]\cdot[\beta]=e=[\beta]\cdot[\alpha]$.

3) For any $[\alpha],[\beta],[\gamma]\in\pi_1(X,x_0)$ we have

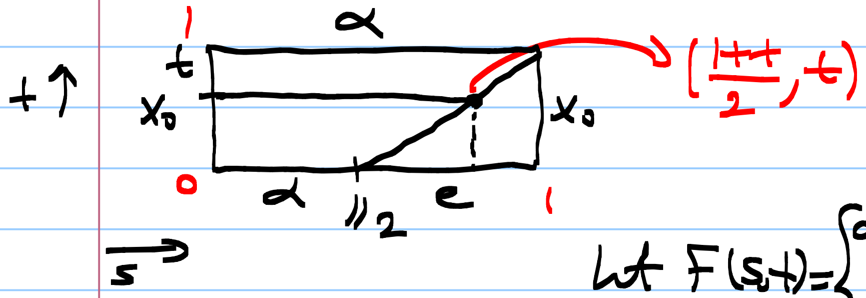
$$([\alpha]\cdot[\beta])\cdot[\gamma]=[\alpha]\cdot([\beta]\cdot[\gamma]).$$

<u>Proof</u>: 1) Let $e:[0,1]\to X$ be the constant loop at x_0: $e(s)=x_0$ for all $s\in[0,1]$.

<u>Claim</u>: The $[e\cdot\alpha]=[\alpha]=[\alpha\cdot e]$

Proof: First let's show that $\alpha . e \sim \alpha$.

$$(\alpha\cdot e)(s)=\begin{cases}\alpha(2s), & 0\le s\le 1/2\\ e(2s-1), & 1/2\le s\le 1\end{cases}=\begin{cases}\alpha(2s), & 0\le s\le 1/2\\ x_0, & 1/2\le s\le 1\end{cases}$$



$$\text{Let } F(s,t)=\begin{cases}\alpha\left(\frac{2s}{1+t}\right), & 0\le s\le \frac{1+t}{2}\\ x_0, & \frac{1+t}{2}\le s\le 1\end{cases}$$

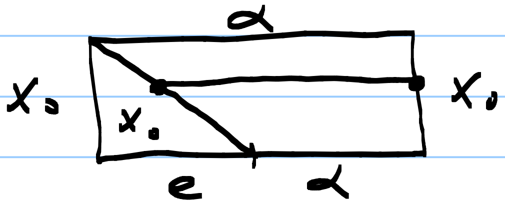
Clearly, $F: I\times I \longrightarrow X$ is continuous and

$$F(s,0)=\begin{cases}\alpha(2s), & 0\le s\le 1/2\\ x_0, & 1/2\le s\le 1\end{cases} = \alpha . e$$

$$F(s,1)=\begin{cases}\alpha(s), & 0\le s\le 1\\ x_0, & s=1\end{cases} = \alpha(s)$$

$$F(0,t)=\begin{cases}x_0, & \cdots\\ x_0, & \cdots\end{cases} = x_0$$ and $F(1,t)=x_0$, for all t.

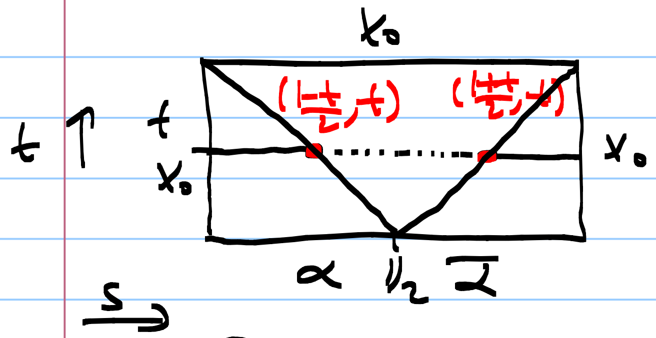
Exercise: Show that $e.\alpha\sim\alpha$ for any $[\alpha]\in\pi_1(X,x_0)$.



2) **Inverse Element:** Given any $[\alpha]\in\pi_1(X,x_0)$

Let $\overline{\alpha}:[0,1]\longrightarrow X$ be given by $\overline{\alpha}(s)=\alpha(1-s)$, for $s\in[0,1]$.

Claim: $\alpha.\overline{\alpha} \sim e \sim \overline{\alpha}.\alpha$



$$F(s,t) = \begin{cases} \alpha(2s), & 0 \leq s \leq \frac{1-t}{2} \\ \alpha(1-t), & \frac{1-t}{2} \leq s \leq \frac{1+t}{2} \\ \alpha(2-2s), & \frac{1+t}{2} \leq s \leq 1. \end{cases}$$

$$F(s,0) = \begin{cases} \alpha(2s), & 0 \leq s \leq 1/2 \\ \alpha(1), & 1/2 \leq s \leq 1/2 \\ \underset{\overline{\alpha}(2s)}{\underset{\shortparallel}{\alpha(2-2s)}} & 1/2 \leq s \leq 1 \end{cases} = \alpha.\overline{\alpha}$$

$$F(s,1) = \begin{cases} \alpha(2s), & s=0 \\ \alpha(0), & 0 \leq s \leq 1 \\ \alpha(0) & s=1 \end{cases} = x_0 = e(s)$$

So, $\alpha.\overline{\alpha} \sim e$.

Exercise: $\overline{\alpha}.\alpha \sim e$.

3) **Associativity:** Let $[\alpha],[\beta],[\gamma] \in \pi_1(X,x_0)$.

Claim: $\alpha.(\beta.\gamma) \sim (\alpha.\beta).\gamma$

Proof:

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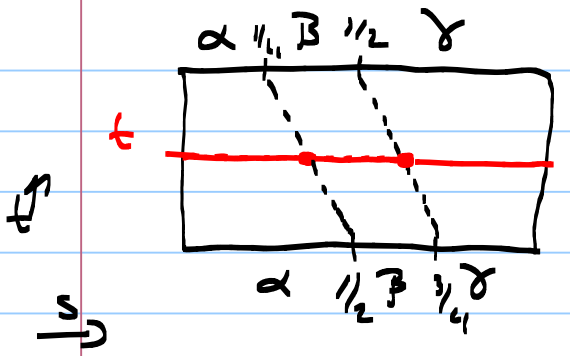
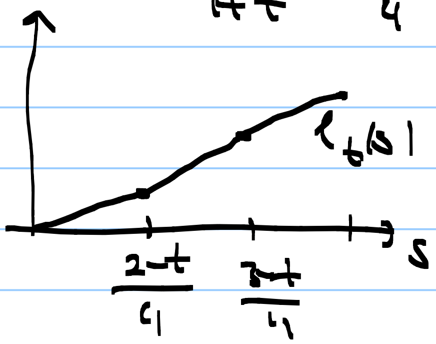


$$\ell_t(s) = \begin{cases} \frac{4s}{2-t}, & 0 \le s \le \frac{2-t}{4} \\ 4s+t-1, & \frac{2-t}{4} \le s \le \frac{3-t}{4} \\ \frac{4s+3t-1}{1+t}, & \frac{3-t}{4} \le s \le 1 \end{cases}$$



Let $F(s,t)$ be the function

$$F(s,t) = \begin{cases} \alpha(\ell_t(s)), & 0 \le s \le \frac{2-t}{4} \\ \beta(\ell_t(s)-1), & \frac{2-t}{4} \le s \le \frac{3-t}{4} \\ \gamma(\ell_t(s)-2), & \frac{3-t}{4} \le s \le 1. \end{cases}$$

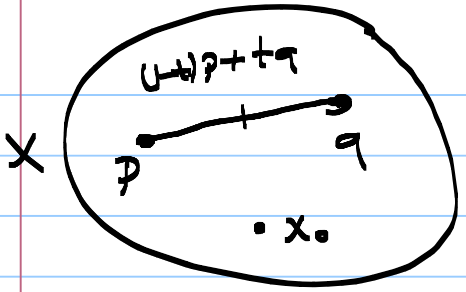
Exercise: Check that $F(s,0) = \alpha\cdot(\beta\cdot\gamma)$

and $F(s,1) = (\alpha\cdot\beta)\cdot\gamma$.

Hence, $\pi_1(X,x_0)$ is a group.

Example: For any convex subset $X \subseteq \mathbb{R}^n$ and any point $x_0 \in X$, $\pi_1(X,x_0) = \langle e \rangle$, the trivial group.

Solution: X is convex means for any two point $p, q \in X \subseteq \mathbb{R}^n$ the line segment $t \longmapsto (1-t)p + tq \in X$, for all $t \in [0,1]$.



Note that if $[\alpha]\in\pi_1(X,x_0)$ then the homotopy

$F: I\times I \to X$, $F(s,t)=(1-t)\alpha(s)+t\cdot x_0$, satisfies

$F(s,0)=\alpha(s)$, $F(s,1)=x_0$, for all $s\in[0,1]$

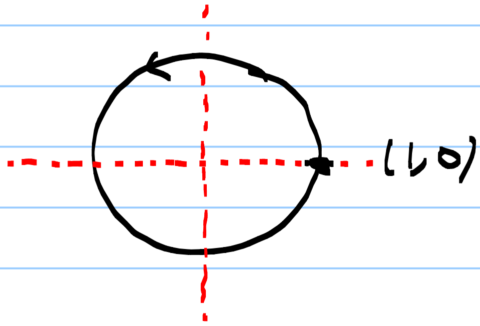
and $F(0,t)=(1-t)\cdot\alpha(0)+t\cdot x_0=(1-t)x_0+tx_0=x_0$
and $F(1,t)=(1-t)\alpha(1)+t\cdot x_0=(1-t)x_0+tx_0=x_0$,

show that $[\alpha]=[e]$ in $\pi_1(X,x_0)$.

Hence, $\pi_1(X,x_0)=(e)$, is the trivial group.

Theorem: Let S^1 be the unit circle in $\mathbb{R}^2$.

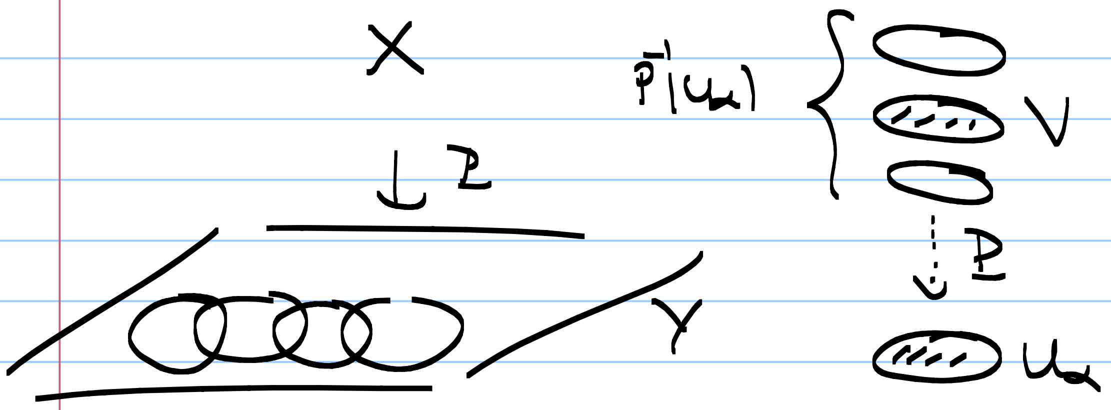
Then $\pi_1(S^1,x_0)$ is isomorphic to the infinite cyclic group $\mathbb{Z}$, where $x_0=(1,0)$.



Proof requires so called the theory of covering spaces.

Definition: Let $P: X\to Y$ be an onto map of topological spaces satisfying the following condition: There is an open cover $\{U_\alpha\}$ of

Y ($Y = \bigcup U_\alpha$) so that each $p^{-1}(U_\alpha)$ is a disjoint union of open subsets of X, each of which is homeomorphic to U_α via p.

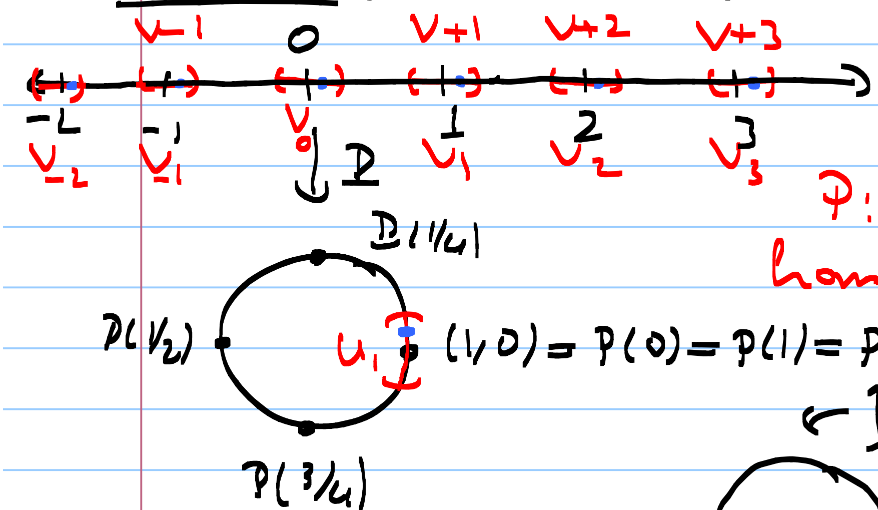


$p: V \to U$ homeomorphism.

The triple $p: X \to Y$ is called a covering space and p the covering space projection (or the covering map).

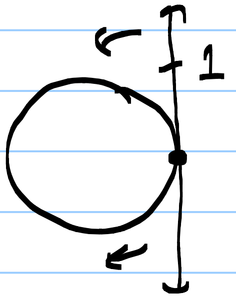
Y: base space, X: total/covering space.

Examples: 1) $p: \mathbb{R} \to S^1$, $p(s) = (\cos 2\pi s, \sin 2\pi s)$



$p: V_k \to U$ is a homeomorphism

$(1,0) = p(0) = p(1) = p(n)$, $n \in \mathbb{Z}$



This is a $\mathbb{Z}$-cover!

$P_{|V_0}: V_0 \to U$, $P^{-1}_{|V_0}: U \to V_0$, $P^{-1}(x,y) = \dfrac{\sin^{-1} x}{2\pi}$

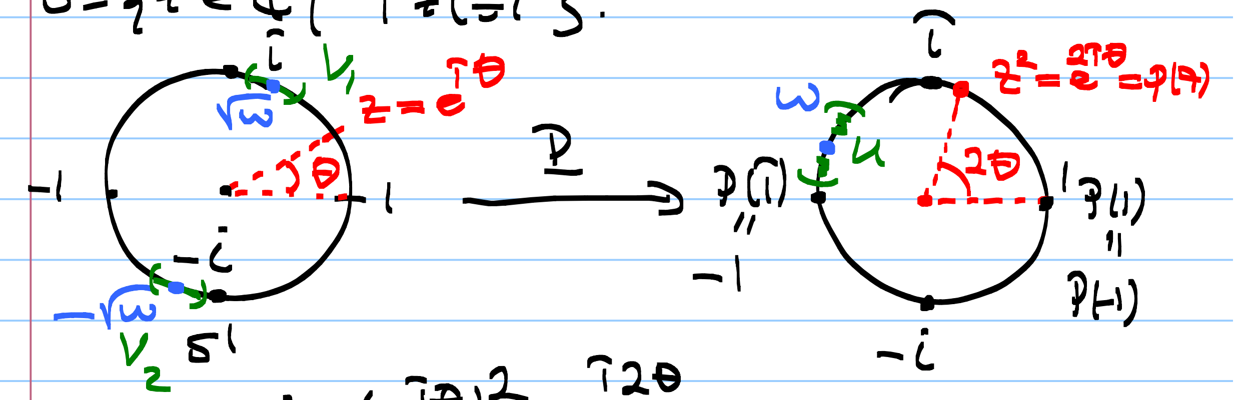
$P: V_5 \to U$, $P^{-1}_{|V_5}: U \to V_5$,

$$P^{-1}_{|V_5}(x,y) = \frac{\sin^{-1} x}{2\pi} + 5$$

2) $P: X = S^1 \longrightarrow S^1 = Y$, $P(z) = z^2$,

$z \in S^1 \subseteq \mathbb{C}$, when S^1 is the unit circle in $\mathbb{C}$.

$S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$.



$P(z) = z^2 = (e^{i\theta})^2 = e^{i2\theta}$

$P^{-1}(w) = \{ \underset{e^{i\theta/2}}{\underset{\shortparallel}{\sqrt{w}}}, \underset{-e^{i\theta/2}}{\underset{\shortparallel}{-\sqrt{w}}} \}$, $w = e^{i\theta}$, $\sqrt{w} = e^{i\frac{\theta}{2}}$

$0 \leq \theta \leq 2\pi$

This is a 2-fold covering (2-cover).

$P_{|V_1}: V_1 \to U$, $P_{|V_2}: V_2 \to U$ are both homeomorphism.

<u>Exercise</u>: Similarly, for any $n = 1, 2, 3, \ldots$

the map $P: S^1 \to S^1$, $P(z) = z^n$ defines an n-fold covering map.

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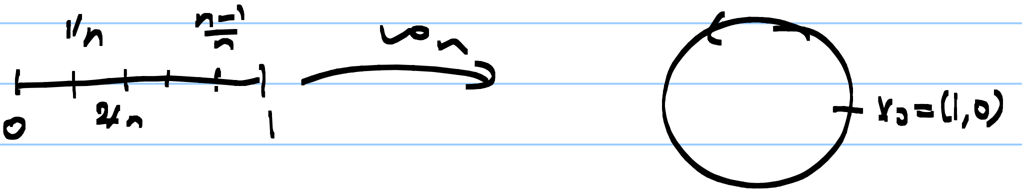
Theorem: The map $\Phi : \mathbb{Z} \to \pi_1(S^1, x_0)$, $x_0 = (1,0)$ sending an integer n to the homotopy class of the loop $\omega_n : [0,1] \to S^1$, $\omega_n(s) = (\cos 2\pi ns, \sin 2\pi ns)$, based at $x_0 = (1,0)$, is an isomorphism.

Proof:



$$\begin{aligned} \mathbb{Z} &\longmapsto \pi_1(S^1, x_0) \\ n &\longmapsto [\omega_n] \end{aligned}$$

Proof has several steps:

i) Consider the covering projection map

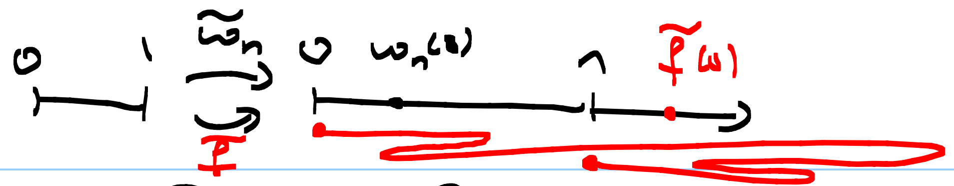
$$P : \mathbb{R} \to S^1, \quad P(s) = (\cos 2\pi s, \sin 2\pi s), \; s \in \mathbb{R}.$$

Let $\tilde{\omega}_n : [0,1] \to \mathbb{R}$, $\tilde{\omega}_n(s) = ns$, $s \in \mathbb{R}$.

Note that
$$\begin{aligned} (P \circ \tilde{\omega}_n)(s) &= P(\tilde{\omega}_n(s)) \\ &= P(ns) \\ &= (\cos 2\pi ns, \sin 2\pi ns) \\ &= \omega_n(s). \end{aligned}$$

Note that $\Phi(n) = [\omega_n]$ can be defined as the homotopy class of the loop $p \circ \tilde{f}$ for any path $\tilde{f}$ in $\mathbb{R}$ from 0 to n, because any such $\tilde{f}$ is homotopic to $\tilde{\omega}_n$, keeping the end points fixed:

$$[\omega_n] = [p \circ \tilde{\omega}_n] = [p \circ \tilde{f}]$$



$\tilde{\omega}_n : [0,1] \to \mathbb{R}$, $\tilde{\omega}_n(s) = ns$
$\tilde{f} : [0,1] \to \mathbb{R}$, $\tilde{f}(0) = 0$, $\tilde{f}(1) = n$

$[p\circ\tilde{\omega}_n] = [p\circ\tilde{f}]$ because $p\circ\tilde{\omega}_n$ and $p\circ\tilde{f}$ are homotopic loops at x_0:

$F : I\times I \longrightarrow S^1$, $F(s,t) = p\big((1-t)\tilde{\omega}_n(s) + t\tilde{f}(s)\big)$, $(s,t)\in I\times I$.

$F(s,0) = p(\tilde{\omega}_n(s)) = \omega(s)$
$F(s,1) = p(\tilde{f}(s))$

$F(0,t) = p((1-t)\cdot 0 + t\cdot 0) = (1,0)$ and
$F(1,t) = p((1-t)\cdot n + t\cdot n) = p(n) = (1,0)$.

ii) Claim: $\Phi : \mathbb{Z} \longrightarrow \pi_1(S^1, x_0)$ is a group homomorphism.

Proof: Must show $\Phi(m+n) = \Phi(m)\cdot\Phi(n)$ or equivalently, $[\omega_{m+n}] = [\omega_m]\cdot[\omega_n]$.

$$\Phi(m)\cdot\Phi(n) = [\omega_m]\cdot[\omega_n] = [\omega_m\cdot\omega_n]$$

$$(\omega_m\cdot\omega_n)(s) = \begin{cases} \omega_m(2s), & 0\le s\le 1/2 \\ \omega_n(2s-1), & 1/2\le s\le 1 \end{cases}$$
$$= \begin{cases} p(\tilde{\omega}_m(2s)), & 0\le s\le 1/2 \\ p(\tilde{\omega}_n(2s-1)), & 1/2\le s\le 1 \end{cases}$$
$$= \begin{cases} p(2ms), & 0\le s\le 1/2 \\ p(n(2s-1)), & 1/2\le s\le 1 \end{cases}$$
$$= \begin{cases} p(2ms), & 0\le s\le 1/2 \\ p(n(2s-1)+m), & 1/2\le s\le 1 \end{cases}$$

$$= P(\alpha(s)), \text{ where } \alpha(s) = \begin{cases} 2ms, & 0 \leq s \leq 1/2 \\ n(2s-1)+m, & 1/2 \leq s \leq 1, \end{cases}$$

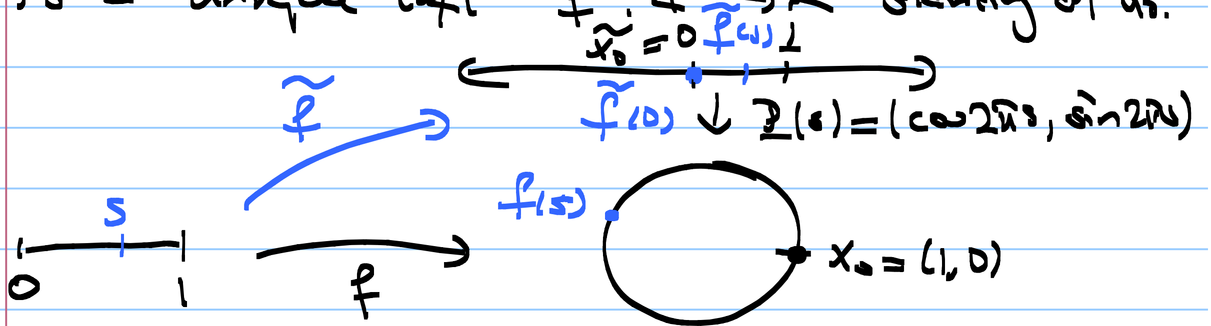
which is a continuous path $\alpha : [0,1] \to \mathbb{R}$ with $\alpha(0) = 0$ and $\alpha(1) = n+m$.

$$\begin{aligned} \text{Hence, } \Phi(m)\cdot\Phi(n) &= [\omega_m \cdot \omega_n] \\ &= [P \circ \alpha] \\ &= [P(\tilde{\omega}_{m+n})] \\ &= [\omega_{m+n}] \\ &= \Phi(m+n). \end{aligned}$$

Hence, $\Phi : \mathbb{Z} \to \pi_1(S^1, x_0)$ is a group homomorphism.

We need to show that Φ is a group isomorphism. In other words, we must show that Φ is one to one and onto.
To do so we need the following facts:

a) For each $f : I \to S^1$ starting at a point x_0 and each $\tilde{x}_0 \in \mathbb{R}$ with $P(\tilde{x}_0) = x_0$ there is a unique lift $\tilde{f} : I \to \mathbb{R}$ starting at $\tilde{x}_0$.



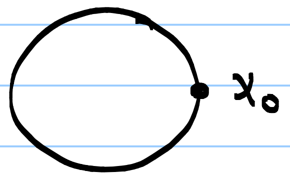
$$(P \circ \tilde{f})(s) = f(s)$$

Note that this fact implies that Φ is onto.

$$\Phi : \mathbb{Z} \longrightarrow \pi_1(S^1, x_0)$$

Let $[f] \in \pi_1(S^1, x_0)$, $f : [0,1] \to S^1$,

$f(0) = x_0 = f(1)$.

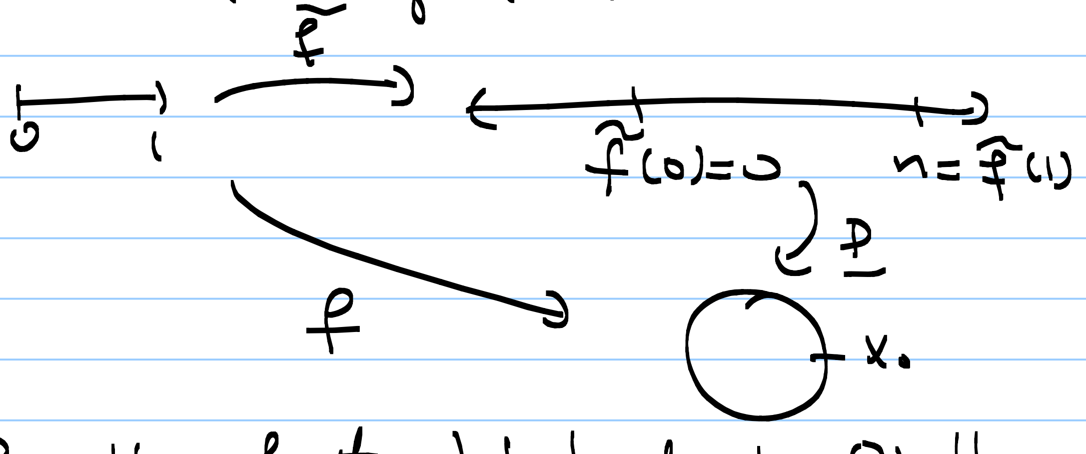


By the fact above there is a lift

$\tilde{f} : [0,1] \longrightarrow \mathbb{R}$ s.t. $p \circ \tilde{f} = f$ and $\tilde{f}(0) = 0$.

$$p(\tilde{f}(1)) = f(1) = x_0 \Rightarrow \tilde{f}(1) \in p^{-1}(x_0) = \mathbb{Z} \subseteq \mathbb{R}$$

Hence, $\tilde{f}(1) = n$ for some $n \in \mathbb{Z}$ and thus $\tilde{f}$ is a path from $[0,1]$ to $\mathbb{R}$ starting at 0 and ending at n.



By the fact stated at step (1) the homotopy class $[f] = [w_n]$. Hence

$$[f] = [w_n] = \Phi(n)$$ so that Φ is onto.

b) Let $F : I \times I \to S^1$ be a homotopy from $f(s) = F(s,0)$ to $g(s) = F(s,1)$ and $\tilde{f} : I \to \mathbb{R}$

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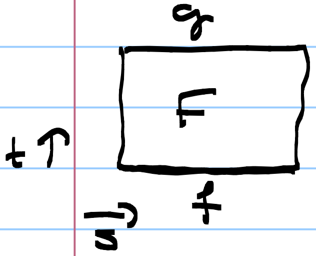
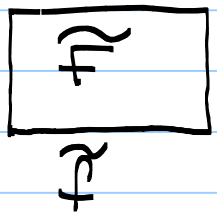
be a lift of $f(s) = F(s,0)$. Then there is a unique lift $\tilde{F} : I \times I \to \mathbb{R}$ of F, i.e.,

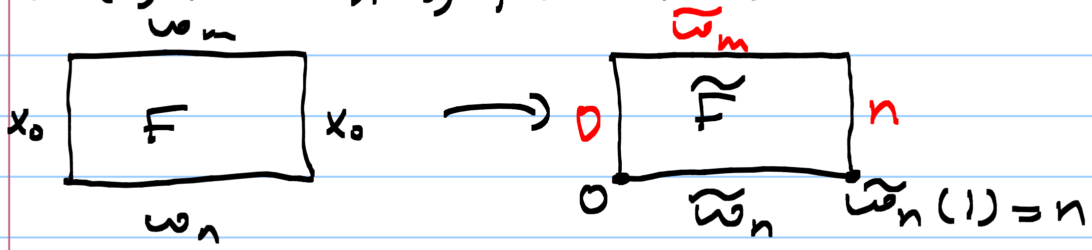
$$(p \circ \tilde{F})(s,t) = F(s,t), \text{ for all } (s,t) \in I \times I$$





Note that this fact proves that Φ is one to one:

To see let $\Phi(n) = \Phi(m)$. Then the loops $\omega_n : I \to S^1$ and $\omega_m : I \to S^1$ are homotopic, say by $F : I \times I \to S^1$. We know that $\tilde{\omega}_n : I \to \mathbb{R}$ is a lift of ω_n. By the fact (b) there is a unique lift $\tilde{F} : I \times I \to S^1$ of F with $\tilde{F}(s,0) = \tilde{\omega}_n(s)$, for all s:



$(1,0) = x_0 = F(0,t) = p(\tilde{F}(0,t)) \Rightarrow \tilde{F}(0,t) \in p^{-1}((1,0)) = \mathbb{Z}$ for all $t \in [0,1]$. $[0,1]$ is connected and $\mathbb{Z}$ is discrete and thus $\tilde{F}(0,t)$ is constant for all $t \in [0,1]$. Since $\tilde{F}(0,0) = \tilde{\omega}_n(0) = 0$ we see that $\tilde{F}(0,t) = 0$ for all $t \in [0,1]$.

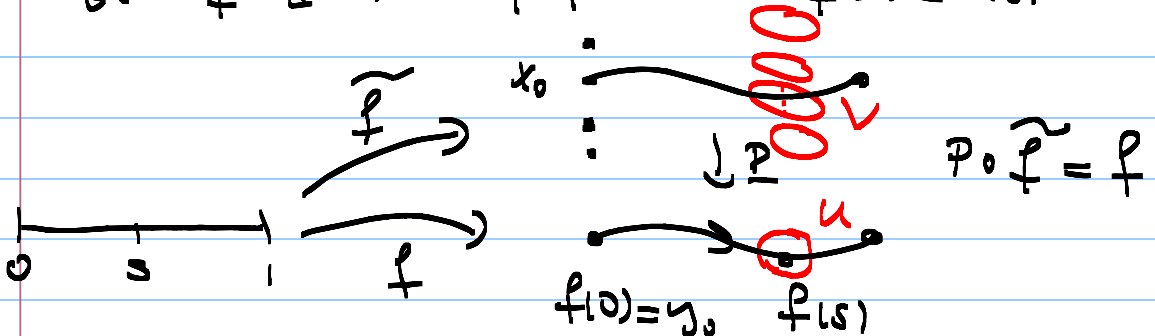
Similarly, $(1,0) = x_0 = F(1,t) = p(\tilde{F}(1,t))$ and thus $\tilde{F}(0,t) \in \mathbb{Z}$, for all $t \in [0,1]$. As above, $\tilde{F}(1,t)$ must be constant and thus $n = \tilde{\omega}_n(1) = \tilde{F}(1,0) = \tilde{F}(1,t)$, for all $t \in [0,1]$. In particular, $\tilde{F}(1,1) = n$.

On the other hand, by the uniqueness of lifts of paths, $\tilde{F}(s,1)=\tilde{\omega}_m(s)$, $s\in[0,1]$, where $F(s,1)=\omega_m(s)$ and $\tilde{\omega}_m(s)$ is the unique lift of $\omega_m(s)$ starting at $\omega_m(0)=0$.

Finally, $n=\tilde{F}(1,1)=\tilde{\omega}_m(1)=m$ and this finishes the proof. ∎

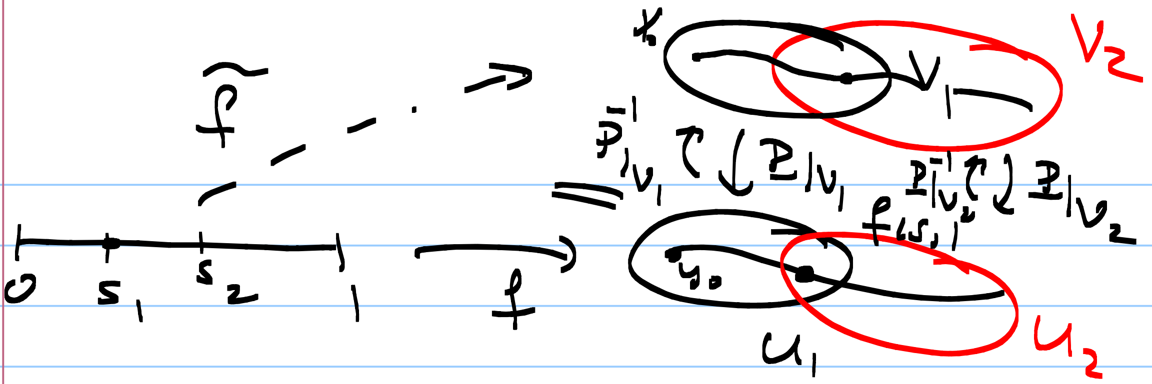
Hence, $\Phi:\mathbb{Z}\longrightarrow\pi_1(S^1,x_0)$ is an isomorphism.

<u>Proof of (a)</u>: Let $P:X\longrightarrow Y$ be a covering space and $f:I\longrightarrow Y$ be a path with $f(0)=y_0$. If $x_0\in P^{-1}(y_0)\subseteq X$ then there is a unique lift $\tilde{f}:I\longrightarrow X$ of f with $\tilde{f}(0)=x_0$.



For any $s\in I$ choose an open subset $U\subseteq Y$ with $f(s)\in U$ so that $P^{-1}(U)$ is a disjoint union of open subsets V's, where each restriction map $P:V\rightarrow U$ is a homeomorphism. Since $[0,1]$ is compact and $P^{-1}(U)$'s form an open cover for $[0,1]$ there is a partition

$0=s_0<s_1<s_2<\dots<s_m=1$ and $U_1,\dots,U_m\subseteq Y$ open subsets so that $f([s_{i-1},s_i])\subseteq U_i$ for all $i=1,\dots,m$. Start with U_1 so that $f([s_0,s_1])=f([0,s_1])\subset U_1$. $f(0)=y_0\in U_1$ and then $x_0\in f^{-1}(y_0)\subseteq f^{-1}(U_1)$. Choose V_1 in $f^{-1}(U_1)$ so that $x_0\in V_1$ and $P:V_1\rightarrow U_1$ is a homeomorphism.



Now define $\tilde f$ on $[0,s_1]=[s_0,s_1]$ as $\tilde f(s)=P^{-1}_{|V_1}(f(s))$.
Then choose V_2 so that $P:V_2\to U_2$ is a homeomorphism and $\tilde f(s_1)\in V_2\cap V_1$. Again define $\tilde f$ on $[s_1,s_2]$ as $\tilde f(s)=P^{-1}_{|V_2}(f(s))$.

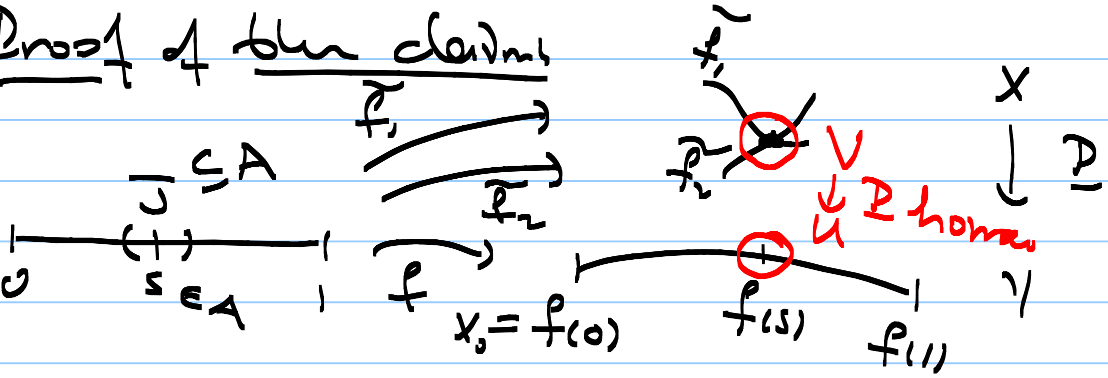
By induction $\tilde f$ is defined on all $[s_0,s_n]=[0,1]$.

Uniqueness: If $\tilde f_1$ and $\tilde f_2$ are two lifts of f starting both at $\tilde f_1(0)=x_0=\tilde f_2(0)$, let

$A=\{s\in I \mid \tilde f_1(s)=\tilde f_2(s)\}$, which is nonempty, since $0\in A$.
Claim: A is both open and closed.

Note that since $[0,1]$ is connected the claim implies that $A=I$ so that $\tilde f_1(s)=\tilde f_2(s)$, for all $s\in A=I$, which yields $\tilde f_1=\tilde f_2$.

Proof of the claim:



$P(\tilde f_1(s))=f(s)=P(\tilde f_2(s))$, for all $s\in J$
Since $P_{|V}:V\to U$ is a homeomorphism

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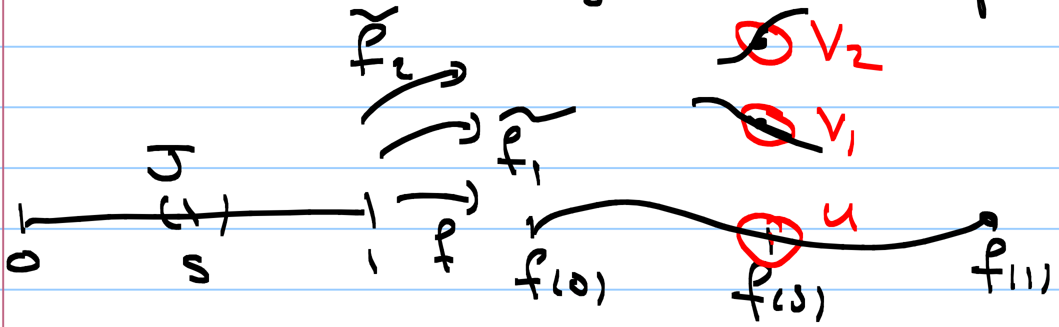
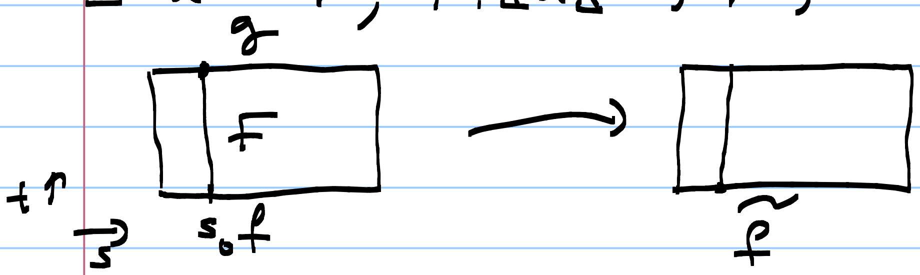
$\hat{f}_1(s) = p^{-1}|_V(p(\hat{f}_1(s))) = p^{-1}|_V(p(\hat{f}_2(s))) = \hat{f}_2(s)$
for all $s \in J$.
Hence, $J \in A$, so that A is open.
A is also closed by a similar argument:



Note that $\hat{f}_1(s) \neq \hat{f}_2(s)$ for all $s \in J$.
Hence, if $s \notin A$ then there is a open subset $s \in J$ of $[0,1]$ so that $J \cap A = \emptyset$.
This implies that $[0,1] \setminus A$ is open and thus A is closed.
This finishes the proof of fact (a).

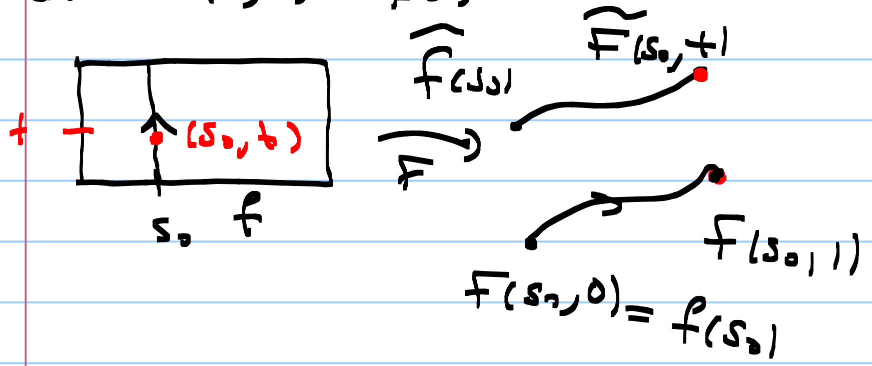
<u>Proof of (b)</u>: Let $F: I \times I \to Y$ be a homotopy and $\tilde{f}: I \to X$ be a lift of $f(s) = F(s,0)$, $s \in I$. Then there is a unique lift $\tilde{F}$ of F so that $\tilde{F}(s,0) = \tilde{f}(s)$.

$p: X \to Y$, $F: I \times I \to Y$, $\tilde{F}: I \times I \to X$



Existence comes from existence part of (a).
For any fixed $s_0 \in I$ consider the path $F(s_0, t)$, $t \in [0,1]$. Then, since $p(\tilde{f}(s_0)) = f(s_0)$ by part (a) there is a unique lift of this path, call $\tilde{F}(s_0, t)$, $t \in [0,1]$.

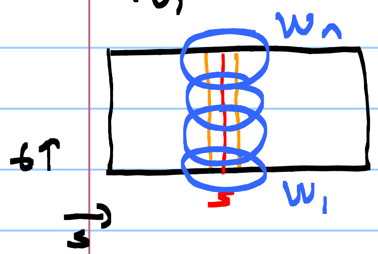
Since $s_0 \in I$ is arbitrary we obtain a function $\tilde{F}$ on $I \times I$ satisfying $p \circ \tilde{F} = F$ and $\tilde{F}(s,0) = \tilde{f}(s)$



Hence such $\tilde{F}$ exists. Uniqueness of $\tilde{F}$ comes from uniqueness part of (a), because the restriction of $\tilde{F}$ to each vertical line segment $\{s_0\} \times I$ is the unique lift of $F(s_0, t)$ starting at $\tilde{f}(s_0)$.

To finish the proof we must show that $\tilde{F}$ is continuous on $I \times I$ (note that we know $\tilde{F}(s_0,t)$, $t \in [0,1]$ is continuous for any fixed s_0).

$\tilde{F}$ is continuous: For any fixed $s \in I$ we can choose open subsets $U_1, \dots, U_n$ in Y and a partition $0 = t_0 < t_1 < \dots < t_n = 1$ of $[0,1]$ so that 1) $F([t_{i-1}, t_i]) \subseteq U_i$, $i = 1, \dots, n$ and 2) $p^{-1}(U_i)$ is a disjoint union of open subsets in X each of which is homeomorphic to U_i via p, say $V_i \subseteq X$, $p|_{V_i} : V_i \to U_i$. Let $W_i = F^{-1}(U_i)$, which is open in I.

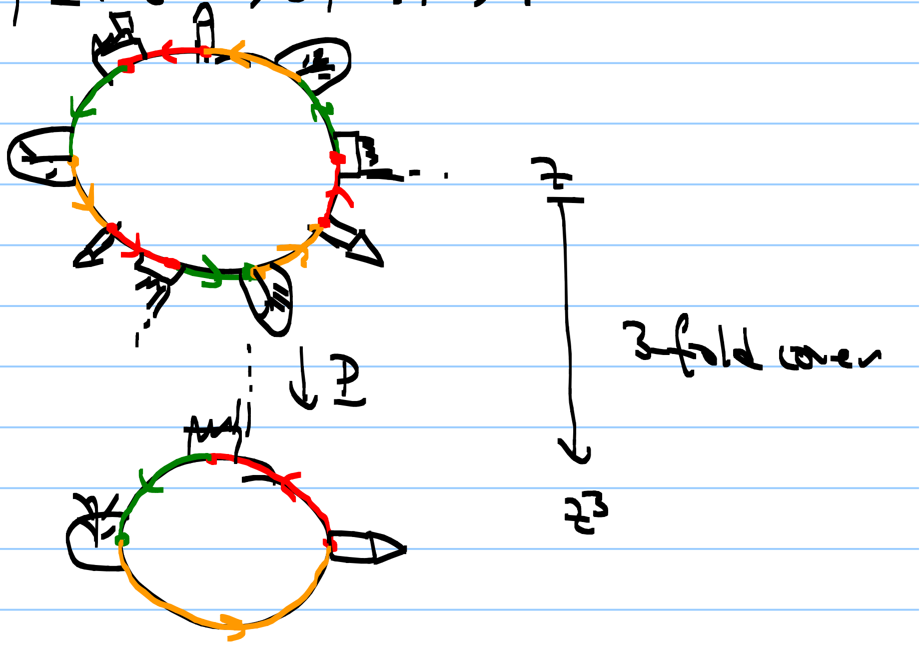


By compactness of I there is some $r_s > 0$ so that $(s - r, s + r) \times I \subseteq W_1 \cup \dots \cup W_n$. Note that for any fixed s, the unique lift of the path $t \mapsto F(s,t)$ which is $t \mapsto \tilde{F}(s,t)$, can also be constructed by patching the paths $t \mapsto p|_{V_i}^{-1}(F(s,t))$, $i = 1, \dots, n$. Finally, since both $p|_{V_i}^{-1}$ and F are continuous $\tilde{F}|_{W_i}$ is continuous for each w_i. This finishes the proof. ∎

Remark: The homotopy lifting results holds more generally: If $p: X \to Y$ is a covering and Z is a space so that we have a homotopy $F : Z \times I \to Y$ with a lift $\tilde{f} : Z \times \{0\} \to X$ of $F|_{Z \times \{0\}}$, then there is a unique lift $\tilde{F}$ of F so that $\tilde{F}(z,0) = \tilde{f}(z)$, $z \in Z$.

So, we've proved that $\pi_1(S^1, x_0) \simeq \mathbb{Z}$.

Examples: 1) $p: S^1 \to S^1$, $z \mapsto z^3$



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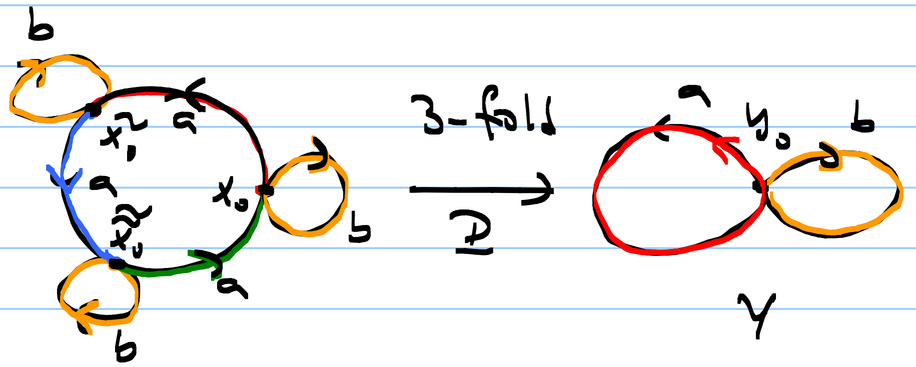

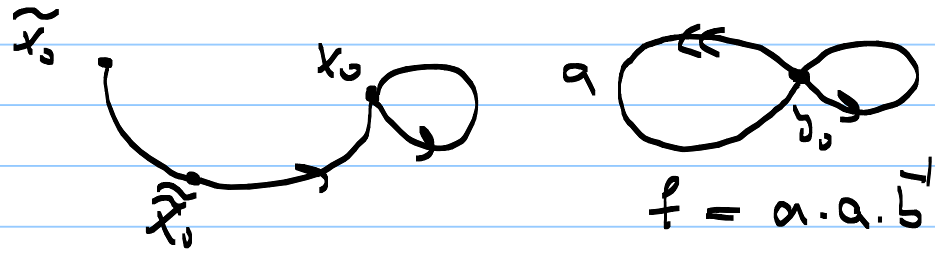

$f = a \cdot a \cdot b^{-1}$

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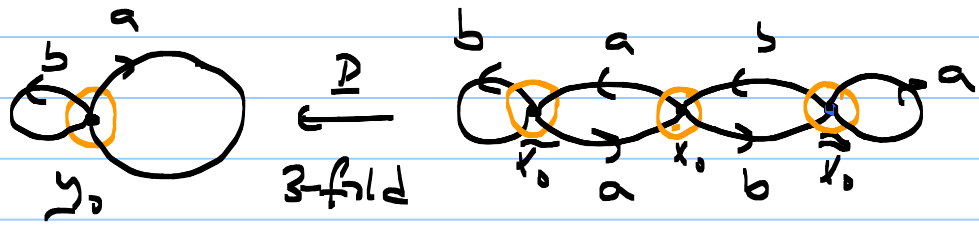
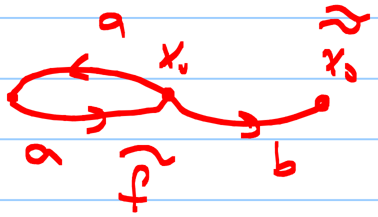
3)



$f \simeq a.a.b$



Induced Homomorphism:

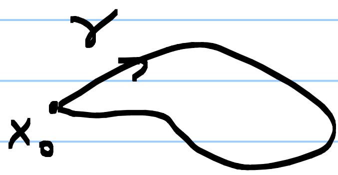
$f: X \to Y$ continuous map of topological spaces. $x_0 \in X$, $y_0 = f(x_0)$. Then f induces a homomorphism

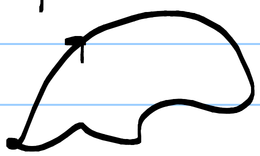
$$f_*: \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

as follows:

$$f([\gamma]) = [f \circ \gamma]$$

$f \circ \gamma$



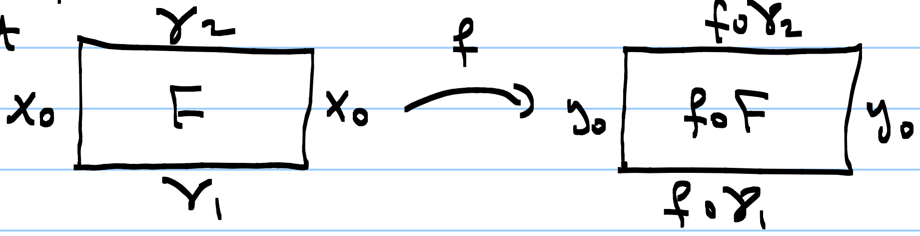


$y_0 = f(x_1)$

Exercise: f_* is really a group homomorphism!

Must check: If $\gamma_1 \sim \gamma_2$ then $f \circ \gamma_1 \sim f \circ \gamma_2$ so that f_* is well defined.

If $\gamma_1 \sim \gamma_2$ then there is $F: I \times I \to X$ so that



Remark: 1) If $f, g: (X, x_0) \to (Y, y_0)$ are homotopic maps through maps mapping x_0 to y_0 then

$f_* = g_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$.

Proof By assumption there is some $F: X \times I \to Y$ so that $F(x,0) = f(x)$, $F(x,1) = g(x)$ and $F(x_0, t) = y_0$.

$f_t(x) = F(x,t)$, $f_0 = f$, $f_1 = g$, $f_t(x_0) = y_0$.

$f_*([\gamma]) = [f\circ\gamma]$, $g_*([\gamma]) = [g\circ\gamma]$

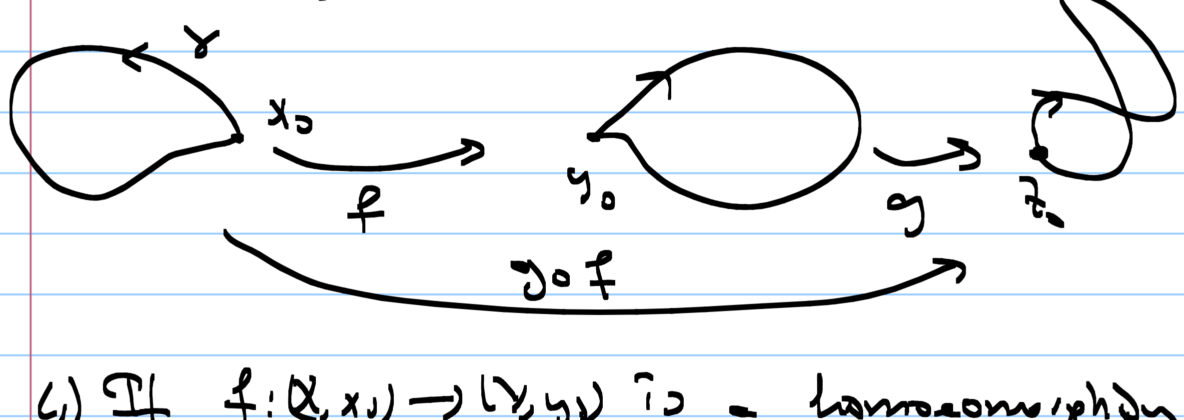
Now the homotopy $t \longmapsto [f_t \circ \gamma]$ takes $[f\circ\gamma]$ to $[g\circ\gamma]$.

2) $f = Id : X \to X$, $f_*([\gamma]) = [f\circ\gamma] = [\gamma]$ so that $f_* = Id_{\pi_1(X, x_0)}$.

3) $f: (X, x_0) \to (Y, y_0)$, $g: (Y, y_0) \longrightarrow (Z, z_0)$ then

$g\circ f : (X, x_0) \longrightarrow (Z, z_0)$ is continuous map so that

$$(g\circ f)_* = g_* \circ f_*$$

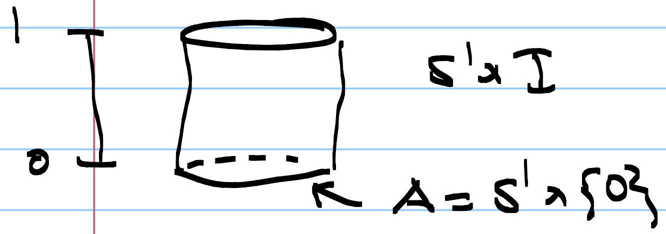


4) If $f: (X, x_0) \to (Y, y_0)$ is a homeomorphism with inverse $g: (Y, y_0) \longrightarrow (X, x_0)$ then $g\circ f = Id_{X, x_0}$ so that $(g\circ f)_* = (Id_{X, x_0})_*$

$g_* \circ f_* = \mathrm{id}_{\pi_1(X, x_0)}$.

Similarly, $f_* \circ g_* = \mathrm{id}_{\pi_1(X, x_0)}$ and thus f_* is an isomorphism.

<u>Definition:</u> A function $r : X \to A$ is called a retraction of X onto $A \subseteq X$ is r is continuous and $r(a) = a$, for all $a \in A$.



$$r : S^1 \times I \longrightarrow A = S^1 \times \{0\}$$
$$(p, t) \longmapsto (p, 0)$$

Note that in this case the composition

$r \circ i : A \to A$, $i : A \hookrightarrow X$ is the inclusion map is identity:

$$(r \circ i)(a) = r(i(a)) = r(a) = a, \quad a \in A.$$

Hence the induced homomorphisms satisfy

$(r \circ i)_* : \pi_1(A, x_0) \to \pi_1(A, x_0)$, $x_0 \in A$, is the identity homomorphism.

So, $r_* \circ i_*$ is identity:

$$\begin{tikzcd}
\pi_1(A, x_0) \arrow[r, "i_*"'] \arrow[rr, bend right, "\mathrm{id}"'] & \pi_1(X, x_0) \arrow[r, "r_*"'] & \pi_1(A, x_0)
\end{tikzcd}$$

So, $r_* \circ i_*$ is identity and thus i_* is injective and r_* is surjective.

Definition: A topological space X is called contractible if the identity function $id : X \to X$ is homotopic to a constant map $c : X \to X$, $c(x) = p$, for all $x \in X$, (for some $p \in X$).

Example: $X = \mathbb{R}^n$, $c : \mathbb{R}^n \to \mathbb{R}^n$, $c(x) = 0$ ($p = 0$). Note that

$F : \mathbb{R}^n \times I \to \mathbb{R}^n$, $F(x,t) = t \cdot x$, is a homotopy from $F(x,1) = x$ to $F(x,0) = 0 = c(x)$ so that X is contractible.

Proposition: If X is a contractible space then $\pi_1(X, x_0) = (e)$, the trivial group.

$Id \underset{h.}{\simeq} c$ $\quad id_* = c_*$, where

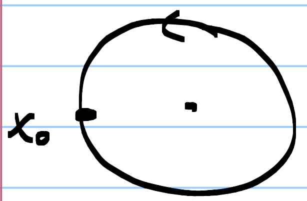
$c_*([\gamma]) = [c \circ \gamma] = [x_0]$ since $c \circ \gamma$ is the constant loop at x_0 (here $c : X \to X$, $c(x) = x_0$ for all $x \in X$).

Example: $D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$

$c : D^n \to D^n$, $c(x) = 0$, $\forall x \in D^n$, shows that D^n is contraction, because we may just take the homotopy $F(x,t) = t \cdot x$ as above.

In particular, $\pi_1(D^n, x_0) = (e)$.

Theorem: There is no retraction $r: D^2 \to \partial D^2 = S^1$.



Proof Assume that such retraction exists. Then consider the composition:

$$\begin{tikzcd}
\partial D^2 = S^1 \arrow[r, "i"] \arrow[rr, bend right, "Id_{S^1}"'] & D^2 \arrow[r, "r"] & \partial D^2 = S^1
\end{tikzcd}$$

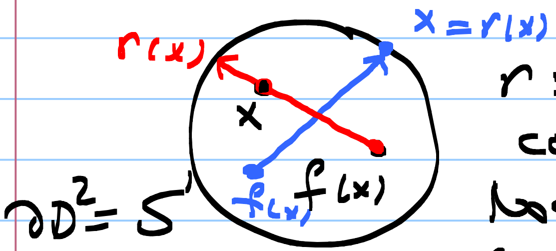
$$r \circ i = Id_{S^1} \Rightarrow r_* \circ i_* = Id_{\pi_1(S^1, x_0)}$$

$$\begin{tikzcd}
\pi_1(S^1, x_0) \arrow[r, "i_*"] \arrow[d, phantom, "\cong" rotate=90] & \pi_1(D^2, x_0) \arrow[r, "r_*"] \arrow[d, phantom, "\shortparallel"] & \pi_1(S^1, x_0) \arrow[d, phantom, "\cong" rotate=90] \\
\mathbb{Z} \arrow[rr] & (0) & \mathbb{Z} \\
n \arrow[rr, mapsto] & & n
\end{tikzcd}$$

This is a contradiction since $i_*(n) = 0$ so that $n = (r_* \circ i_*)(n) = r_*(i_*(n)) = r_*(0) = 0$, for all $n \in \mathbb{Z}$.

Corollary Any map $f: D^2 \to D^2$ has a fixed point.

Proof: Assume on the contrary that such f exists: $f(x) \neq x$, for all $x \in D^2$



$r: D^2 \to S^1$ is a continuous function (Exercise). Note that if $x \in S^1$ then $r(x) = x$. Hence, $r: D^2 \to S^1$ is a retraction, which is a contradiction to the previous result. This finishes the proof. ∎

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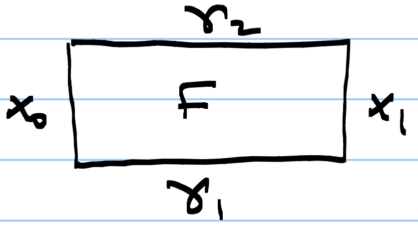
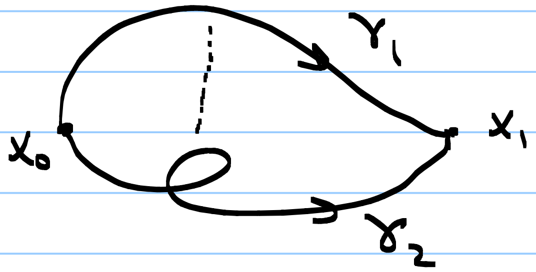
Definition: A space X is called simply connected if X is path connected and $\pi_1(X,x_0)=(e)$ for any (thus for all) $x_0\in X$.

Proposition: A space X is simply connected if for any two points x_0, x_1 of X and any two paths γ_1 and γ_2 joining x_0 to x_1 are homotopic through paths joining x_0 to x_1.

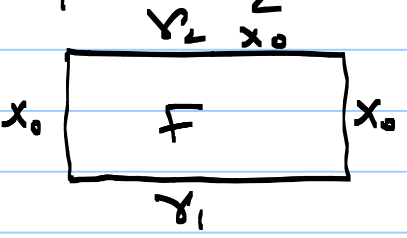
Proof: ($\Leftarrow$) So we assume that if γ_1 and γ_2 are two paths joining any two points x_1, x_1 then there is a homotopy joining γ_1 to γ_2 keeping the end points fixed:

$F: I\times I \to X$, $F(s,0)=\gamma_1(s)$, $F(s,1)=\gamma_2(s)$
$F(0,t)=x_0$, $F(1,t)=x_1$



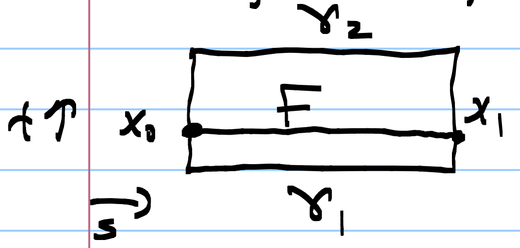


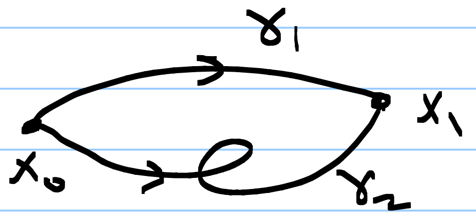
must show: $\pi_1(X,x_0)=(e)$.
Just take $x_0=x_1$ and $\gamma_2: I\to X$ the constant path at x_0. Then if γ_1 is any loop at x_0 by the assumption there is a homotopy F taking γ_1 to γ_2 keeping the end points fixed:



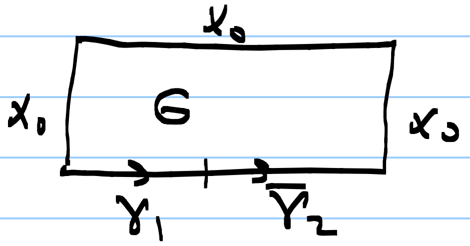
Hence $[\gamma_1]=e$ in $\pi_1(X,x_0)$, so that X is simply connected.

(⇒) Now assume that X is simply connected. <u>must show</u>: If $x_0, x_1 \in X$ and γ_1 and γ_2 are two paths joining x_0 to x_1 then there ∃ a homotopy taking γ_1 to γ_2, keeping the end points fixed.

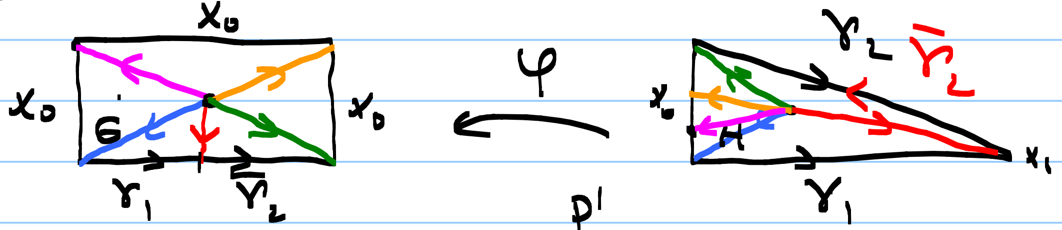




Since X is simply connected the path $\gamma_1\overline{\gamma_2}$ is homotopic to the constant path at x_0. So there is a homotopy $G: I\times I \to X$ given by



We construct a simplicial homeomorphism φ as follows:



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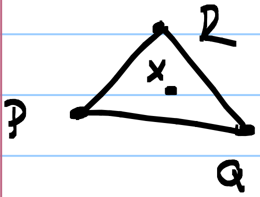
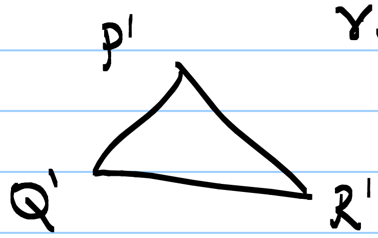




$x = t_0 P + t_1 Q + t_2 R \xmapsto{\varphi} \varphi(x) = t_0 P' + t_1 Q' + t_2 R'$

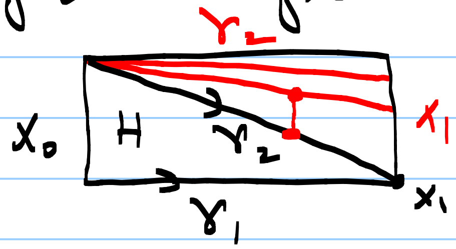
$t_i \geq 0$

$\Sigma t_i = 1$ $\qquad$ φ is a homeomorphism

t_i is uniquely determined

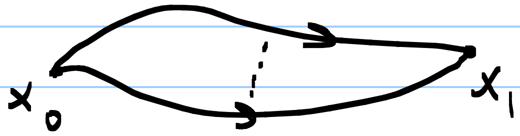
(Barycentric coordinates of x))

$H \doteq G \circ \varphi$. Finally, we define $F: I \times I \to X$ using the diagram below:

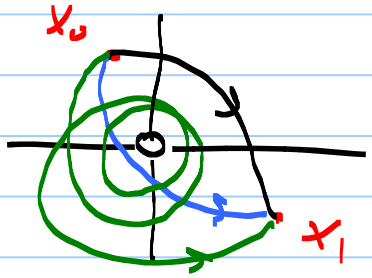


This finishes the proof. ∎

Remark: So a space X is simply connected if X is connected and any two points are connected by a unique path up to homotopy keeping the end points fixed.



Example $\mathbb{R}^2 \setminus \{(0,0)\}$ is not simply connected.



Definition: A map $f: X \to Y$ is called a homotopy equivalence if there is another map $g: Y \to X$ so that

1) $g \circ f$ is homotopic to id_X, and

2) $f \circ g$ is homotopic to id_Y.

In this case, g is called a homotopy inverse to f and we say that the spaces X and Y are homotopy equivalent.

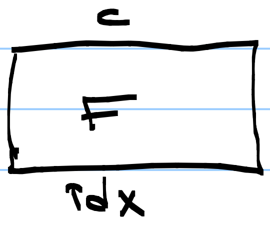
Remark: Being homotopy equivalent is an equivalence relation among topological spaces.

Example: 1) If X is contractible to a point say x_0 then X and $\{x_0\}$ are homotopy equivalent spaces.

$i: \{x_0\} \longrightarrow X$ inclusion map

$c: X \longrightarrow \{x_0\}$ constant function

$F: X \times I \longrightarrow X$, $F(x,0) = x$, $F(x,1) = x_0$, for all $x \in X$.



$c \circ i : \{x_0\} \longrightarrow \{x_0\}$, $c \circ i = Id_{\{x_0\}}$

$i \circ c : X \longrightarrow X$, $x \mapsto x_0$, which is homotopic to Id_X via F. Hence, c and i are homotopy inverses of each other.

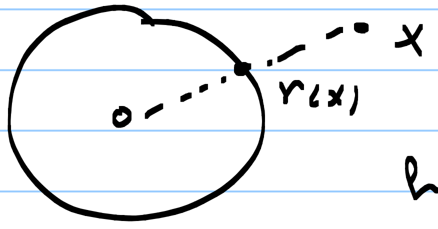
2) $X = \mathbb{R}^n \setminus \{0\}$, $Y = S^{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \Sigma x_i^2 = 1\}$ unit sphere in $\mathbb{R}^n$.

Claim: X and Y are homotopy equivalent.

Proof $i: Y = S^{n-1} \longrightarrow \mathbb{R}^n \setminus \{0\} = X$ the inclusion map

$x \longmapsto x$

$r: \mathbb{R}^n \setminus \{0\} \longrightarrow S^{n-1}$, $x \longmapsto r(x) = \frac{x}{\|x\|}$



Exercise: i and r are homotopy inverses of each other.

Video 17

Proposition: If $f: X \to Y$ is a homotopy equivalence, $x_0 \in X$ and $y_0 = f(x_0)$ then $f_*: \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$ is an isomorphism.

Idea: $f: X \to Y$ homotopy equivalence $\Longrightarrow$ $g: Y \to X$ s.t. $f \circ g \sim id_Y$ and $g \circ f \sim id_X$.

$\Longrightarrow$ $(f \circ g)_* = (id_Y)_* = Id_{\pi_1(Y, y_0)}$ and

$(g \circ f)_* = (id_X)_* = Id_{\pi_1(X, x_0)}$

$f_* \circ g_* = Id$ and $g_* \circ f_* = Id$ so that f_* is an isomorphism.

Remark: Note that proof is not complete since $(g \circ f)(x_0)$ may not be x_0!

Proposition: Let (X, x_0) and (Y, y_0) be path connected based spaces. Then $(X \times Y, (x_0, y_0))$ is path connected and the fundamental group $\pi_1(X \times Y, (x_0, y_0))$ is isomorphic to the product of groups $\pi_1(X, x_0) \times \pi_1(Y, y_0)$.

Proof: Consider the map $\varphi: \pi_1(X) \times \pi_1(Y) \to \pi_1(X \times Y)$ given by

$\varphi([\gamma_1], [\gamma_2]) \to [(\gamma_1, \gamma_2)]$, where

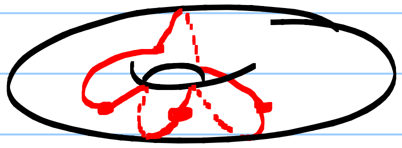
$\gamma_1: I \to X$ and $\gamma_2: I \to Y$ are loops at x_0 and y_0, respectively.

$(\gamma_1,\gamma_2): I \to X\times Y$, $(\gamma_1,\gamma_2)(s) = (\gamma_1(s),\gamma_2(s))$.

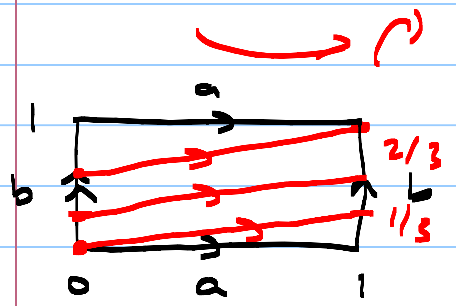
Claim: φ is a group isomorphism.

Proof: Exercise! Hint: Think of projections $P_X : X\times Y \to X$ and $P_Y : X\times Y \to Y$.

Corollary: $T^2 = S^1\times S^1$ 2-torus.



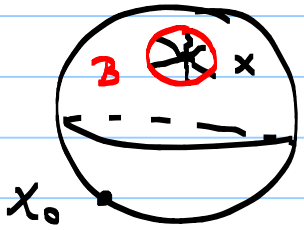
$$\pi_1(T^2) \simeq \pi_1(S^1)\times\pi_1(S^1)$$
$$\simeq \mathbb{Z}\times\mathbb{Z}$$
$$(1,\ 3)$$



Remark: An element (m,n) of $\pi_1(T^2)$ is represented by a simple loop if and only if $(m,n) = 1$.

Proposition: $\pi_1(S^n, x_0) = (e)$ if $n\geq 2$.

Proof: Let $x\in S^n$, $x\neq x_0$. Choose a small ball B in S^n around x so that $x_0\notin B$.

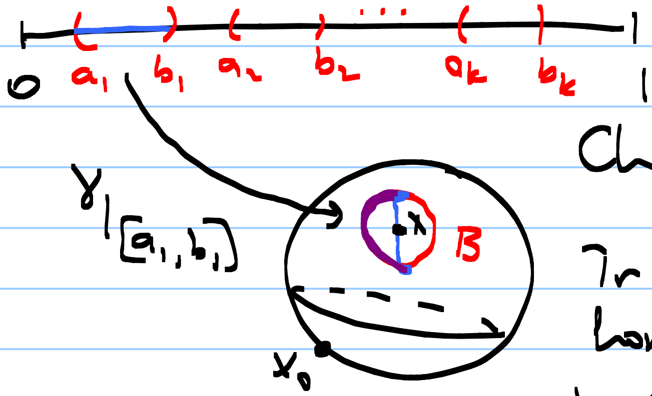


Let $\gamma: I\to S^n$ be a path at x_0. Since γ is continuous the inverse image $\gamma^{-1}(B)$ is an open subset of I not including 0 and 1 in I, because $\gamma(0)=\gamma(1)=x_0$.

Note that $\gamma^{-1}(B)$ is a disjoint union of open intervals in $(0,1)$. On the other hand, $\gamma^{-1}(x)$ is closed in I and since I is compact $\gamma^{-1}(x)$ is also compact.

Clearly, $\gamma^{-1}(x) \subseteq \gamma^{-1}(B)$ (since $x \in B$). Since $\gamma^{-1}(x)$ is compact finitely many components of $\gamma^{-1}(B)$ covers $\gamma^{-1}(x)$.

$\gamma^{-1}(B) = \bigcup_{i \in \mathbb{N}} (a_i, b_i) \Rightarrow \gamma^{-1}(x) \subseteq (a_1, b_1) \cup \cdots \cup (a_k, b_k)$ for some k.



Change each $\gamma|_{[a_i, b_i]}$ in the ball via a homotopy we may map each $[a_i, b_i]$ to the boundary ∂B. In particular, new path γ' will never pass through the point x.

So, $[\gamma'] = [\gamma]$ in $\pi_1(S^n, x_0)$ and

$$\gamma' : I \to \underset{\underset{\mathbb{R}^n}{\cong}}{S^n \setminus \{x\}} \subseteq S^n.$$

Since $\mathbb{R}^n$ is contractible γ' is homotopic to the constant path at x_0. Then $[\gamma] = [\gamma'] = e$ in $\pi_1(S^n, x_0)$. This finishes the proof. ■

Remark: Find the place in the above proof where we used the assumption that $n \geq 2$.

Video 1P

Seifert-Von Kampen's Theorem:

Assume that X is a path connected space and $X = U \cup V$, where U, V and $U \cap V$ are path connected open subsets. Then the homomorphism $(x_0 \in U \cap V)$

$$\Phi : \pi_1(U, x_0) * \pi_1(V, x_0) \longrightarrow \pi_1(X, x_0)$$
$$([\gamma_1], [\gamma_2]) \longmapsto i_{U_*}([\gamma_1]) \cdot i_{V_*}([\gamma_2])$$

is onto and its kernel $\ker \Phi$ is generated by all the elements of the form

$$(i_{U_*} \circ j_{U_*})(\omega)(i_{V_*} \circ j_{V_*})(\omega^{-1}), \text{ where } \omega \in \pi_1(U \cap V, x_0).$$

$$i_U : U \longrightarrow U \cup V = X, \quad i_V : V \longrightarrow U \cup V = X$$

$$J_U : U \cap V \longrightarrow U, \quad J_V : U \cap V \longrightarrow V \quad \text{inclusion maps}$$

<u>Notation</u>: $\pi_1(X, x_0) \simeq \pi_1(U, x_0) \underset{\pi_1(U \cap V, x_0)}{*} \pi_1(V, x_0)$

Amalgamated product over $U \cap V$.

$$\begin{tikzcd}
 & \pi_1(U) \arrow[dr, "i_{U_*}"] & \\
\pi_1(U \cap V) \arrow[ur, "j_{U_*}"] \arrow[dr, "j_{V_*}"'] & & \pi_1(U \cup V) = \pi_1(X) \\
 & \pi_1(V) \arrow[ur, "i_{V_*}"'] &
\end{tikzcd}$$

Proof of the Seifert-Van Kampen's Theorem:

$X = U \cup V$, U, V open subsets, where U, V and $U \cap V$ are all path connected.
Let $x_0 \in X$ be a ~~base~~ point.

Let $\gamma : [0,1] \to X$ be a loop at x_0.

The open set $\gamma^{-1}(U)$ is a disjoint union of open intervals in $[0,1]$.

Claim: All but finitely many components (a,b) of $\gamma^{-1}(U)$ are contained in $\gamma^{-1}(V)$.

Proof: Assume on the contrary that there is an infinite sequence (a_n, b_n) of connected components of $\gamma^{-1}(U)$ so that $\gamma((a_n,b_n)) \not\subseteq V$. Choose $c_n \in (a_n, b_n)$ with $\gamma(c_n) \notin V$, for each $n \in \mathbb{N}$.

Since (a_n,b_n) is a connected component of $\gamma^{-1}(U)$ we see that $\gamma(a_n), \gamma(b_n) \notin U$ and thus $\gamma(a_n), \gamma(b_n) \in X \setminus U = V \setminus U$.

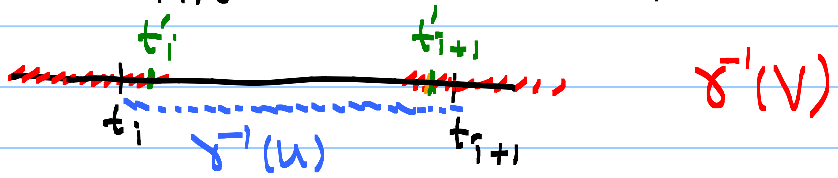
$\bigcup (a_n, b_n) \subseteq [0,1]$ and thus $\lim (b_n - a_n) = 0$. On the other hand the infinite set $\{a_n, b_n \mid n \in \mathbb{N}\}$ has an accumulation point, say $x_0 \in [0,1]$. Passing to a subsequence if necessary we may assume that $\lim a_n = x_0$. Hence, $\lim b_n = x_0$. $\gamma(a_n) \in X \setminus U$, for all n and $X \setminus U$ is closed. Then $\gamma(x_0) = \lim \gamma(a_n) \in X \setminus U$. So, $x_0 \in \gamma^{-1}(V)$. Since $\gamma^{-1}(V)$ is open there is some $\epsilon > 0$ with $(x_0 - \epsilon, x_0 + \epsilon) \subseteq \gamma^{-1}(V)$.

Since $\lim a_n = x_0 = \lim b_n$ there is some $n_0 \in \mathbb{N}$ with $a_{n_0}, b_{n_0} \in (x_0 - \epsilon, x_0 + \epsilon)$. It follows that $c_n \in [a_{n_0}, b_{n_0}] \subseteq (x_0 - \epsilon, x_0 + \epsilon) \subseteq \gamma^{-1}(V)$, which implies $\gamma(c_{n_0}) \in V$, a contradiction.

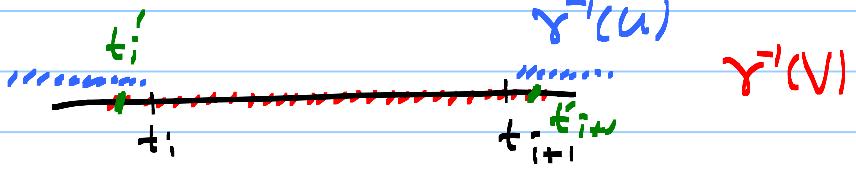
This finishes the proof of the claim. ▲

Since the end points of each component (a,b) of $\gamma^{-1}(U)$ lie in $\gamma^{-1}(V)$ (i.e., $\gamma(a), \gamma(b) \notin U$) and $\gamma(0) = \gamma(1) = x_0 \in V$, there is a partition $0 = t_0 < t_1 < t_2 < \cdots < t_n = 1$ for $[0,1]$ so that $\gamma(t_i) \in V$ for all i, and $\gamma([t_i, t_{i+1}]) \subseteq U$ or $\gamma([t_i, t_{i+1}]) \subseteq V$, for all $i = 0, \ldots, n-1$.

If $\gamma([t_i, t_{i+1}]) \subseteq U$ but $\{\gamma(t_i), \gamma(t_{i+1})\} \notin U$, since $\gamma(t_i), \gamma(t_{i+1}) \in V$ we may replace t_i and t_{i+1} with t_i' and t_{i+1}', respectively, so that $\gamma([t_i', t_{i+1}']) \subseteq U$ and $\gamma(t_i'), \gamma(t_{i+1}') \in V$.



Similarly, if $\gamma([t_i, t_{i+1}]) \subseteq V$ and $\{\gamma(t_i), \gamma(t_{i+1})\} \notin U$ we may replace t_i and t_{i+1} with t_i' and t_{i+1}', respectively, so that $\gamma([t_i', t_{i+1}']) \subseteq V$ and $\{\gamma(t_i'), \gamma(t_{i+1}')\} \subseteq U$.



Finally, we have the following: For each $i = 0, \ldots, n$, $\gamma(t_i) \in U \cap V$ and $\gamma([t_i, t_{i+1}]) \subseteq U$ or $\gamma([t_i, t_{i+1}]) \subseteq V$, for all $i = 0, \ldots, n-1$.

Choose a path α_i in $U \cap V$ joining x_0 to $\gamma(t_i)$.
Then

$$\underbrace{\gamma|_{[t_0,t_1]} \cdot \bar{\alpha}_1}_{} \cdot \underbrace{\alpha_1 \gamma|_{[t_1,t_2]} \cdot \bar{\alpha}_2}_{} \cdot \underbrace{\alpha_2 \cdot \gamma|_{[t_2,t_3]} \cdot \bar{\alpha}_3}_{}$$

$$\cdots \cdot \bar{\alpha}_k \cdot \underbrace{\alpha_k \cdot \gamma|_{[t_k,t_{k+1}]} \cdot \bar{\alpha}_{k+1}}_{} \cdots \cdot \bar{\alpha}_{n-1} \cdot \underbrace{\alpha_{n-1} \cdot \gamma|_{[t_{n-1},t_n]}}_{}$$

$\gamma|_{[t_0,t_1]} \cdot \bar{\alpha}_1$, $\alpha_{n-1} \cdot \gamma|_{[t_{n-1},t_n]}$ and all

$\alpha_k \cdot \gamma|_{[t_k,t_{k+1}]} \cdot \bar{\alpha}_{k+1}$ are loops at x_0

lying completely in U or V.

This proves that the homomorphism $\Phi_0 : \pi_1(U) * \pi_1(V) \to \pi_1(X)$ is onto.

Clearly this induces an onto homomorphism

$$\Phi : \pi_1(U) \underset{\pi_1(U\cap V)}{*} \pi_1(V) \to \pi_1(X).$$

To prove that Φ is also injective we need to introduce so called equivalence of factorizations of elements of $\pi_1(X)$.

Let $[f]=[f_1]\cdot[f_2]\cdots[f_n]$, $[f_i]\in\pi_1(U)$ or $[f_i]\in\pi_1(V)$ be a factorization of an element $[f]\in\pi_1(X)$. Consider the following moves:

1) If $[f_i]$ and $[f_{i+1}]$ belong to $\pi_1(U)$ or $\pi_1(V)$ simultaneously we may replace $[f_i]\cdot[f_{i+1}]$ by $[f_i \cdot f_{i+1}]$.

2) If some $[f_i]\in\pi_1(U)$ and $\pi_1(U\cap V)$ then we may regard $[f_i]$ in $\pi_1(V)$.

If one can pass from one factorization of an element to another factorization of the same element by applying finitely many of the above moves, we'll call these two factorizations are equivalent.

Claim: Any two factorization of an element are equivalent.

Note that this claim proves the injectivity of Φ.

Proof of the claim: Let $[f_1]\cdots[f_k]=[f_1']\cdots[f_l']$ be two factoriation of an element of $\pi_1(X)$. Let $F: I\times I \to X$ be the homotopy from $f_1\cdots f_k$ to $f_1'\cdots f_l'$. Since $I\times I$ is compact and $X=U\cup V$ we may divide $I\times I$ into subrectangles $R_1, \dots, R_{mn}$ as below so that $F(R_i)\subseteq U$ or $F(R_i)\subseteq V$ for each $i=1,\dots,mn$

Let γ_i be the path seperating $R_1,\dots,R_i$ from $R_{i+1},\dots,R_{mn}$.

Each corner of any R_i lies in $U\cap V$. For any corner v choose a path g_v from x_0 to v, lying in the path connected set $U\cap V$.

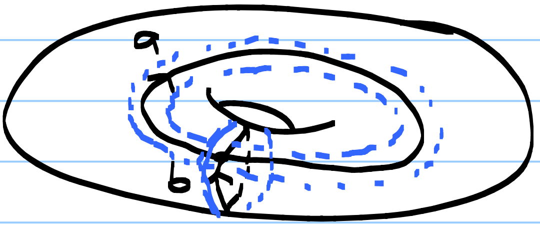
Note that each $F|_{\gamma_i}$ is homotopic to the following factorization as explained in case $i=2m-1$.

$$F|_{\gamma_{2m-1}} \sim (F|_1\cdot\bar{g}_{v_1})\cdot(g_{v_1}\cdot F|_2\cdot\bar{g}_{v_2})\cdot(g_{v_2}\cdot F|_3\cdot\bar{g}_{v_3})\cdot \dots \cdot(g_{v_{m-1}}\cdot F|_m\cdot\bar{g}_{v_m})\cdot$$
$$(g_{v_m}\cdot F|_{m+1})$$

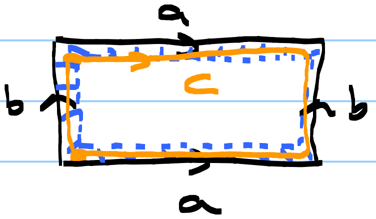
Note that each passage from γ_i to γ_{i+1} is a move. For example passage from γ_{2m-1} to γ_{2m} replaces the part $(g_{v_{m-1}}\cdot F|_m\cdot\bar{g}_{v_m})\cdot(g_{v_m}\cdot F|_{m+1})$ of $F|_{\gamma_{2m-1}}$ with $(g_{v_{m-1}}\cdot F|_{m+2})$, which is a move of type 1.

Note that $F|_{\gamma_0} \sim f_1 \cdots f_k$ and $F|_{\gamma_{mn}} \sim f_1' \cdots f_\ell'$ and this finishes the proof.

Some Applications:



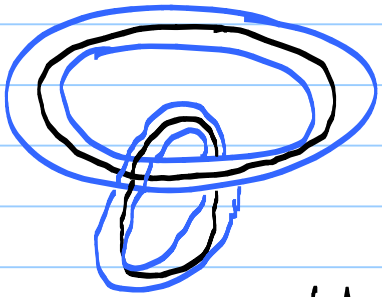
1) $T^2 = S^1 \times S^1$



$T^2 = U \cup V$

U: interior of the rectangle.

V = a neighborhood of $a \cup b$.



$U \cap V = V \setminus (a \cup b)$

U is contractible $\Rightarrow \pi_1(U) = (e)$

V is homotopy equivalent to $a \cup b$.

$$\Rightarrow \pi_1(V) = \pi_1(a \cup b) = \pi_1(S^1 \vee S^1) \overset{(*)}{=} Fr(a,b) = \langle a, b \mid - \rangle$$

U∩V is homotopy equivalent to the circle c.

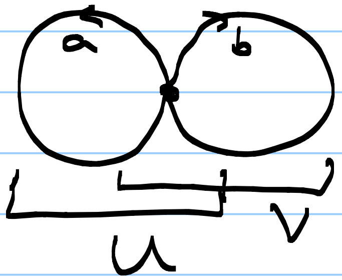
$$\begin{tikzcd}
{} & \pi_1(U) \arrow[dr] & {} & e \\
\pi_1(U \cap V) \arrow[ur] \arrow[dr] \arrow[urrr, bend left] & \| \ (e) & \pi_1(U \cup V) = \pi_1(T^2) & {} \\
\text{is} & \pi_1(V) \arrow[ur] & {} & {} \\
\underline{\underline{\langle c \rangle}} \arrow[rrr, bend right] & \text{is} & {} & \underline{bab^{-1}a^{-1}} = e \\
{} & \langle a, b \mid - \rangle & {} & ba = ab
\end{tikzcd}$$

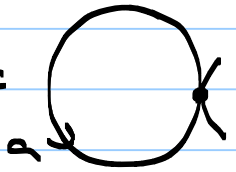
$$\pi_1(T^2) = \pi_1(U) \underset{\pi_1(U \cap V)}{*} \pi_1(V) = \langle a, b \mid bab^{-1}a^{-1} \rangle$$

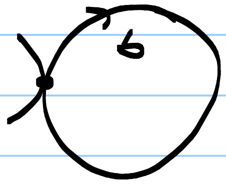
$$= \langle a, b \mid ab = ba \rangle = \mathbb{Z} \times \mathbb{Z}$$

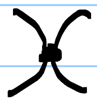
2) Wedge of circles:

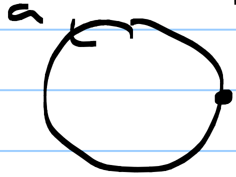
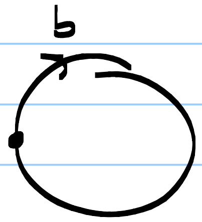
$X = S^1 \vee S^1$



$U =$ 

$V =$ 

$U \cap V =$  $\underset{h.e.}{\simeq} \cdot$

$U \underset{h.e.}{\simeq}$  $= S^1$, $V \underset{h.e.}{\simeq}$  $= S^1$

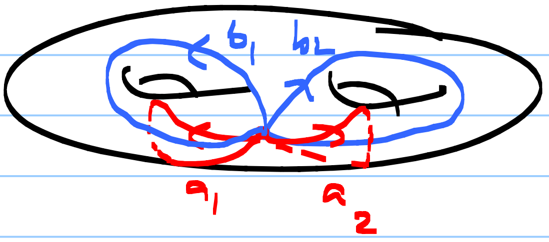
$$\begin{tikzcd} & \pi_1(V) \arrow[dr] & \\ \pi_1(U \cap V) \arrow[ur] \arrow[dr] & & \pi_1(U \cup V) \\ & \pi_1(U) \arrow[ur] & \end{tikzcd}$$

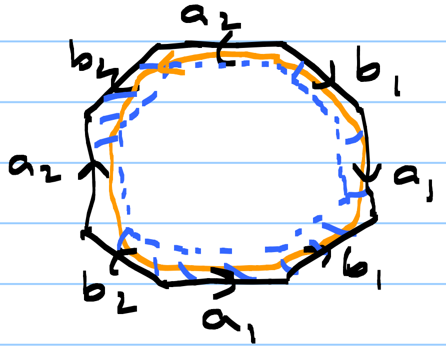
$$\begin{tikzcd} & \overset{\langle b \rangle}{\overset{\shortparallel}{\mathbb{Z}}} \arrow[dr] & \\ (e) \arrow[ur] \arrow[dr] & & \pi_1(U \cup V) = \langle a, b \mid - \rangle \\ & \underset{\langle a \rangle}{\underset{\shortparallel}{\mathbb{Z}}} \arrow[ur] & \end{tikzcd}$$

$= Fr_2$.

Similarly, $\pi_1(\bigvee_n S^1) = Fr_n$.

3) $X = \Sigma_2$





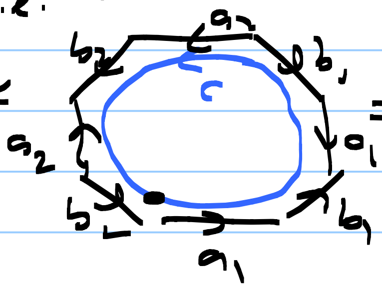
U: inside region

V: neighborhood of a_i, b_i's.

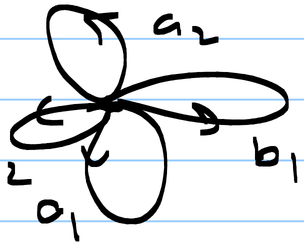
$U \cap V \underset{h.e.}{\simeq} c$

$U \underset{h.e.}{\simeq} \{pt\} \Rightarrow \pi_1(U) = (e)$

$V \simeq$



$=$



$= \bigvee_4 S^1$

$\pi_1(V) = Fr_4 = \langle a_1, b_1, a_2, b_2 \mid - \rangle$

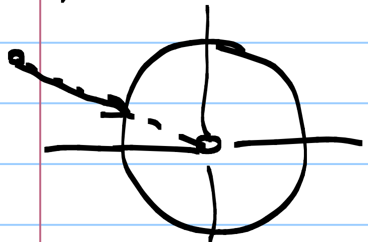
$U \cap V \underset{h.e.}{\simeq} c \Rightarrow \pi_1(U\cap V) = \pi_1(S^1) = \langle c \rangle \simeq \mathbb{Z}.$

$$\begin{tikzcd}
 & \pi_1(U) \arrow[dr] \arrow[d, phantom, "\shortparallel"] & & e \\
\pi_1(U\cap V) \arrow[ur] \arrow[dr] \arrow[d, phantom, "\wr S"] & (e) & \pi_1(U\cup V) & \\
\mathbb{Z} \arrow[d, phantom, "\shortparallel"] & \pi_1(V) \arrow[ur] \arrow[d, phantom, "\shortparallel"] & & a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1} \\
\langle c \rangle \arrow[uuurrr, bend left, |->] \arrow[urrr, bend right, |->] & \langle a_1, b_1, a_2, b_2 \mid - \rangle & &
\end{tikzcd}$$

$\pi_1(\Sigma_2) = \pi_1(U \cup V) = \langle a_1, b_1, a_2, b_2 \mid a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1} \rangle$

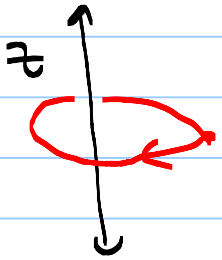
4) $X = \mathbb{R}^2 \setminus \{(0,0)\} \longrightarrow S^1$ homotopy equivalent

$x \longmapsto \frac{x}{\|x\|}$



$\pi_1(\mathbb{R}^2 \setminus \{(0,0)\}) \simeq \pi_1(S^1) \simeq \mathbb{Z}.$

5) $Y = \mathbb{R}^3 \setminus \{z\text{-axis}\} = \{(x,y,z) \in \mathbb{R}^3 \mid x \neq 0 \text{ or } y \neq 0\}$



$\pi_1(\mathbb{R}^3 \setminus \{z\text{-axis}\}) \simeq \mathbb{Z}.$

$P: \mathbb{R}^3 \setminus \{z\text{-axis}\} \longrightarrow \mathbb{R}^2 \setminus \{(0,0)\}$

$(x,y,z) \longmapsto (x,y)$

P is a homotopy equivalence with inverse

$$Q : \mathbb{R}^2 \setminus \{(0,0)\} \longrightarrow \mathbb{R}^3 \setminus \{z\text{-axis}\}$$
$$(x,y) \longmapsto (x,y,0)$$

$$P \circ Q : \mathbb{R}^2 \setminus \{(0,0)\} \longrightarrow \mathbb{R}^2 \setminus \{(0,0)\} \Rightarrow P\circ Q = \mathrm{Id}.$$
$$(x,y) \longmapsto (x,y)$$

$$Q \circ P : \mathbb{R}^3 \setminus \{z\text{-axis}\} \longrightarrow \mathbb{R}^3 \setminus \{z\text{-axis}\}$$
$$(x,y,z) \longmapsto (x,y,0)$$

$Q\circ P$ is homotopic to the $\mathrm{Id}_{\mathbb{R}^3 \setminus \{z\text{-axis}\}}$

$$F : \mathbb{R}^3 \setminus \{z\text{-axis}\} \times [0,1] \longrightarrow \mathbb{R}^3 \setminus \{z\text{-axis}\}$$

$$F((x,y,z),t) = (x,y,tz).$$

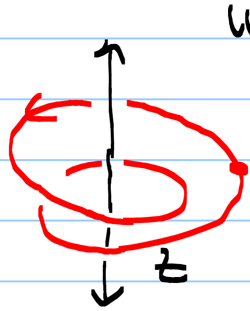
$$F((x,y,t),0) = (x,y,0) = Q\circ P.$$

$$F((x,y,z),1) = (x,y,z) = \mathrm{Id}_{\mathbb{R}^3(z\text{-axis})}(x,y,z).$$

Hence, P and Q are homotopy inverses.

$$\mathbb{R}^3 \setminus \{z\text{-axis}\} \underset{h.e.}{\simeq} \mathbb{R}^2 \setminus \{(0,0)\}.$$

Hence, $\pi_1(\mathbb{R}^3 \setminus \{z\text{-axis}\}) \simeq \mathbb{Z}$ also.



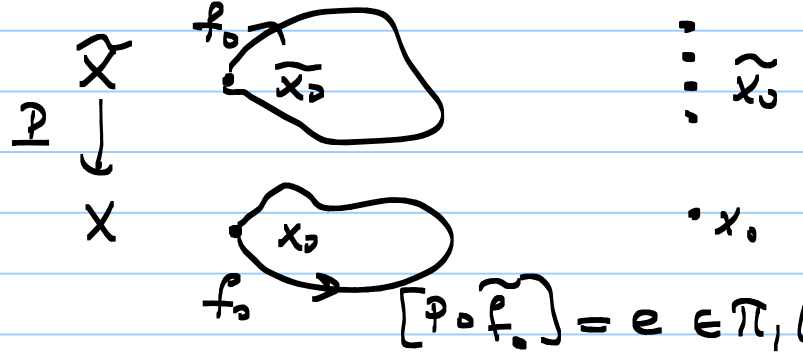
Video 19

Galois Teory of Covering Spaces

Aim: Construct a correspondence between covering spaces of a space X and the subgroups of the fundamental group $\pi_1(X)$.

Proposition: Let $p:(\tilde{X},\tilde{x}_0)\to(X,x_0)$ be a covering space. Then the homomorphism $p_*:\pi_1(\tilde{X},\tilde{x}_0)\to\pi_1(X,x_0)$ is injective. The image of p_* consists of the loops in X based at x_0, whose lifts to $\tilde{X}$ starting at $\tilde{x}_0$ are loops.

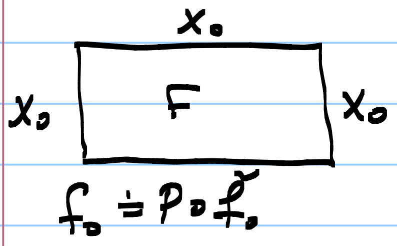
Proof: Let $\tilde{f}_0: I\to\tilde{X}$ be a loop at $\tilde{x}_0$ so that $p_*([\tilde{f}_0])=e$ in $\pi_1(X,x_0)$.



$[p\circ\tilde{f}_0]=e\in\pi_1(X,x_0)$.

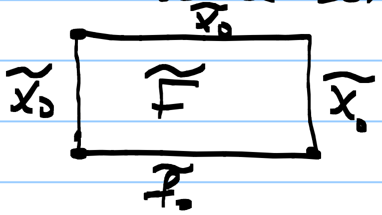
must show: $[\tilde{f}_0]=e\in\pi_1(\tilde{X},\tilde{x}_0)$.

Since $p\circ\tilde{f}_0$ is homotopic to the constant loop then is a homotopy $F:I\times I\to X$ so that



Since $\tilde{f}_0$ is a lift of f_0 the homotopy F lifts to a homotopy $\tilde{F}:I\times I\to\tilde{X}$

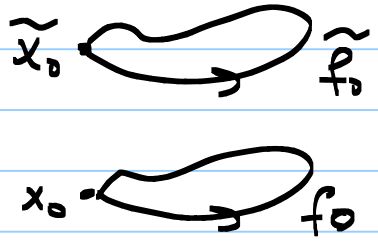
satisfying



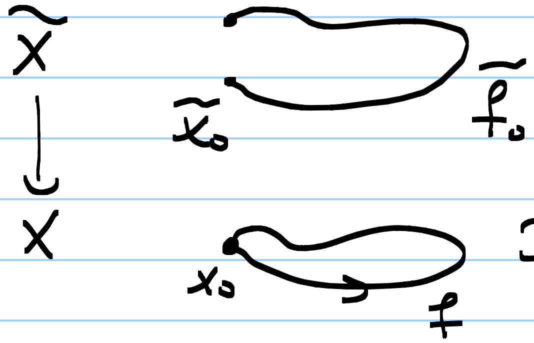
so that

$\tilde F$ gives a homotopy from $\tilde f_0$ to the constant loop at $\tilde x_0$. Hence, $[\tilde f_0] = e$ in $\pi_1(\tilde X, \tilde x_0)$. In other words, p_* is injective.

For the second statement let $[f_0] = p_*[\tilde f_0]$.

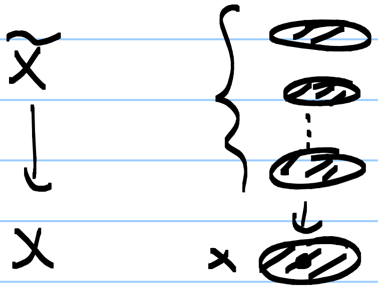


Then $\tilde f_0$ is the unique lift of f_0 starting at $\tilde x_0$. Since $\tilde f_0$ is a loop at $\tilde x_0$ we conclude that elements of $p_*(\pi_1(\tilde X, \tilde x_0))$ are elements on $\pi_1(X, x_0)$ represented by loops at x_0, whose unique lifts to $\tilde x_0$ are also loops. ■



Is $[f] \in H \doteq p_*(\pi_1(\tilde X, \tilde x_0))$?

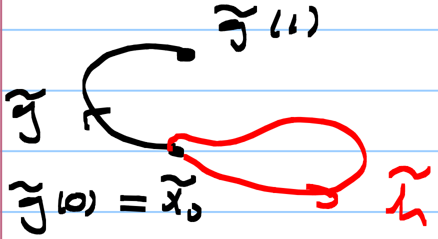
<u>Definition</u>: Let $p: \tilde X \to X$ be a covering space. For any $x \in X$ the cardinality of $p^{-1}(x)$ is called the "number of sheets" of the covering above x.

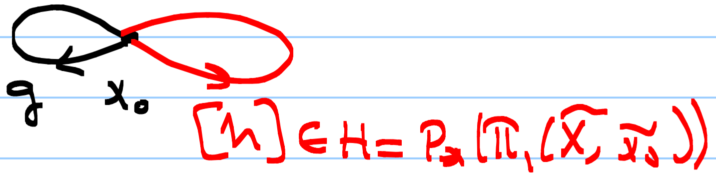


<u>Proposition</u>: The number of sheets $|p^{-1}(x_0)|$ of a covering $p: (\tilde X, \tilde x_0) \to (X, x_0)$ of path connected spaces is equal to the index

of the subgroup $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ in $\pi_1(X, x_0)$.

Proof: Let g be a loop at x_0 and $\tilde{g}$ be its unique lift to $\tilde{X}$ starting at $\tilde{x}_0$. If $[h] \in H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ and $\tilde{h}$ is the unique lift of h starting at $\tilde{x}_0$ then the end points $\tilde{g}(1)$ and $(\tilde{h}\cdot\tilde{g})(1)$ are the same:





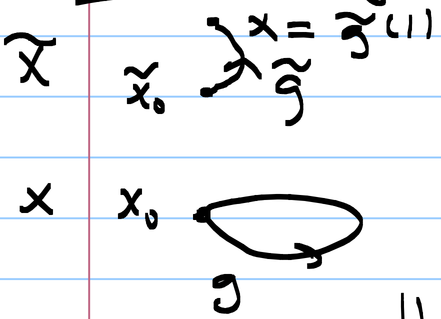
$$h\cdot g \to \tilde{h}\cdot\tilde{g} \Rightarrow (\tilde{h}\cdot\tilde{g})(1) = \tilde{g}(1).$$

So, we get a well defined function:

$$\Phi : \{H[g] \mid [g] \in \pi_1(X, x_0)\} \longrightarrow p^{-1}\{x_0\} \text{ by}$$

$$\Phi(H[g]) = \tilde{g}(1).$$

Φ is surjective:



$x \in p^{-1}\{x_0\}$, then $\exists$ a element $[g] \in \pi_1(X, x_0)$ so that

$$\Phi(H[g]) = \tilde{g}(1) = x.$$

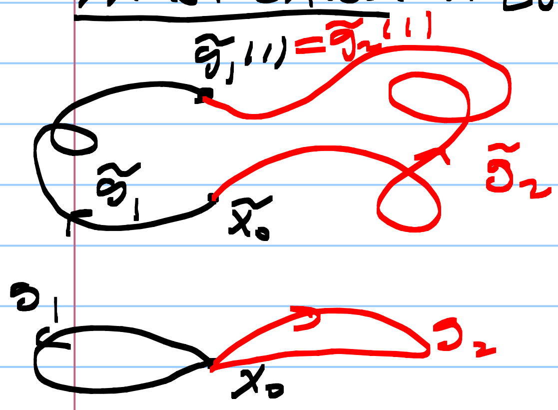
Hence, Φ is onto.

Video 20

Φ is injective: Assume that

$\Phi(H[g_1]) = \Phi(H[g_2])$, for some $[g_i] \in \pi_1(X,x_1)$.

<u>must show:</u> $H[g_1] = H[g_2]$.



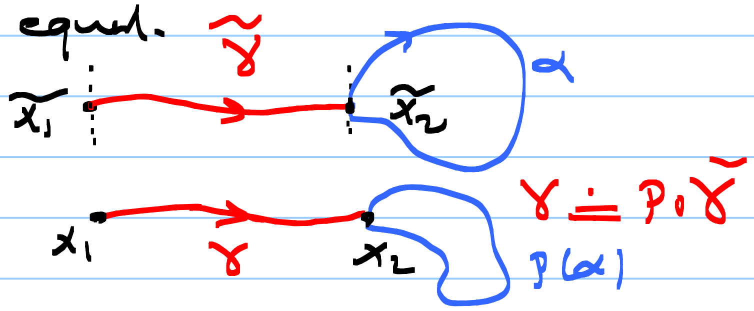
Note that $\tilde g_1 \overline{\tilde g_2}$ is a loop at $\tilde x_0$.
Hence, $p_*[\tilde g_1 \overline{\tilde g_2}] \in H$.

However, $p_*([\tilde g_1 \cdot \overline{\tilde g_2}]) = [g_1 \cdot \bar g_2] = [g_1][\bar g_2] \in H$

$[g_1]\cdot[g_2]^{-1} \in H \Leftrightarrow H[g_1] = H[g_2]$.

Hence, Φ is one to one. ∎

<u>Remark:</u> If $p:\tilde X \to X$ is a covering projection of path connected spaces then for any x_1, x_2 in X the cardinalities $p^{-1}(x_1)$ and $p^{-1}(x_2)$ are equal.



$|p^{-1}(x_1)| = [\pi_1(X,x_1):H_1]$, $H_1 = p_*(\pi_1(X,\tilde x_1))$

and $|p^{-1}(x_2)| = [\pi_1(X,x_2):H_2]$, $H_2 = p_*(\pi_1(X,\tilde x_2))$.

$$\begin{tikzcd}
T_{\tilde{\gamma}} : \pi_1(\tilde{X}, \tilde{x}_1) \arrow[r, "\simeq"] \arrow[d, "p_*"] & \pi_1(\tilde{X}, \tilde{x}_2) \arrow[d, "p_*"] & \text{is an isom.} \\
T_{\gamma} : \pi_1(X, x_1) \arrow[r, "\simeq"] & \pi_1(X, x_2) & \text{is an isom.}
\end{tikzcd}$$

Since the diagram is commutative the index $[\pi_1(X,x_1) : H_1] = [\pi_1(X,x_2) : H_2]$.

Hence, $|p^{-1}(x_1)| = [\pi_1(X,x_1) : H_1)$

$$= [\pi_1(X,x_2) : H_2]$$

$$= |p^{-1}(x_2)|.$$

This common cardinality $|p^{-1}(x_1)|$ is called the degree of the covering.

Proposition: (Lifting Criterion)

Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering space, $f : (Y, y_0) \to (X, x_0)$ a continuous map, where $f(y_0) = x_0$. Assume that Y is path connected and locally path connected. Then f has a lift

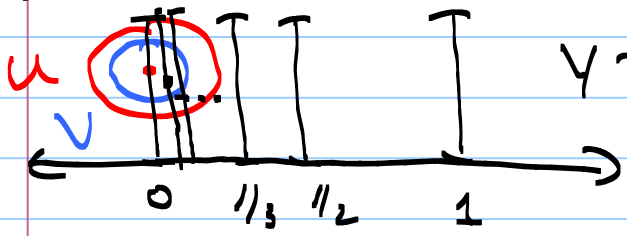
$$\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x}_0) \text{ if and only if}$$

$$f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0)).$$

Remark: Locally path connected: If $y \in Y$ and $y \in U \subseteq Y$ open subset then there is another open subset V in Y with $y \in V \subseteq U$

so that V is path connected.

Example: $Y = I \times \{0, 1/n \mid n = 1, 2, \dots\} \cup \{x\text{-axis}\}$



Y is path connected but not locally path connected.

Proof: First assume that f has a lift $\tilde{f}$: $f : (Y, y_0) \to (X, x_0)$, $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ so that $p \circ \tilde{f} = f$.

$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & (X, x_0) \end{tikzcd} \Rightarrow \begin{tikzcd} & \pi_1(\tilde{X}, \tilde{x}_0) \arrow[d, "p_*"] \\ \pi_1(Y, y_0) \arrow[ur, "\tilde{f}_*"] \arrow[r, "f_*"'] & \pi_1(X, x_0) \end{tikzcd}$$

Both diagram are clearly commutative. In particular, $f_* = p_* \circ \tilde{f}_*$. Hence,

$$f_*(\pi_1(Y, y_0)) = (p_* \circ \tilde{f}_*)(\pi_1(Y, y_0))$$
$$= p_*(\tilde{f}_*(\pi_1(Y, y_0)))$$
$$\subseteq p_*(\pi_1(\tilde{X}, x_0)),$$ since

$\tilde{f}_*(\pi_1(Y, y_0)) \subseteq \pi_1(\tilde{X}, \tilde{x}_0)$.

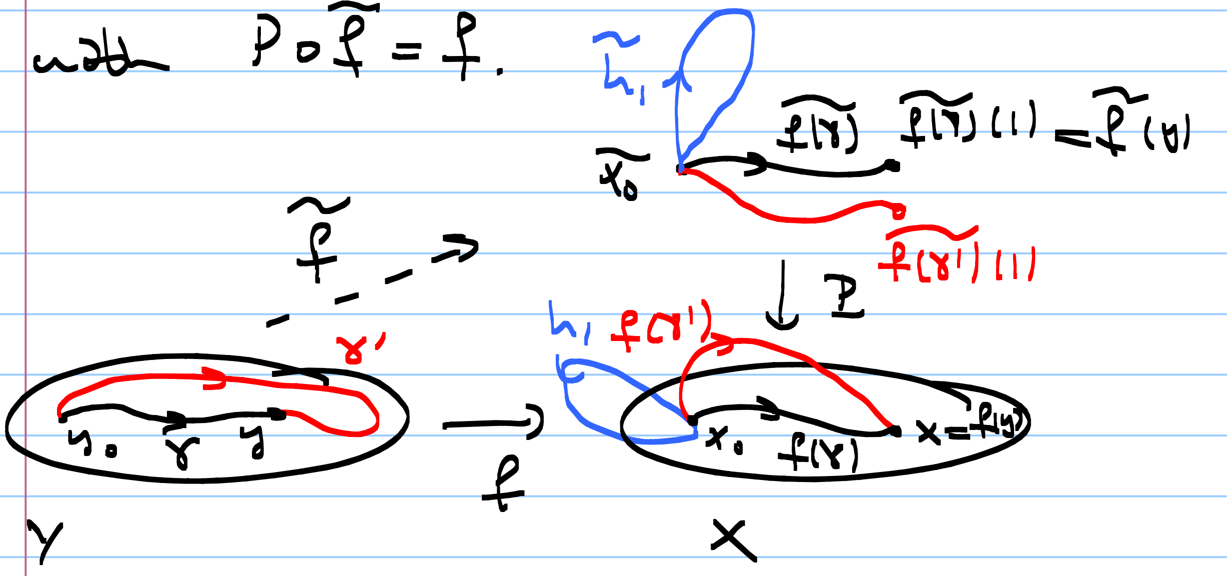
Hence, $f_*(\pi_1(Y, y_0)) \subseteq H \doteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.

This finishes the proof of one direction.

Video 21

Now assume that $f_*(\pi_1(Y,y_0)) \subseteq H = p_*(\pi_1(\tilde X,\tilde x_0))$.

must construct: A lift $\tilde f : (Y, y_0) \to (\tilde X, \tilde x_0)$ with $p \circ \tilde f = f$.



Define $\tilde f(y_0)$ as $\tilde x_0$, so that $(p \circ \tilde f)(y_0) = p(\tilde x_0) = x_0$.

For any other point $y \in Y$ choose a path γ in Y joining y_0 to y and define $\tilde f(y)$ as the end point of the unique lift of $f(\gamma)$ to $\tilde X$ starting at $\tilde x_0$:

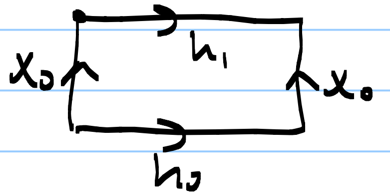
$$\tilde f(y) \doteq \widetilde{f(\gamma)}(1).$$

Well definedness of $\tilde f(y)$: If γ' is another path in Y joining y_0 to y then we must show that

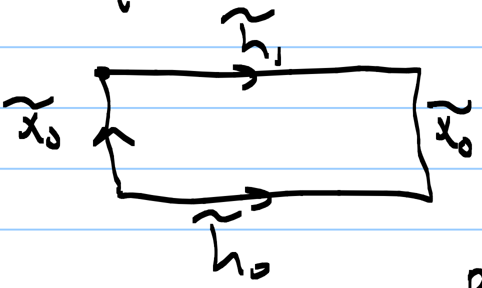
$$\widetilde{f(\gamma')}(1) = \widetilde{f(\gamma)}(1).$$

Note that $f(\gamma')\overline{f(\gamma)}$ is a loop at x_0 and thus define a homotopy class, say $[h_1]$ in $\pi_1(X, x_0)$. Since $[h_1] \in f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde X, \tilde x_0))$ there is a class $[\tilde h_1] \in \pi_1(\tilde X, \tilde x_0)$ so that $[h_0] = p_*([\tilde{h}_1])$. Let $h_1 = p(\tilde{h}_1)$ so that $[h_0] = [h_1]$.

In particular, there is a homotopy $h_t : I \times I \longrightarrow X$ so that



Since h_1 has already a lift $\tilde{h}_1$ starting at $\tilde{x}_0$ the homotopy h_t has a unique lift to $\tilde{h}_t : I \times I \longrightarrow \tilde{X}$ so that



Hence, $\tilde{h}_0$ is a loop at $\tilde{x}_0$.

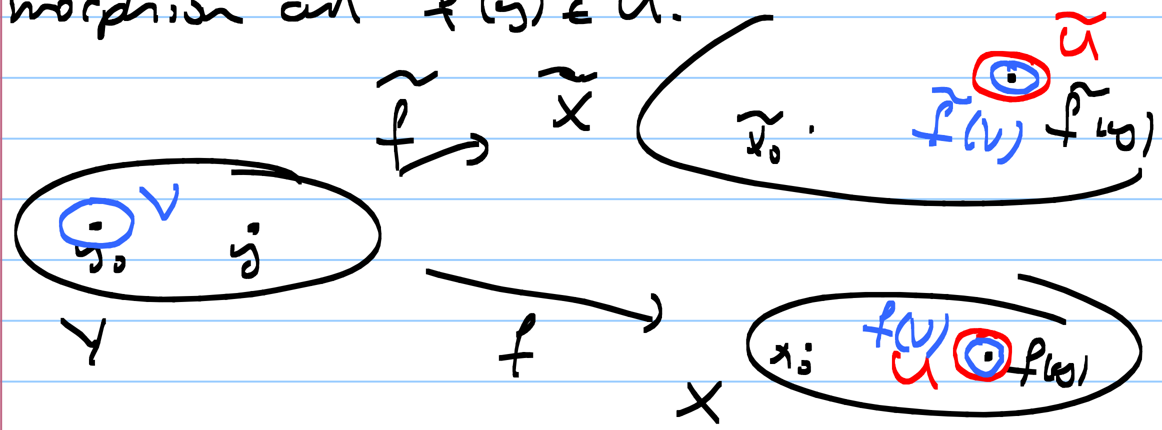
However,

$$p(\tilde{h}_0) = h_0 = f(\gamma')\, f(\bar{\gamma})$$

and thus the unique lift $\widetilde{f(\gamma')} \cdot \widetilde{f(\bar{\gamma})}$ of $f(\gamma') f(\bar{\gamma})$ is a loop.

Then $\widetilde{f(\gamma')}(1) = \widetilde{f(\gamma)}(1)$, so that $\tilde{f}$ is well defined.

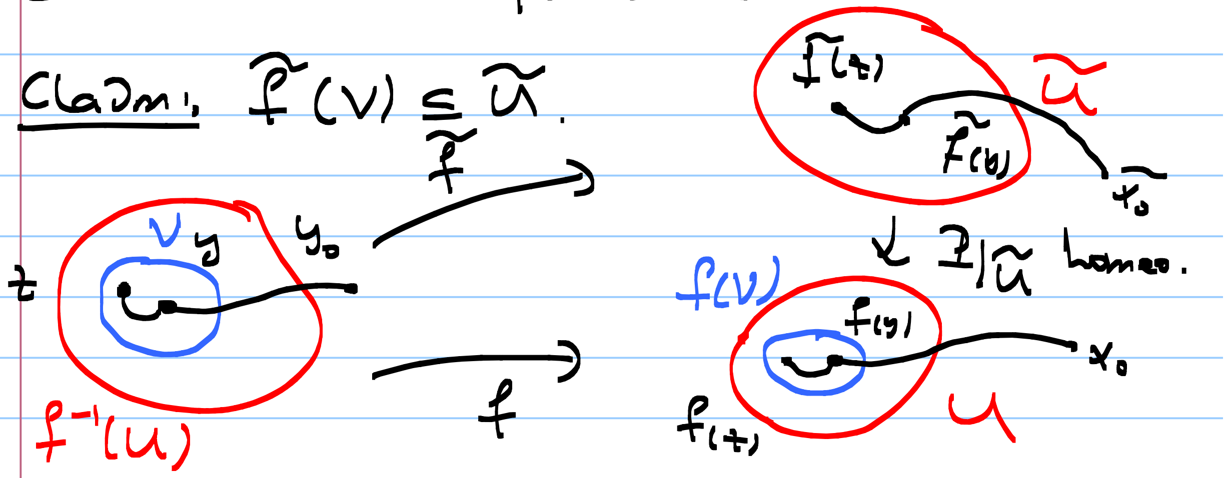
<u>$\tilde{f}$ is continuous</u>: Let $y \in Y$ and $U \subseteq X$ an an open subset containing $f(y)$ such that there is some $\tilde{U} \subseteq \tilde{X}$ with $p : \tilde{U} \to U$ homeomorphism and $\tilde{f}(y) \in \tilde{U}$.



<u>must find</u>: An open subset V in Y with $y \in V$ and $\tilde{f}(V) \subseteq \tilde{U}$.

Since Y is locally path connected there is an open subset V so that $y \in V$, V is path connected and $f(V) \subseteq U$.

Claim: $\tilde{f}(V) \subseteq \tilde{U}$.



This finishes the proof. ∎

Proposition: Given a covering space $p: \tilde{X} \to X$ and a map $f: Y \to X$ with two lifts $\tilde{f}_1$ and $\tilde{f}_2$ from Y to $\tilde{X}$ that agree at one point, then if Y is connected these two lifts agree on all of Y.

Proof

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, bend left, "\tilde{f}_1"] \arrow[ur, "\tilde{f}_2"'] \arrow[r, "f"'] & X \end{tikzcd}$$

$p \circ \tilde{f}_i = f, i = 1, 2.$

By assumption there is some $y_0 \in Y$ so that $\tilde{f}_1(y_0) = \tilde{f}_2(y_0)$. Then the subset

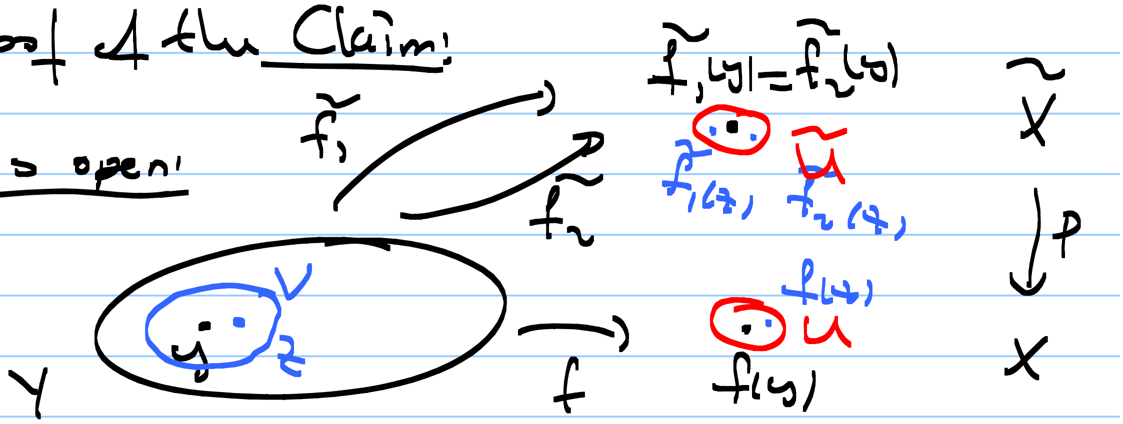
$$A = \{ y \in Y \mid \tilde{f}_1(y) = \tilde{f}_2(y) \}$$ is not empty, since $y_0 \in A$.

Claim: A is both open and closed in Y.

Note that since $A \neq \emptyset$ and Y is connected the claim implies that $A=Y$ so that $\tilde{f}_1 = \tilde{f}_2$.

Proof of the Claim:

A is open:



Choose V open subset in Y with $y \in Y$ so that $\tilde{f}_1(V) \subseteq \tilde{U}$ and $\tilde{f}_2(V) \subseteq \tilde{U}$. In particular, $\tilde{f}_1(z), \tilde{f}_2(z)$ lies in $\tilde{U}$ and $p(\tilde{f}_i(z)) = f(z)$. Since $p: \tilde{U} \to U$ is a homeomorphism $\tilde{f}_1(z) = \tilde{f}_2(z)$.

In particular, $V \subseteq A$ so that $\ast$ is open.

Exercise: Show that A is closed.

This finishes the proof. ∎

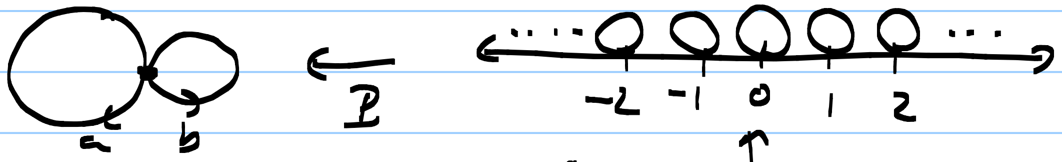
Video 22

Classification of Covering Spaces:

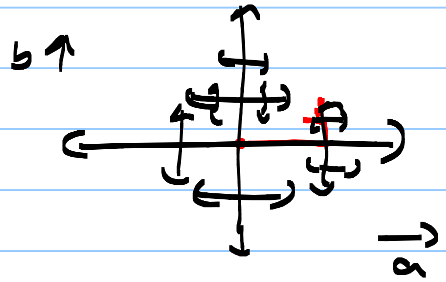
A covering $p:\tilde{X}\to X$ is called universal covering if $\tilde{X}$ is simply connected, i.e., $\tilde{X}$ is path connected and $\pi_1(\tilde{X},\tilde{x}_0)=(e)$.

Ex: $p:\mathbb{R}\to S^1$, $p(t)=(\cos 2\pi t, \sin 2\pi t)$, $t\in\mathbb{R}$, $\mathbb{R}$ connected and $\pi_1(\mathbb{R})=(e)$. Hence, $p:\mathbb{R}\to S^1$ is "the" universal covering.

Ex $X=S^1\vee S^1$



$aba^{-1}b^2$



Aim: To construct the universal covering of any given space.

Theorem: Let X be a path connected, locally path connected and semilocally simply connected space. Then X has a universal covering space.

- Locally path connected: $x\in U\subseteq X$ open subset. Then there a path connected open subset V s.t. $x\in V\subseteq U$.

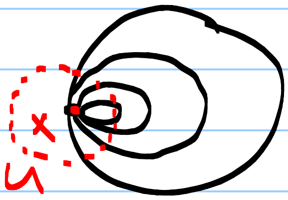
- Locally simply connected: $x\in U\subseteq X$ open subset. Then there is a simply connected open subset V s.t. $x\in V\subseteq U$.

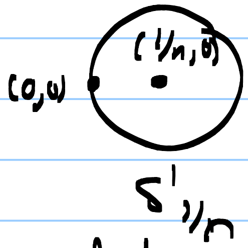
– Semilocally simply connected: $x \in U \subseteq X$ open subset. Then there is an open subset V so that $x \in V \subseteq U$ and the map

$$\pi_1(V,x) \to \pi_1(U,x) \text{ is trivial.}$$

Ex $X \subseteq \mathbb{R}^2$

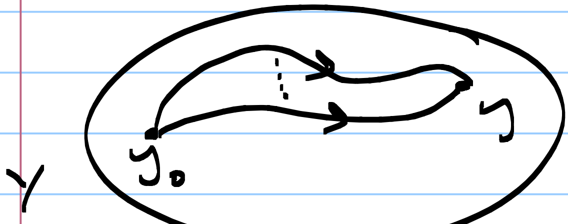
$$X = \bigvee_{n=1}^{\infty} S^1_{1/n}$$





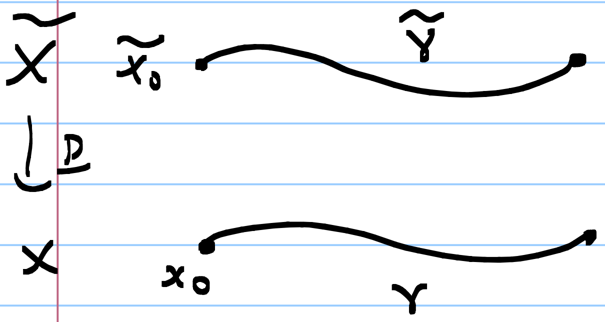
X is not semilocally simply connected.

Proof. Idea: Recall that if Y is simply connected and $y_0 \in Y$, then there is a 1-1 correspondence between the points of Y and the homotopy classes of paths joining y_0 to points of Y, where homotopies fixing the end points.



$y \longleftrightarrow$ Homotopy classes of paths

If $p : \tilde{X} \to X$ is the universal covering then

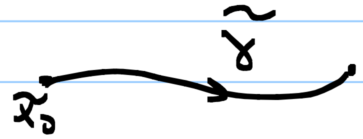


$\tilde{\gamma}$ is the unique lift of γ starting at $\tilde{x}_0$.

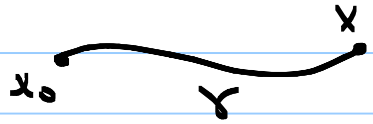
So let's define $\tilde{X}$ as the homotopy classes of paths in X starting at x_0.

$\tilde{X} = \{ [\gamma] \mid \gamma(0) = x_0 \}$, where homotopies fixing the end points

Projection map:



$\tilde{x} = \tilde{\gamma}(1)$



$p(\tilde{x}) = \gamma(1) = x$

$p : \tilde{X} \longrightarrow X$, $p([\gamma]) = \gamma(1) = x$

must show:

1) $\tilde{X}$ has a topology so that $p: \tilde{X} \to X$ is a covering space.

2) $\tilde{X}$ is simply connected.

Note that $p : \tilde{X} \to X$ is clearly onto.

Let's put a topology on $\tilde{X}$:

Let $\mathcal{U}$ denote the collection of path connected open subsets $U \subseteq X$ such that

$$\pi_1(U) \longrightarrow \pi_1(X) \text{ is trivial.}$$

If $U \in \mathcal{U}$ and $V \subseteq U$ is any other path connected open subset then

$\pi_1(V) \longrightarrow \pi_1(U) \longrightarrow \pi_1(X)$ is still trivial.

Hence, $V \in \mathcal{U}$.

<u>Claim:</u> $\mathcal{U}$ is a basis for the topology on X.

Proof: i) $x \in X$, $x \in W$ open then there is some $U \subseteq X$ open s.t. $x \in U \subseteq W$ and
$\pi_1(U) \longrightarrow \pi_1(W)$ is trivial.
$\Rightarrow \pi_1(U) \longrightarrow \pi_1(W) \longrightarrow \pi_1(X)$ is trivial.

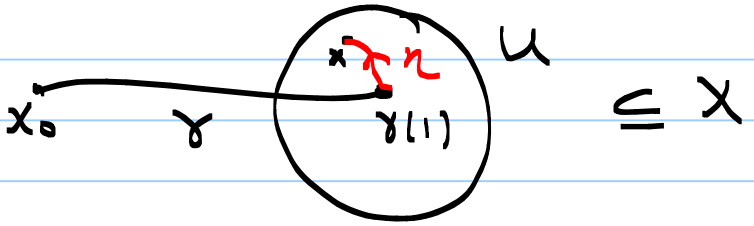
ii) $x \in X$, $x \in U_1 \cap U_2$, $U_i \in \mathcal{U}$. Since X is locally path connected then is a path connected subset $x \in U \subseteq U_1 \cap U_2$. In particular, $x \in U \subseteq U_1$ and $U_1 \in \mathcal{U}$. So, $U \in \mathcal{U}$.

Hence, $\mathcal{U}$ is a basis for the topology on X.

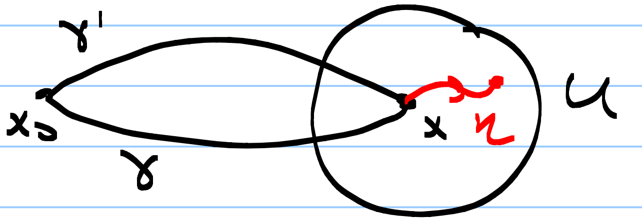
Now let's define a basis for a topology on $\tilde{X}$:

If $U \in \mathcal{U}$ and $[\gamma] \in \tilde{X}$ with $\gamma(1) \in U$, then define

$$U_{[\gamma]} = \{[\gamma \cdot \eta] \mid \eta \text{ is a path in } U \text{ with } \eta(0) = \gamma(1)\}.$$



<u>Observations:</u> 1) If $\gamma' \in [\gamma]$ then $U_{[\gamma']} = U_{[\gamma]}$.



$[\gamma \cdot \eta] = [\gamma' \cdot \eta]$

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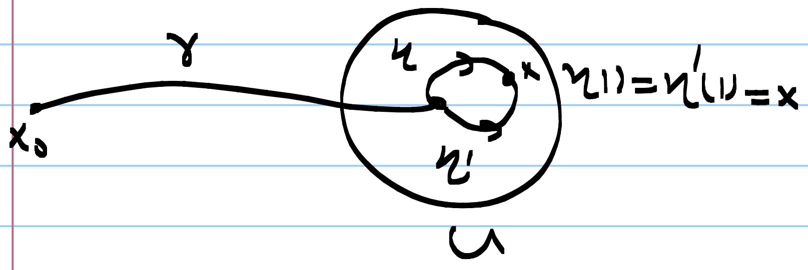
Hence, $U_{[\gamma]}$ depends only on the homotopy class $[\gamma]$.

2) Since U is path connected the map

$P: U_{[\gamma]} \longrightarrow U$, $[\gamma\cdot\eta] \longmapsto \eta(1)$ is onto.

3) Moreover, $P: U_{[\gamma]} \longrightarrow U$ is also injective.

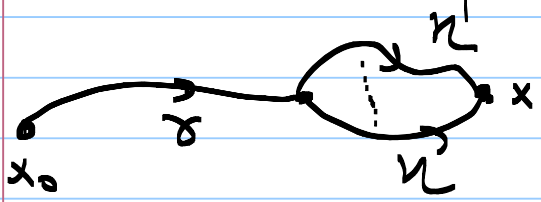


Assume that $P([\gamma\cdot\eta]) = P([\gamma\cdot\eta'])$.

must show: $[\gamma\cdot\eta] = [\gamma\cdot\eta']$ in $\tilde{X}$.

$\pi_1(U) \longrightarrow \pi_1(X)$ is trivial by the choice of U.

Hence η and η' are homotopic in X by a homotopy keeping the end point fixed.



Hence, $\gamma\cdot\eta$ is homotopic to $\gamma\cdot\eta'$ via a homotopy keeping the end points fixed.

Conclusion: $P: U_{[\gamma]} \longrightarrow U$ is a bijection.

Claim: The collection $\tilde{\mathcal{U}} = \{U_{[\gamma]} \mid U \in \mathcal{U}, [\gamma] \in \tilde{X}\}$ is a basis for a topology on $\tilde{X}$.

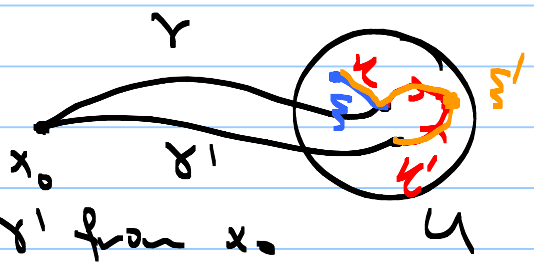
Proof of the claim: Exercise.

Observation: 1) $P : U_{[\gamma]} \longrightarrow U$ is a homeomorphism

2) P is continuous since $\widetilde{X}$ is covered by $U_{[\gamma]}$'s and $P|_{U_{[\gamma]}}$ is a homeomorphism.

3) P is a covering map. If $U \in \mathcal{U}$ then

$$P^{-1}(U) = \dot{\bigcup}\, U_{[\gamma]}$$



For any two path γ, γ' from x_0 to some points in U either $U_{[\gamma]} = U_{[\gamma']}$ or $U_{[\gamma]} \cap U_{[\gamma']} = \phi$.

If $[\gamma \cdot \eta] = [\gamma' \cdot \eta']$ then for any ξ there is some ξ' with

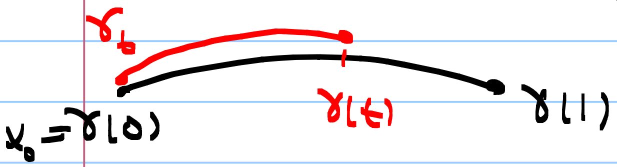
$$[\gamma \cdot \xi] = [\gamma' \cdot \xi'].$$

Hence, $U_{[\gamma]} = U_{[\gamma']}$.

$\widetilde{X}$ is connected: Take any point $[\gamma] \in \widetilde{X}$.

So $\gamma : [0,1] \longrightarrow X$ is a path so that $\gamma(0) = x_0$. For any $t \in [0,1]$ define the path

$$\gamma_t : [0,1] \longrightarrow X, \quad \gamma_t(s) = \begin{cases} \gamma(s) & 0 \leq s \leq t \\ \gamma(t) & t \leq s \leq 1 \end{cases}$$

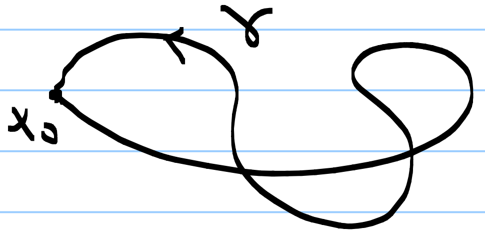


Note that $t \mapsto [\gamma_t]$ is a path in $\tilde{X}$ joining $[\gamma]$ to the constant path at x_0.

<u>Finally, we must show that $\tilde{X}$ is simply connected.</u>

Since the homomorphism $P_\# : \pi_1(\tilde{X}, [x_0^*]) \to \pi_1(X, x_0)$ is injective, ($[x_0^*]$ is the homotopy class of the constant path at x_0) it is enough to show that the image of $P_\#$ is trivial.

Let $[\gamma]$ be in the image of $P_\#$. So γ is a loop at x_0.



Then the path $t \mapsto [\gamma_t]$ is a lift of γ to the covering $\tilde{X} \to X$, because $P([\gamma_t]) = \gamma_t(1) = \gamma(t)$

Since $t \mapsto [\gamma_t]$ is a loop at $[x_0^*]$ the end points of this path must be same.

$t = 0 \Rightarrow [\gamma_0] = [x_0^*]$

$t = 1 \Rightarrow [\gamma_1] = [\gamma]$.

$[\gamma] = [x_0^*]$ in $\tilde{X}$, in other words γ is homotopic to the constant path at x_0 via a homotopy keeping the end points fixed.

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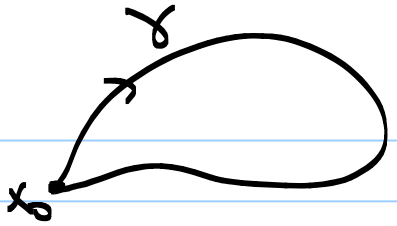


$[\gamma] = e$ in $\pi_1(X, x_0)$.

Here, Ker $p_\#$ is trivial so that $\pi_1(\tilde{X}, \tilde{x}_0) = (e)$.

Remark: For more detail see pages 100–112 of my Math 537 notes.

Definition: Let $p_1 : \tilde{X}_1 \to X$ and $p_2 : \tilde{X}_2 \to X$ be two covering spaces. We'll say that these covering spaces are isomorphic if there is a homeomorphism $\varphi : \tilde{X}_1 \to \tilde{X}_2$ so that the diagram below is commutative.

$$\begin{tikzcd}
\tilde{X}_1 \arrow[rr, "\varphi"] \arrow[dr, "p_1"'] & & \tilde{X}_2 \arrow[dl, "p_2"] \\
& X \arrow[u, phantom, "\circlearrowright"] &
\end{tikzcd}$$

$p_1 = p_2 \circ \varphi$

or $p_1 \circ \varphi^{-1} = p_2$.

Proposition: If exists universal cover of a space is unique up to isomorphism.

Indeed the above proposition is a consequence of a more general result:

Proposition: Let $(\tilde{X}_1, \tilde{x}_1)$ and $(\tilde{X}_2, \tilde{x}_2)$ be two connected coverings for a space (X, x_0) so that

$p_{1*}(\pi_1(\tilde{X}_1, x_0)) = H = p_{2*}(\pi_1(\tilde{X}_2, \tilde{x}_2))$. Then the two covering are isomorphic.

Proof:

$$\begin{tikzcd}
\tilde{X}_1 \arrow[rr, dashed, "\varphi"] \arrow[dr, "p_1"'] & & \tilde{X}_2 \arrow[dl, "p_2"] \\
& X \arrow[u, phantom, "\circlearrowright"] &
\end{tikzcd}$$

$p_i(\tilde{x}_i) = x_0$

must show: $\exists\, \varphi : X_1 \to X_2$ homeomorphism so that $p_1 = p_2 \circ \varphi$

$$\begin{tikzcd}
 & \tilde{X}_2 \arrow[dl, bend right, "\phi"'] \arrow[d, "p_2"] & \underline{\underline{\tilde{x}_2}} \\
\underline{\underline{\tilde{x}_1}} \quad \tilde{X}_1 \arrow[ur, "\varphi"] \arrow[r, "p_1"'] & X & x_0
\end{tikzcd}$$

Since $p_{1*}(\pi_1(\tilde{X}_1, \tilde{x}_1))$ $= p_2(\pi_1(\tilde{X}_2, \tilde{x}_2))$ by the lifting criterion

there is a unique lift $\varphi : \tilde{X}_1 \to \tilde{X}_2$ so that $\varphi(\tilde{x}_1) = \tilde{x}_2$. By symmetry there is a lift $\phi : \tilde{X}_2 \to \tilde{X}_1$ so that $\phi(\tilde{x}_2) = \tilde{x}_1$.

Note that $p_1 = p_2 \circ \varphi$ and $p_2 = p_1 \circ \phi$.

$$\begin{tikzcd}
\tilde{X}_1 \arrow[rr, "\phi\circ\varphi"] \arrow[dr, "p_1"'] & \circlearrowleft & \tilde{X}_1 \arrow[dl, "p_1"] \\
 & X &
\end{tikzcd}$$

commutative.

$$p_1 \circ (\phi \circ \varphi) = (p_1 \circ \phi) \circ \varphi = p_2 \circ \varphi = p_1.$$

Hence $\phi \circ \varphi$ is a lift of
$$\begin{tikzcd}
 & \tilde{X}_1 \arrow[d, "p_1"] \\
\tilde{X}_1 \arrow[ur, "\phi\circ\varphi"] \arrow[r, "p_1"'] & X
\end{tikzcd}$$
p_1 to the covering $\tilde{X}_1$. However, $\text{id} : \tilde{X}_1 \to \tilde{X}_1$ is also a lift. Finally, by the uniqueness of lifts we see that $\phi \circ \varphi = \text{id}_{\tilde{X}_1}$. Similarly, $\varphi \circ \phi = \text{id}_{\tilde{X}_2}$

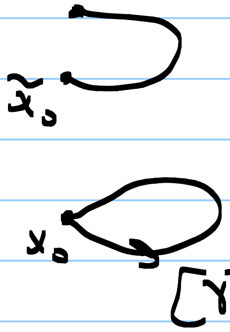
This finishes the proof. ■

Remark: Taking $H = (e)$ we see that any two simply connected covering spaces of a given space are isomorphic. Hence, a path connected, locally path connected and semilocally simply connected space has a unique universal covering, up to isomorphism.

Proposition: Suppose that X is p.c., l.p.c. and s.l.s.c. Then for every subgroup H of $\pi_1(X,x_0)$ there is a covering (unique upto isomorphism) $p: X_H \to X$ such that $p_*(\pi_1(X_H,\tilde{x}_0)) = H$, for a suitably chosen base point $\tilde{x}_0 \in X_H$.

Proof: The main observation is the following: A loop γ lifts to a loop in the covering space if and only if $[\gamma]$ belongs to $H = p_*(\pi_1(X_H, x_0))$.

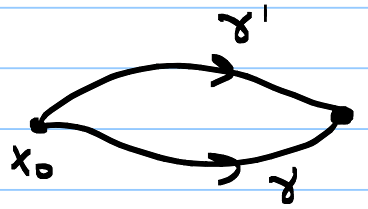
$$\begin{tikzcd} \tilde{X}, \tilde{x}_0 \arrow[d, "p"'] \\ X, x_0 \end{tikzcd}$$



We'll construct X_H as a quotient of the universal covering say $\tilde{X} \to X$.

$$\begin{tikzcd} \tilde{X} \arrow[d, "p"'] \\ X \end{tikzcd}$$

Let $[\gamma], [\gamma'] \in \tilde{X}$ so that $p([\gamma]) = \gamma(1) = \gamma'(1) = p([\gamma'])$. We let $[\gamma] \sim [\gamma']$ in $\tilde{X}$ if and only if $[\gamma \cdot \bar{\gamma}'] \in H = p_*(\pi_1(\tilde{X}, \tilde{x}_0^*)) \leq \pi_1(X, x_0)$.

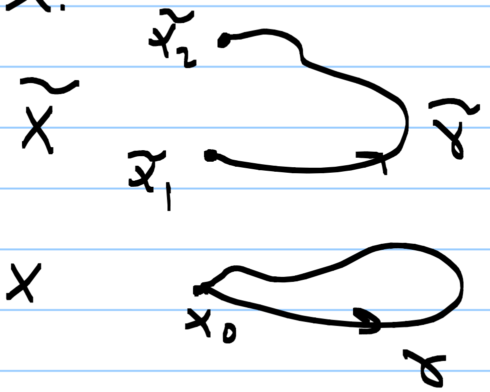


So let X_H be the quotient space $\tilde{X}/\sim$, where $\sim$ is defined as above. Note that for any basic neighborhood U in X two components $\tilde{U}_i$, $\tilde{U}_j$ of $p^{-1}(U)$ are either identified by a homeomorphism or no points of $\tilde{U}_i$ and $\tilde{U}_j$ are identified.

Note that X_H is still a covering space.
By the construction $p_*(\pi_1(X_H, y_0)) = H$.
This finishes the proof. ∎

Remark: $\tilde{X} \to X$ universal covering.

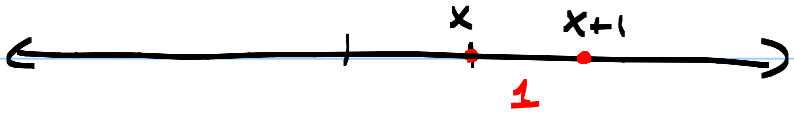
$G = \pi_1(X, x_0)$. Then G acts on $\tilde{X}$ so that $\tilde{X}/G = X$.



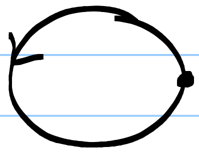
$[\gamma]\cdot\tilde{x}_1 = \tilde{x}_2$.

Since $\tilde{X}$ is connected any two points $\tilde{x}_1, \tilde{x}_2$ above any x_0 is connected by a path $\tilde{\gamma}$ so that $p \circ \tilde{\gamma} = \gamma$ is a loop at x_0 and thus $\tilde{x}_2 = [\gamma]\cdot\tilde{x}_1$

Example: $p: \mathbb{R} \longrightarrow S^1$, $p(t) = (\cos 2\pi t, \sin 2\pi t)$



$S^1 = \mathbb{R}/\mathbb{Z}$
$x \sim x+1$



$\pi_1(S^1) \simeq \mathbb{Z} = G$
$1 \in \mathbb{Z}$

$1 \cdot x = x+1$. Similarly, $n \in \pi_1(S^1)$, $n \cdot x = x+n$.

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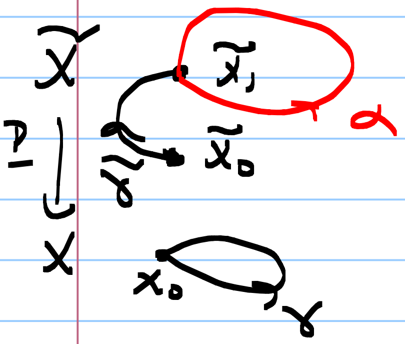
Theorem: Let X be a path connected, locally path connected and semilocally simply connected space. Then there is a bijection between the set of base point preserving isomorphism classes of path connected covering spaces

$$p:(\tilde X,\tilde x_0)\longrightarrow(X,x_0)$$

and the set of subgroups of $\pi_1(X,x_0)$ obtained by associating the subgroup $p_*(\pi_1(\tilde X,\tilde x_0))$ to the covering space $(\tilde X,\tilde x_0)$.

If base points are ignored, the correspondence gives a bijection between isomorphism classes of path connected covering spaces $p:\tilde X\to X$ and conjugacy classes of subgroups of $\pi_1(X,x_0)$.

Proof: First part already proved. So we just need to prove the second statement:



Namely, we need to show that changing the base point within $p^{-1}(x_0)$ corresponds exactly to changing the $p_*(\pi_1(\tilde X,\tilde x_0))$ to a conjugate subgroup.

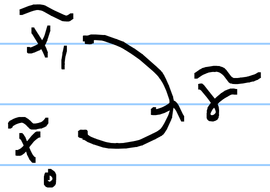
$$H=p_*(\pi_1(\tilde X,\tilde x_0)),\quad H_1=p_*(\pi_1(\tilde X,\tilde x_1))$$

$$\begin{tikzcd}
\pi_1(\tilde X,\tilde x_0)=\{ \arrow[d, "p_*"'] & {[\tilde\gamma.\alpha.\bar{\tilde\gamma}]} \arrow[d, mapsto] & \mid [\alpha]\in\pi_1(\tilde X,\tilde x_1)\} \arrow[d, equal] \\
H & {[\gamma].p_*([\alpha])[\gamma]^{-1}\in[\gamma]H_1[\gamma]^{-1}} & H_1
\end{tikzcd}$$

$$\Rightarrow H=[\gamma]H_1[\gamma]^{-1}$$

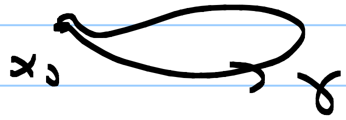
Conversely, if H and H_1 are conjugate sub-groups of $\pi_1(X,x_0)$ then

$$H = [\gamma] H_1 [\gamma]^{-1}, \text{ for some } [\gamma] \in \pi_1(X,x_1).$$



$(\tilde{X},\tilde{x}_0)$, $(\tilde{X},\tilde{x}_1)$

<u>**Exercise:**</u> Find an isomorphism $\varphi:(\tilde{X},\tilde{x}_0) \to (\tilde{X},\tilde{x}_1)$ so that $p = p \circ \varphi$.



$$\begin{tikzcd}
(\tilde{X},\tilde{x}_0) \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & (\tilde{X},\tilde{x}_1) \arrow[dl, "p"] \\
& (X,x_0) &
\end{tikzcd}$$

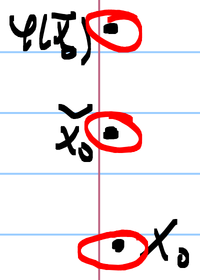
<u>**Remark:**</u> Taking $H=(e) < \pi_1(X,x_0)$ which is conjugate to itself. Then the simply connected covering we constructed before is unique up to isomorphism and thus we may call it the universal covering space.

Deck transformations and Group Actions:

If $p: \tilde{X} \to X$ is a covering space the isomorphisms of $p: \tilde{X} \to X$ are called deck transformations of the covering space.

$$\mathrm{Deck}(p: \tilde{X} \to X) = \{\varphi: \tilde{X} \to \tilde{X} \mid p = p \circ \varphi\}$$

$$\begin{tikzcd}
\tilde{X} \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & \tilde{X} \arrow[dl, "p"] \\
& X &
\end{tikzcd}
\qquad
\begin{tikzcd}
\tilde{x}_0 \arrow[dr] & & \varphi(\tilde{x}_0) \arrow[dl] \\
& x_0 &
\end{tikzcd}$$



Clearly, $\mathrm{Deck}(p: \tilde{X} \to X)$ is a group.

Claim: $|\mathrm{Deck}(p: \tilde{X} \to X)| \leq |p^{-1}(x_0)|$

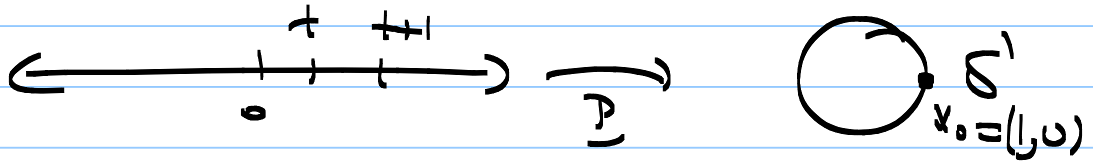
Proof: Note that any deck transformation φ is a lift of $\mathrm{Id}: X \to X$:

$$\begin{tikzcd}
\tilde{X} \arrow[r, "\varphi"] \arrow[d, "p"'] & \tilde{X} \arrow[d, "p"] \\
X \arrow[r] & X
\end{tikzcd}$$

x_0

Indeed, φ is a lift of p and we know that any lift is uniquely determined by its image at a single point. This finishes the proof. ∎

Ex: $p: \mathbb{R} \to S^1$, $p(t) = (\cos 2\pi t, \sin 2\pi t)$

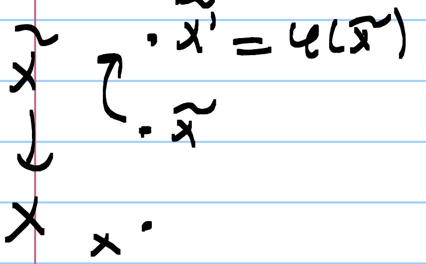


$\mathrm{Deck}(p: \mathbb{R} \to S^1) \simeq \mathbb{Z} = p^{-1}(x_0)$

$\varphi_n \longleftrightarrow n$

$\varphi_n : \mathbb{R} \to \mathbb{R}, \quad \varphi_n(t) = t+n$

<u>Definition:</u> A covering space $p: \tilde{X} \to X$ is called normal (regular) if for each $x \in X$ and each pair of points $\tilde{x}, \tilde{x}'$ over x there is a deck transformation taking $\tilde{x}$ to $\tilde{x}'$.



<u>Proposition:</u> Let $p: (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a path connected covering space of a path connected, locally path connected space X, and let H be the subgroup $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \leq \pi_1(X, x_0)$. Then

a) This covering is normal if and only if H is a normal subgroup of $\pi_1(X, x_0)$.

b) The group of Deck transformations

$\mathrm{Deck}(p: \tilde{X} \to X)) \doteq G(\tilde{X})$ is isomorphic to the quotient $N(H)/H$, where $N(H)$ is the normalizer of H in $\pi_1(X, x_0)$.

In particular, $G(\tilde{X})$ is isomorphic to $\pi_1(X, x_0)/H$ if the covering is normal.

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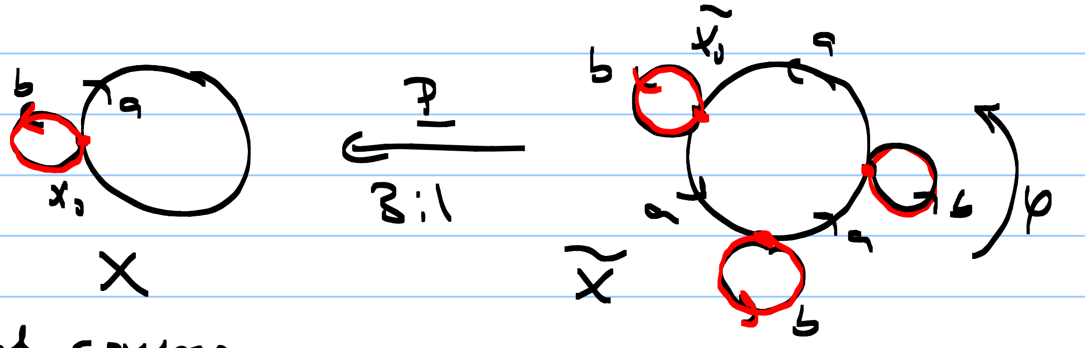
If further, $(\tilde{X},\tilde{x}_0)$ is the universal cover then $G(X) \simeq \pi_1(X,x_0)/_{H=(e)} \simeq \pi_1(X,x_0)$.

Corollary Assume the above setup. Then the cover $\tilde{X} \to X$ is normal (regular) if and only if for every $[\gamma] \in \pi_1(X,x_0)$ we have either all lifts of γ are loops or all lifts of γ are non loops.

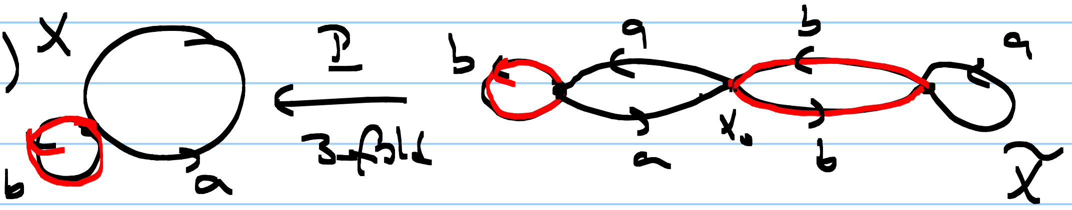
Examples: 1) Normal 3-fold cover.



Normal covering

$$\mathrm{Deck}(\tilde{X},X) = G(\tilde{X}) \simeq \mathbb{Z}/_3 = \langle \varphi \rangle$$

$\varphi : \tilde{X} \to \tilde{X}$, $2\pi/3$ radian rotation

2)


Non normal 3-fold covering

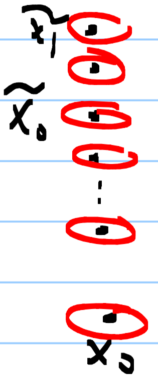
$G(\tilde{X}) = (e)$

Definition: A normal covering $p:\tilde X\to X$ is called a Galois covering.

Group actions and Coverings:

$\tilde X \xrightarrow{p} X$ regular (Galois) cover.

$G(\tilde X) = \mathrm{Deck}(p:\tilde X\to X)$ is a group with cardinality $|p^{-1}(x_0)|$.



Note that in this case the quotient space

$$\tilde X/_{G(\tilde X)} = \tilde X/_{\tilde x\sim g(\tilde x),\, g\in G(\tilde X)}$$

is clearly homeomorphic to X.

This is because each preimage $p^{-1}(x)$ is a single G orbit and thus the projection map $p:\tilde X\to X$, which is a local homeomorphism becomes a 1-1 and onto local homeomorphism which is nothing but a true homeomorphism.

$$\begin{tikzcd}
\tilde X \arrow[r, "p"] \arrow[d, "\pi"'] & X \\
\tilde X/G \arrow[ur, "f"'] &
\end{tikzcd}$$

local homo. + onto

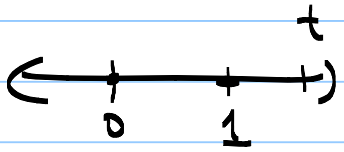
f 1-1, onto, local homo. $\Rightarrow f$ homo.

$\tilde X/G = X/_{\tilde x\sim\tilde x'}$ iff $p(\tilde x)=p(\tilde x')$

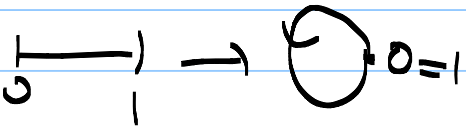
Example: 1) $p: \mathbb{R} \longrightarrow S^1$, $p(t) = (\cos 2\pi t, \sin 2\pi t)$

$G \simeq \mathbb{Z} = \langle \varphi_n \rangle$ $\varphi_n: \mathbb{R} \to \mathbb{R}$, $t \mapsto t+n$

$\mathbb{R}/G \simeq \mathbb{R}/_{t \sim t+1}$

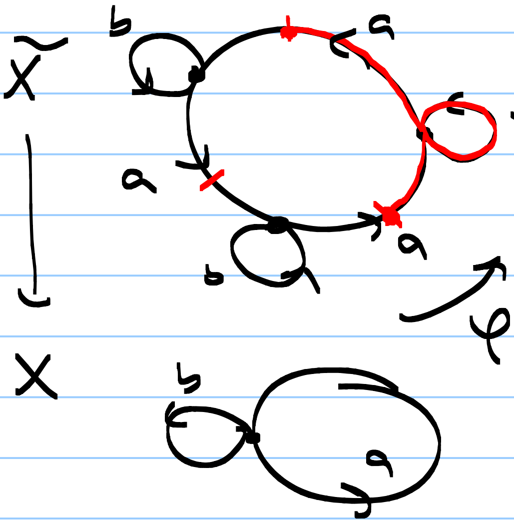


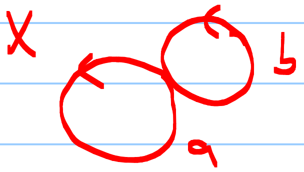
$\simeq [0,1]/_{0\sim 1}$



$\simeq S^1$

2) $\tilde{X}$

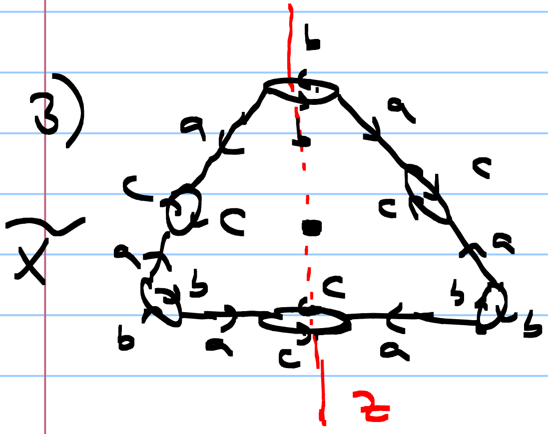


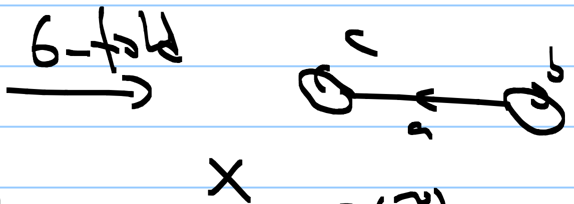


$G = \langle \varphi \rangle \simeq \mathbb{Z}_3$

φ: $2\pi/3$-radian rotation

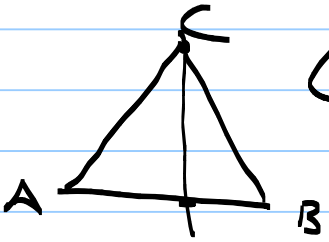
3)





$G(\tilde{X}) = \langle \varphi, \psi \rangle$
$= S_3$

φ: $2\pi/3$-radian rotation
ψ: π-radian rotation about z-axis.



$\langle \varphi, \psi \rangle \simeq S_3$: symmetries of an equilateral triangle.

$S_3 = \{e, (AB), (AC), (BC), (ABC),$
$(ACB)\}$

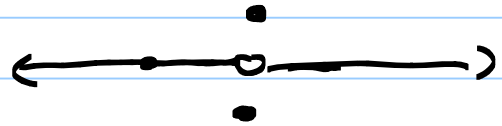
$\tilde{X}/S_3 \simeq X$ This is a regular S_3-covering

For a proof of the above proposition and following Corollary see pages 121–126 of my Math 537 Lecture Notes. Also see Videos 35 and 36 for Math 537.

Example of a non-Hausdorff Covering Space

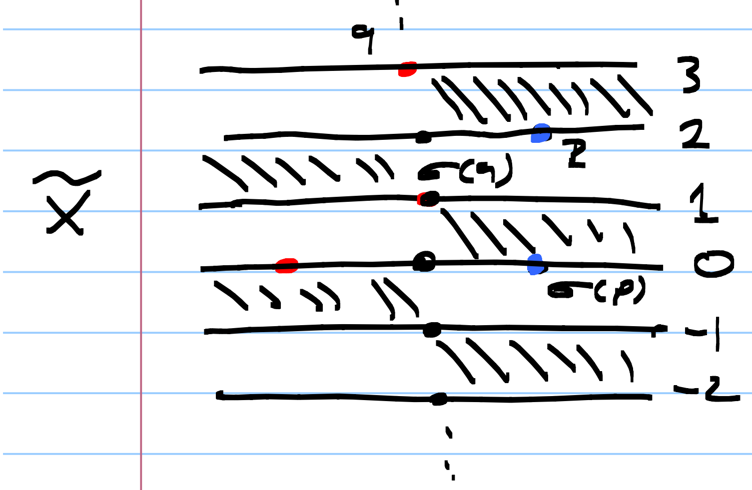
X = real line with double origin

$X = \mathbb{R} \times \{\pm 1\} / (x,-1) \sim (x,1),\ x \neq 0.$



$\pi_1(X) \simeq \mathbb{Z}$. To show this we construct the universal cover $\tilde{X}$ of X.

$\tilde{X} = \mathbb{R} \times \mathbb{Z} / \sim$ $\quad (x,n) \sim (x,n+1)$
if and only if
$(x > 0$ and n is even$)$ or $(x < 0$ and n is odd$)$



$\mathrm{Deck}(\tilde{X} \to X) \simeq \mathbb{Z}$
$\qquad\qquad\qquad\qquad\ \ \ \| $
$\qquad\qquad\qquad\quad\ \ \langle \sigma \rangle$

$\sigma(x,n) = (x,n+2)$

$\tilde{X} / \underset{\mathbb{Z}}{\underset{\wr\wr}{\langle \sigma \rangle}} = X$

Video 27

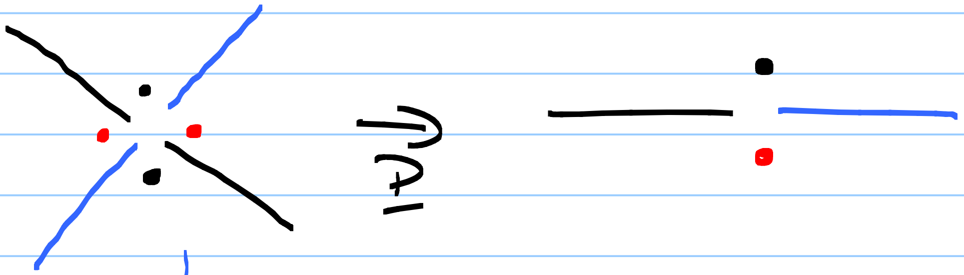
$\tilde{X}$ is the universal cover: $\pi_1(\tilde{X}) = (e)$

$\pi_1(\tilde{X}) \simeq \mathrm{Deck}(\tilde{X} \to X) = \langle \sigma \rangle \simeq \mathbb{Z}$

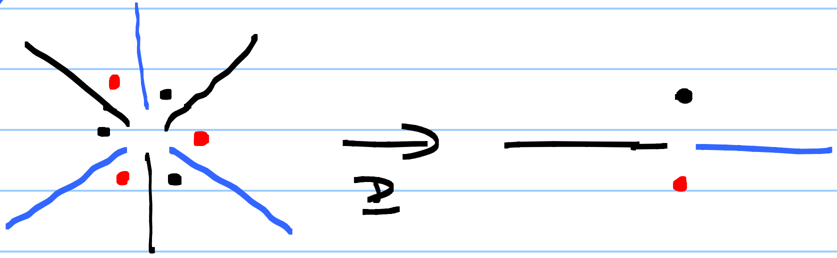
$H < \pi_1(\tilde{X})$ proper $H = n\mathbb{Z} \leq \mathbb{Z}$

$$\begin{tikzcd} X_H = ? \arrow[d] \\ X \end{tikzcd}$$

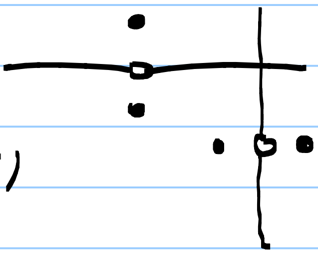
n=2



n=3



Example $X_1 = X \vee X$



$$\pi_1(X_1) \simeq \pi_1(X) * \pi_1(X)$$
$$\simeq \mathbb{Z} * \mathbb{Z}$$
$$\simeq F_2$$

(Problem 2, in Exercise Sheet 3)

Differential Equations, Galois Theory and Covering Spaces

$x^2y'' + (x+1)y' + 3y = 0 \quad x\in\mathbb{R}, \quad y=y(x)$

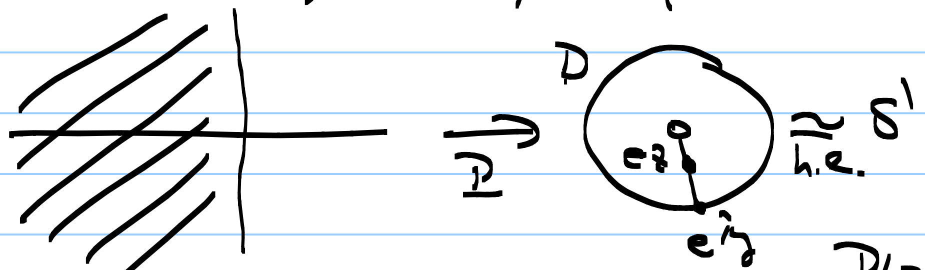
$y'' + \frac{x+1}{x^2}y' + \frac{3}{x^2}y = 0, \quad x\neq 0,$

$x=0$, regular singular point for the equation.

$e^t = x \longrightarrow y = y(x) = y(e^t) \qquad x = e^t \neq 0$

$y' + \frac{z+1}{z^2}y' + \frac{3}{z^2}y = 0, \quad y = y(z), \quad z = e^t$

$P: \tilde{D} \longrightarrow D, \quad D = \{z\in\mathbb{C} \mid 0<|z|<1\}$



$P(z) = e^z$
$= e^x e^{iy}$

$\tilde{D} = \{z\in\mathbb{C} \mid z = x+iy, \ x<0\}$

P is covering map, $\tilde{D}$ is contractible, $\pi_1(\tilde{D}) = (e)$.

So, $P: \tilde{D} \longrightarrow D$ is the universal cover.

Aim: Understand the nature of the solutions (if they exist) near $z=0$.
To do so we replace z with e^t. In other words, we pull back the equation on D to $\tilde{D}$.

Video 28

Theorem: Let $D \subseteq \mathbb{C}$ be a simply connected region and P, Q be analytic functions on D. Then for any $z_0 \in D$ and complex numbers $\alpha, \beta \in \mathbb{C}$ the differential equation

$$\frac{d^2w}{dz^2} + P(z)\frac{dw}{dz} + Q(z)w = 0 \quad \text{has a}$$

unique solution $w = \phi(z)$ defined on D so that $\phi(z_0) = \alpha$ and $\phi'(z_0) = \beta$.

Sketch of Proof

1) Convert the equation to a 1st order system

$$w'' + P(z)w' + Q(z)w = 0$$

$$w_1 = w \quad \Rightarrow w_1' = w' = w_2$$

$$w_2 = w' \quad \Rightarrow w_2' = w'' = -Pw' - Qw$$
$$= -Qw_1 - Pw_2$$

$$W = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \quad W' = \begin{bmatrix} w_2 \\ -Qw_1 - Pw_2 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1 \\ -Q & -P \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$\Rightarrow W' = AW, \text{ where } A = \begin{bmatrix} 0 & 1 \\ -Q & -P \end{bmatrix}.$$

$$W' - AW = 0$$

2) Convert the differential equation to an integral equation:

$$y = y(t)$$

$$y'(t) = f(t,y), \quad y(t_0) = y_0$$

$$y(t) - y(t_0) = \int_{t_0}^{t} y'(s)\,ds$$

$$= \int_{t_0}^{t} f(s, y(s))\,ds$$

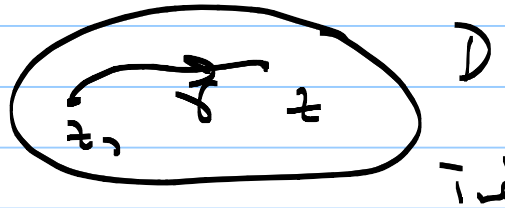
$$\boxed{\Rightarrow y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\,ds}$$

In our case, $W'(z) = A(z)W(z)$, $W(z_0) = \begin{bmatrix}\alpha\\ \beta\end{bmatrix}$

$$W(z) - W(z_0) = \int_{z_0}^{z} W'(\xi)\,d\xi$$

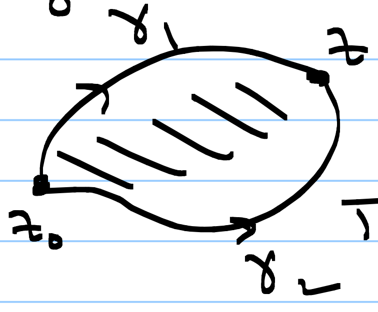
$$= \int_{z_0}^{z} \begin{bmatrix} w_1'(\xi) \\ w_2'(\xi) \end{bmatrix} d\xi$$

Here we take the integral along any path γ from z_0 to z.



Here we assume that $w_i(z)$ is an analytic function and thus the integral

$\int_\gamma w_i'(\xi)\,d\xi$ is path independent.



$$\int_{\gamma_1 - \gamma_2} w_i'(\xi)\,d\xi = 0 \quad \text{(Cauchy Theorem)}$$

The integral of an analytic function along a closed loop is zero.

<u>Remark</u>: Indeed, Morera's Theorem tells that if a function has trivial integral along any closed path then it is analytic.

$$w(z) = \begin{bmatrix}\alpha\\ \beta\end{bmatrix} + \int_{z_0}^{z} A(z)\, w(z)$$

3) Solve the Integral Equation using Picard Iteration:

Let $w_0(z) = \begin{bmatrix}0\\ 0\end{bmatrix}$ the constant function.

Then set $w_1(z) = \begin{bmatrix}\alpha\\ \beta\end{bmatrix} + \int_{z_0}^{z} A(\xi)\, w_0(\xi)\, d\xi$

$\Rightarrow w_1(z) = \begin{bmatrix}\alpha\\ \beta\end{bmatrix}$

Let $w_2(z) = \begin{bmatrix}\alpha\\ \beta\end{bmatrix} + \int_{z_0}^{z} A(\xi)\, w_1(\xi)\, d\xi$

Similarly, let $w_n(z) = \begin{bmatrix}\alpha\\ \beta\end{bmatrix} + \int_{z_0}^{z} A(\xi)\, w_{n-1}(\xi)\, d\xi$

4) The sequence $w_n(z)$ converges in a neighborhood of z_0. Then we extend this solution to whole of D by 'analytic continuation'.

Continuous Functions on Covering Spaces

(14th Week in Kugel's Book)

$D \subseteq \mathbb{C}$ a region, $C^0(D)$ the ring of continuous functions on D.

$f, g \in C^0(D)$, $(f \pm g)(p) = f(p) \pm g(p)$

$$(f \cdot g)(p) = f(p) \cdot g(p)$$

$$(f/g)(p) = f(p)/g(p), \text{ provided that } g(p) \neq 0 \ \forall p \in D.$$

Now let's take a covering space of topological spaces

$p : X \to Y$, $C_0(X)$, $C_0(Y)$ the rings of continuous function of X and Y, respectively.

Note that there is a natural map

$$p^* : C_0(Y) \to C_0(X), \ f \longmapsto f \circ p$$

$$\begin{tikzcd} X \arrow[r, "p"'] & Y \arrow[r, "f"'] & \mathbb{R}/\mathbb{C} \end{tikzcd}$$

Proposition p^* is an injective ring homomorphism.

Proof $$\underbrace{p^*(f+g)}(x) = (f+g)(p(x))$$
$$= f(p(x)) + g(p(x))$$
$$= (p^* f)(x) + (p^* g)(x)$$
$$= \underbrace{(p^* f + p^* g)}(x), \ \forall x \in X$$

Hence, $\Phi^*(f+g) = \Phi^* f + \Phi^* g$.

Similarly, $\Phi^*(f\cdot g) = \Phi^*(f)\ \Phi^*(g)$, so that Φ^* is a ring homomorphism.

For injectivity, let $\Phi^*(f) = 0$. Hence,

$0 = \Phi^*(f)(x) = f(\Phi(x))$ for all $x \in X$.

Recall that $\Phi: X \to Y$ is an onto map. Hence, $f(y) = 0$, for all $y \in Y$. Thus, f is the zero function, $f = 0$, so that Φ^* is injective. ∎

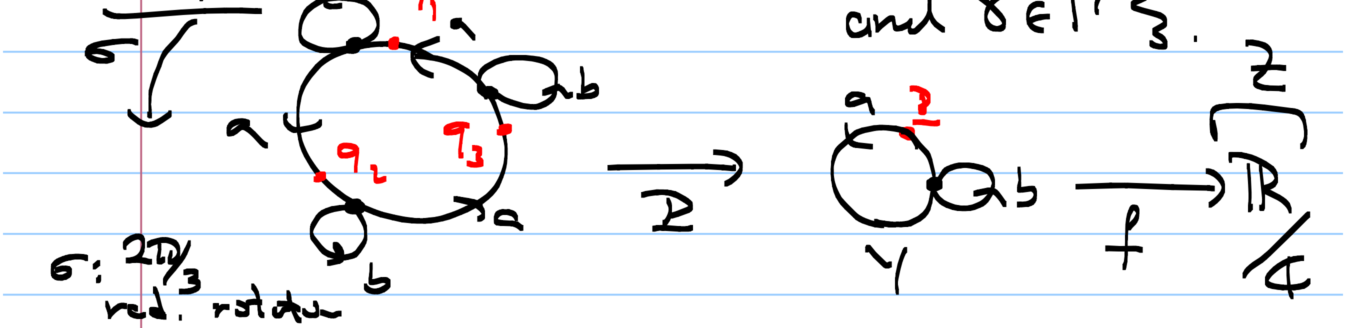
$$X \xrightarrow{\Phi} Y \implies C_0(Y) \xrightarrow{\Phi^*} C_0(X)$$

Proposition: Assume that $\Phi: X \to Y$ is a (normal) Galois covering with deck transformation group $\Gamma = \mathrm{Deck}(X \xrightarrow{\Phi} Y)$. Then the image of the ring homomorphism $\Phi^*(C_0(Y))$ is isomorphic to the subring of Γ-invariant functions on X:

$$\Phi^*(C_0(Y)) = C_0(X)^{\Gamma}$$

$$:= \{ g \in C_0(X) \mid g(\gamma(x)) = g(x), \forall x \in X \text{ and } \gamma \in \Gamma \}.$$

Example:



Video 29

$g = (p \circ f)(q_1) = f(p)$, $\sigma(q_1) = q_2$
$\sigma^2(q_1) = q_3$

$\Gamma = \langle \sigma \rangle = \{ id, \sigma, \sigma^2 \} \cong \mathbb{Z}/_3$

$g \circ \sigma = g$, $g \circ \sigma^2 = g$

$p^* f = g$ is Γ-invariant.

Proof: $p : X \longrightarrow Y = X/_\Gamma = X/_\sim$

$x_1 \sim x_2 \iff x_2 = \gamma(x_1)$, for some $\gamma \in \Gamma$.

$Y = \{ [x] \mid x \in X \}$, $p^{-1}([x]) = \{ \gamma(x) \mid \gamma \in \Gamma \}$.

$$\begin{tikzcd} x \arrow[d, mapsto] & X \arrow[r, "f"] \arrow[d, "p"'] & \mathbb{R}/\mathbb{C}, Z \\ {[x]} & Y = X/_\Gamma \arrow[ur, "g"'] & \end{tikzcd}$$

$[x] = \{ \gamma(x) \mid \gamma \in \Gamma \}$

must prove: $p^* C^0(Y) = C^0(X)^\Gamma$

$\subseteq$: $g : Y \to Z$, $g(p(x)) = f(x)$

$g(\overbrace{p(\gamma(x))}) = f(\gamma(x)) \Rightarrow g(p(x)) = f(\gamma(x))$

$\Rightarrow f(x) = f(\gamma(x))$, for all $x \in X, \gamma \in \Gamma$.

$$\begin{tikzcd} X \arrow[rr, "\gamma"] \arrow[dr, "p"'] & & X \arrow[dl, "p"] \\ & Y & \end{tikzcd}$$

$p = p \circ \gamma$

$\Rightarrow f \in C^0(X)^\Gamma = \{ h \in C^0(X) \mid f \circ \gamma = f \ \forall \gamma \in \Gamma \}$

$p^* C^0(Y) \subseteq C^0(X)^\Gamma$.

$\supseteq$: Now let $f \in C^0(X)^\Gamma$. Then $(f \circ \gamma)(x) = f(x)$ for all $x \in X$.

$\Rightarrow f(\gamma(x)) = f(x)$, $\gamma \in \Gamma$.

$\Rightarrow f$ is constant on the Γ-orbits.

$$\begin{tikzcd} x \arrow[d, mapsto] & X \arrow[r, "f"] \arrow[d, "\exists"'] & \mathbb{R}, \mathbb{C}, \mathbb{Z} \\ {[x]} & X/\Gamma \arrow[ur, "\exists g"'] & \end{tikzcd}$$

s.t. $f(x) = g(P(x))$

$[x] = \{\gamma(x) \mid \gamma \in \Gamma\}$

So $C^0(X)^\Gamma \subseteq P^*(C^0(Y))$

$\Longrightarrow C^0(X)^\Gamma = P^*(C^0(Y))$

∎

Remark: Since $P^*: C^0(Y) \to C^0(X)$ is injective we often identify $C^0(Y)$ with its image $P^*(C^0(Y))$. Hence, we may regard $C^0(Y)$ as a subring of $C^0(X)$.

Example: $S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1\}$.

$\mathbb{RP}^2 = S^2/\mathbb{Z}_2 = S^2/(x,y,z) \sim (-x,-y,-z)$

$[x:y:z] = \{(x,y,z), (-x,-y,-z)\}$

$\Gamma = \mathbb{Z}_2 = \langle \sigma \rangle$, $\sigma: S^2 \to S^2$, $\sigma(x,y,z) = (-x,-y,-z)$.

$$\begin{tikzcd} (x,y,z) \arrow[d, mapsto] & S^2 \arrow[r, "f"] \arrow[d] & \mathbb{R} \\ {[x:y:z]} & \mathbb{RP}^2 = S^2/\sim \arrow[ur, "g"'] & \end{tikzcd}$$

$f(x,y,z) = g([x,y,z])$

$f(-x,-y,-z) = g([x,y,z])$

$\Rightarrow f(x,y,z) = f(-x,-y,-z)$, $\forall (x,y,z) \in S^2$.

Which (polynomial) functions f satisfy

$$f(-x,-y,-z) = f(x,y,z), \forall (x,y,z) \in S^2?$$

$x^2, y^2, z^2, xy, xz, yz$

$R(S^2) = \mathbb{R}[x,y,z]/(x^2+y^2+z^2-1)$ the ring of polynomial functions on S^2.

$$R(\mathbb{R}P^2) = \mathbb{R}[x^2, y^2, z^2, xy, xz, yz] / (x^2+y^2+z^2-1)$$

$$R(\mathbb{R}P^2) = R(S^2)^{\Pi} \subseteq R(S^2)$$

Complex Functions on $\mathbb{C}$ and Surfaces

$D \subseteq \mathbb{C}$ domain, $f : D \to \mathbb{C}$ analytic (holomorphic) if f has continuous derivative:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

f analytic, for any $z_0 \in D$, f has a power series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad a_n = \frac{f^{(n)}(z_0)}{n!}$$

$f(z) = \frac{1}{z}$ is <u>not</u> analytic at $z = 0$.

A function $f : D_0 \to \mathbb{C}$ is called meromorphic on D if $f : D_0 \to \mathbb{C}$ is analytic and $D \setminus D_0$ is a finite set.

Example $f(z) = \frac{1}{z} + \frac{1}{z-1} + \frac{5}{z+2}$, $f(z)$ is analytic on $\mathbb{C}$ excepts on the points $\{0, 1, -2\}$. Hence, f is a meromorphic function on $\mathbb{C}$.

Definition The set of meromorphic functions on a domain D is denoted as $K(D)$.

Similarly, the set of holomorphic functions on D is denoted as O(D).

Remark 1) O(D) is a clearly a ring.

2) K(D) is a field, because any analytic function has at most finitely many zeros.

$f \in K(D)$, $f \in O(D_0)$, $D_0 = D \setminus \{z_1, \dots, z_k\}$

$1/f \in O(D_1)$, $D_1 = D_0 \setminus \{w_1, \dots, w_l\}$, $f(w_i) = 0$

Ex. $f(z) = \dfrac{z(z^2+2)}{(z-1)(z+3)} \in K(\mathbb{C})$

$$1/f(z) = \frac{(z-1)(z+3)}{z(z^2+2)} \in K(\mathbb{C})$$

Ex $f_1(z) = e^z \in O(\mathbb{C})$ $\quad f_1(z) = \displaystyle\sum_{n=0}^{\infty} \frac{z^n}{n!}$

$f_2(z) = e^{1/z} \in K(\mathbb{C})$ $\quad f_2(z) = \displaystyle\sum_{n=0}^{\infty} \frac{1}{\underline{\underline{z^n}}\, n!}$

A meromorphic function that can be written as the ratio of two holomorphic functions is said to have poles as singularities.

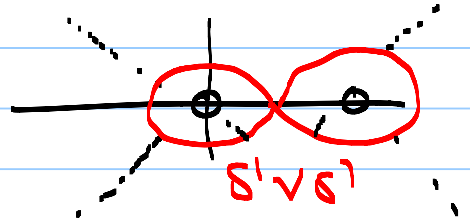
Ex $f(z) = \dfrac{z^2(z+2)}{(z-1)e^z}$

On the other hand, $e^{1/z}$ cannot be written as the ratio of two holomorphic functions, because $e^{1/z}$ has an essential singularity at zero.

Video 30

What about holomorphic/meromorphic functions on covering spaces?

$D \subseteq \mathbb{C}$ domain, $D = \mathbb{C} \setminus \{0,1\}$



$D \underset{h.e.}{\simeq} S^1 \vee S^1$

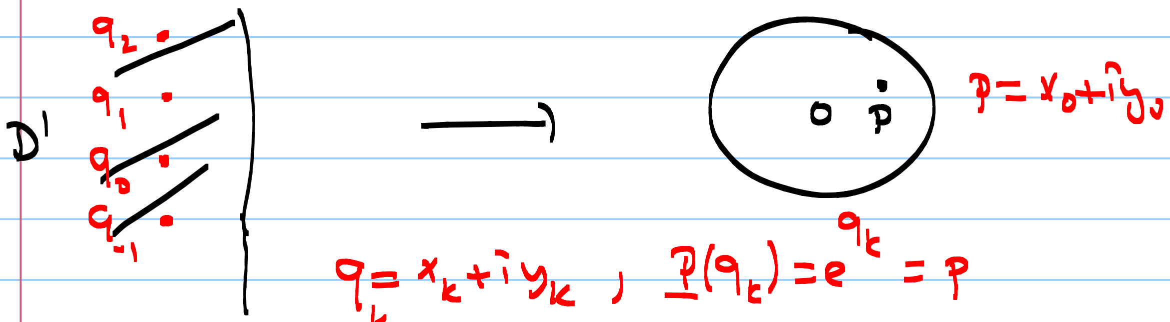
$\pi_1(D) = F_2$

Let $D' \xrightarrow{p} D$ be a covering space.

Remark: Recall that $p : D' \longrightarrow D$, when

$D' = \{z = x+iy \mid x<0\}$, $D = \{z \in \mathbb{C} \mid 0<|z|<1\}$.

$p : D' \longrightarrow D$, $p(z) = e^z$



$q_k = x_k + iy_k$, $p(q_k) = e^{q_k} = p$

$q_0 = x_0 + iy_0$, $q_1 = x_0 + i(y_0 + 2\pi)$, $q_2 = x_0 + i(y_0 + 4\pi)$

$q_{-1} = x_0 + i(y_0 - 2\pi)$ $\boxed{e^{2\pi i} = 1}$

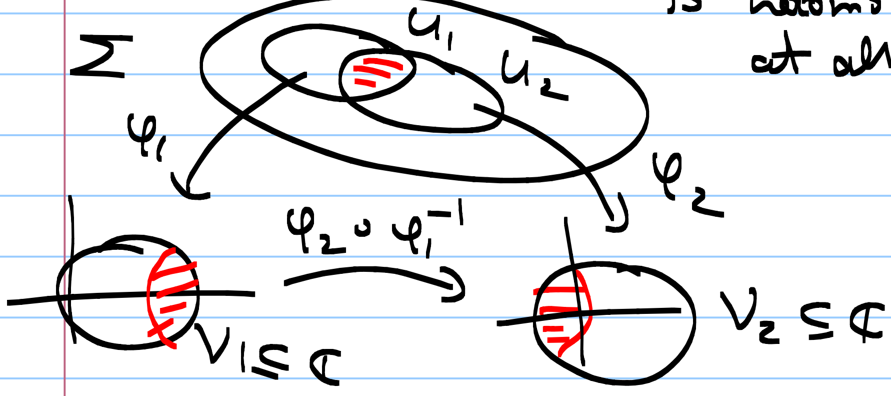
Function Theory on Covering Spaces (15th Week)

Complex Surfaces (Riemann Surface)

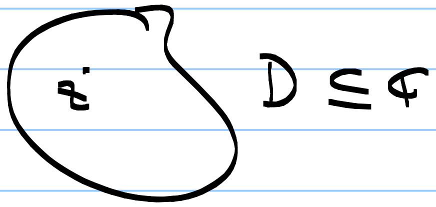
A complex surface Σ is a topological space so that for every point $p \in \Sigma$ there is a open subset $p \in U \subseteq \Sigma$ and a homeomorphism $\varphi : U \to V$, where $V \subseteq \mathbb{C}$ is an open subset satisfying the following condition:
Whenever we have two such coordinate patches $\varphi_1 : U_1 \to V_1$ and $\varphi_2 : U_2 \to V_2$ the composition

$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$$

is holomorphic (i.e. cmplx diff at all points of $\varphi_1(U_1 \cap U_2)$).

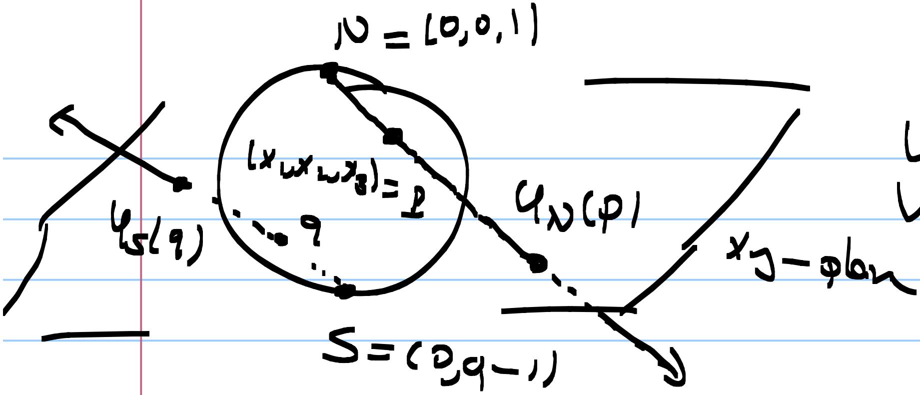


Examples: 1) Any open subset $D \subseteq \mathbb{C}$ is a complex surface (Riemann Surface).



Take $U = D = V$ and $\varphi : U \to V$ the identity map: $\varphi(z) = z$

2) Riemann Sphere: $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$.



$U_N = S^2 \setminus \{N\}$
$U_S = S^2 \setminus \{S\}$

$$\varphi_N : U_N \longrightarrow \mathbb{R}^2 = \mathbb{C}, \quad \varphi_N(x_1,x_1,x_3) = \left(\frac{x_1}{1-x_3}, \frac{x_2}{1-x_3}\right)$$
$$= \frac{x_1}{1-x_3} + i\,\frac{x_2}{1-x_3}$$

$$\varphi_S : U_S \longrightarrow \mathbb{R}^2 = \mathbb{C}, \quad \varphi_S(x_1,x_2,x_3) = \left(\frac{x_1}{1+x_3}, \frac{x_2}{1+x_3}\right)$$
$$= \frac{x_1}{1+x_3} - i\,\frac{x_2}{1+x_3}$$

$U_N = S^2 \setminus \{N\}, \quad U_S = S^2 \setminus \{S\}, \quad U_N \cup U_S = S^2$

$$\varphi_N^{-1} : \mathbb{R}^2 = \mathbb{C} \longrightarrow S^2, \quad \varphi_N^{-1}(y_1+iy_2) = \left(\frac{2y_1}{1+\|y\|^2}, \frac{2y_2}{1+\|y\|^2}, \frac{\|y\|^2-1}{1+\|y\|^2}\right)$$

$$\varphi_S^{-1} : \mathbb{R}^2 = \mathbb{C} \longrightarrow S^2, \quad \varphi_S^{-1}(y_1+iy_2) = \left(\frac{2y}{1+\|y\|^2}, \frac{-2y_2}{1+\|y\|^2}, \frac{1-\|y\|^2}{1+\|y\|^2}\right),$$

when $\|y\|^2 = y_1^2 + y_2^2$.

$$(\varphi_S \circ \varphi_N^{-1})(y_1+iy_2) = \varphi_S\Big(\underset{\substack{\shortparallel \\ x_1}}{\frac{2y_1}{1+\|y\|^2}}, \underset{\substack{\shortparallel \\ x_2}}{\frac{2y_2}{1+\|y\|^2}}, \underset{\substack{\shortparallel \\ x_3}}{\frac{\|y\|^2-1}{1+\|y\|^2}}\Big)$$

$$= \frac{x_1}{1+x_3} - i\,\frac{x_2}{1+x_3}$$

$$= \frac{2y_1/(1+\|y\|^2)}{2\|y\|^2/(1+\|y\|^2)} - i\,\frac{2y_2/(1+\|y\|^2)}{2\|y\|^2/(1+\|y\|^2)}$$

$$= \frac{y_1 - i y_2}{\|y\|^2}$$

$z = y_1 + i y_2$, $\|z\|^2 = \|y\|^2 = y_1^2 + y_2^2$.

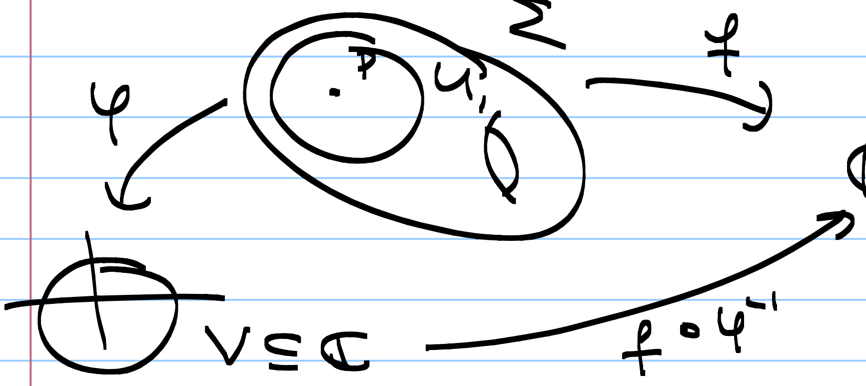
Thus, $(\varphi_S \circ \varphi_N^{-1})(z) = \dfrac{\bar{z}}{\|z\|^2} = \dfrac{\bar{z}}{z \cdot \bar{z}} = \dfrac{1}{z}$, which is holomorphic.

Thus, the sphere S^2 in $\mathbb{R}^3$ has a Riemann-Surface structure.

Functions on Riemann-Surfaces

Σ Riemann Surface

$f : \Sigma \to \mathbb{C}$



If $f \circ \varphi^{-1}$ is complex differentiable then we say that f is (complex) differentiable.

Remark: If $\varphi_1 : U_1 \to V_1$ and $\varphi_2 : U_2 \to V_2$ are two intersecting coordinate patches on the surface Σ then we have

$$f \circ \varphi_1^{-1} = (f \circ \varphi_2^{-1}) \circ (\varphi_2 \circ \varphi_1^{-1}).$$

Hence, $f \circ \varphi_1^{-1}$ is holomorphic if and only if

$f\circ\varphi_2^{-1}$ is holomorphic.

A function $f:\Sigma\to\mathbb{C}$ is called meromorphic if $f\circ\varphi^{-1}$ is meromorphic for any coordinate patch.

Example: $S^2=\mathbb{CP}^1$

$\mathcal{O}(\mathbb{CP}^1)$: holomorphic function on $\mathbb{CP}^1$ consists of constant functions only.

Proof: Assume that $f:S^2=\mathbb{CP}^1\to\mathbb{C}$ analytic function.

Then $f\circ\varphi_N^{-1}:\mathbb{C}\to\mathbb{C}$ is holomorphic, i.e., entire function.

So we can write $f\circ\varphi_N^{-1}$

$$f\circ\varphi_N^{-1}(z)=\sum_{n=0}^{\infty}a_nz^n,\text{ for some } a_n\in\mathbb{C}.$$

Similarly, $f\circ\varphi_S^{-1}(z)=\sum_{n=0}^{\infty}b_nz^n$, for some $b_n\in\mathbb{C}$.

However, $f\circ\varphi_N^{-1}(z)=(f\circ\varphi_S^{-1})(\varphi_S\circ\varphi_N^{-1})(z)$

$$=(f\circ\varphi_S^{-1})(1/z)$$

$$\Rightarrow\sum_{n=0}^{\infty}a_nz^n=\sum_{n=0}^{\infty}b_n\frac{1}{z^n},\text{ for all } z\neq0.$$

$\Rightarrow a_n=b_n=0$ if $n>0$ and $a_0=b_0$.

Hence, f is a constant. ∎

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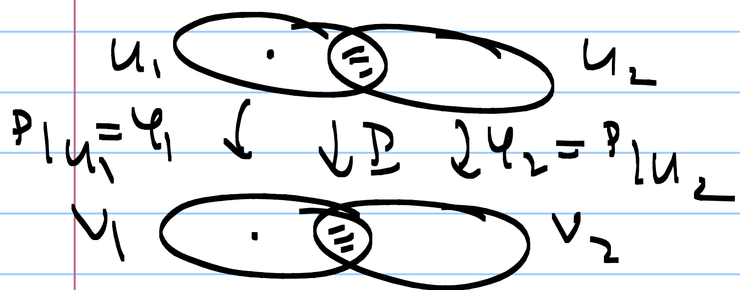
Functions on Covering Spaces: $D \subseteq \mathbb{C}$ domain (open connected)

$p: D' \to D$ covering space.

Complex Structure on D':

$$\begin{tikzcd} x \in D' \arrow[d, "p"] \\ p(x) \in D \subseteq \mathbb{C} \end{tikzcd}$$

For any $x \in D'$ there is clearly an open subset V so that $p: V \to p(V) = U$ is a homeomorphism.



Here, $\varphi_2 \circ \varphi_1^{-1} = (p|_{U_2}) \circ (p|_{U_1})^{-1}$

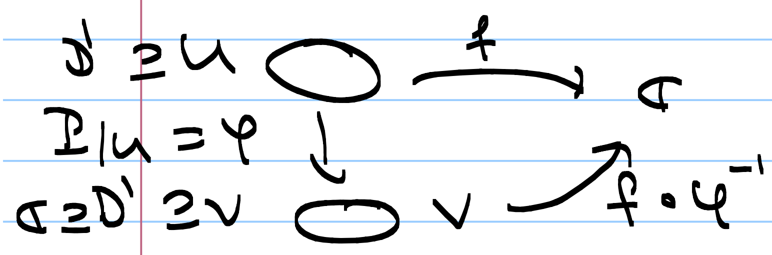
$$= (p|_{U_1 \cap U_2}) \circ (p|_{U_1 \cap U_2})^{-1} = id|_{V_1 \cap V_2}$$

$V_1 \cap V_2 \subseteq D \subseteq \mathbb{C}$ open subset

Thus, we get a Riemann Surface structure on D'.

A function $f: D' \to \mathbb{C}$ is holomorphic if for any $p \in D'$ the composition

$$f \circ \varphi^{-1}: V \to \mathbb{C}$$



Similarly, a function $f: D' \to \mathbb{C}$ is meromorphic if $f \circ \varphi^{-1}$ is meromorphic for any coordinate patch.

Let $P: D' \to D \subseteq \mathbb{C}$ Galois covering space with deck transformation group Γ.

Recall that $C^0(D')^{\Gamma} = P^*(C^0(D))$

$P^*(C^0(D)) = \{ P \circ f \mid f: D \to \mathbb{C}$ continuous funct$\}$

$C^0(D')^{\Gamma} = \{ f \in C^0(D') \mid f \circ \gamma = f, \forall \gamma \in \Gamma \}$.

$\gamma : D' \to D'$

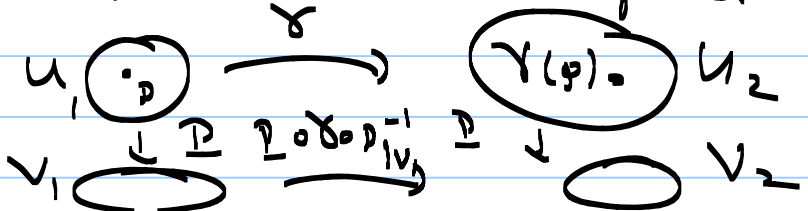
Similar result holds for holomorphic and meromorphic functions:

__Proposition:__ Let $P: D' \to D$ be a Galois covering with Galois group Γ. If $\mathcal{O}(D)$, $\mathcal{O}(D')$, $K(D)$ and $K(D')$ denote the holomorphic and meromorphic functions on D and D', respectively, then

$$\mathcal{O}(D')^{\Gamma} = P^*(\mathcal{O}(D)) \text{ and } K(D')^{\Gamma} = P^*(K(D))$$

__Proof:__ Note that elements of the Galois group $\gamma \in \Gamma$ acts holomorphically on D':

$\gamma : D' \to D'$ holomorphic.

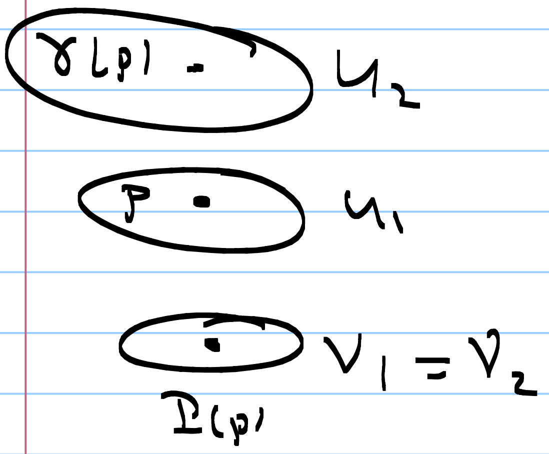


$$\begin{tikzcd} D' \arrow[rr, "\gamma"] \arrow[dr, "p"'] & & D' \arrow[dl, "p"] \\ & D & \end{tikzcd}$$

$p \circ \gamma = p$

Hence, $p \circ \gamma \circ p^{-1}|_{V_1} = p \circ p^{-1}|_{V_1} = \mathrm{id}_{V_1}$

which is clearly analytic. Hence, γ is holomorphic.



Exercise: Finish the proofs.

Remark: Recall that $p^*: C^0(D) \to C^0(D')$ was injective and thus we may identify $p^*(C^0(D))$ with its image $p^*(C^0(D))$ in $C^0(D')$:

$$C^0(D) = C^0(D')^{\Gamma}.$$

Similarly, we regard $\mathcal{O}(D)$ and $K(D)$ as subring/subfield of $\mathcal{O}(D')$ and $K(D')$, respectively, and write

$$\mathcal{O}(D) = \mathcal{O}(D')^{\Gamma} \text{ and } K(D) = K(D')^{\Gamma}.$$

Differential Equations (16th Week)

Recall that we've seen the following theorem:

Theorem: Let $D \subseteq \mathbb{C}$ be a simply connected domain, P, Q holomorphic functions on D. A homogeneous linear differential equation

$$(\#) \quad \frac{d^2w}{dz^2} + P(z)\frac{dw}{dz} + Q(z)w = 0$$ with initial

conditions at $z = z_0 \in D$, given by

$(*)$ $w(z_0) = \alpha$, $\frac{dw}{dz}(z_0) = \beta$ has a unique holomorphic solution w on D satisfying both $(\#)$ and $(*)$.

Remark: Assume that $w_1, \dots, w_m$ are analytic solutions of $(\#)$. Then for any $\lambda_1, \dots, \lambda_m \in \mathbb{C}$ the function

$\lambda_1 w_1 + \lambda_2 w_2 + \dots + \lambda_m w_m$ is still a solution of $(\#)$. In particular, the set of solutions of $(\#)$ is a vector space $V_\#$ contained in $\mathcal{O}(D)$.

Proposition: The map $\Phi : V_\# \longrightarrow \mathbb{C}^2$ given by

$\Phi(w) = (w(z_0), w'(z_0))$, is a linear isomorphism.

Proof: Φ is linear: If $w_1, w_2 \in V_\#$ and

$c_1, c_2 \in \mathbb{C}$, then $c_1 w_1 + c_2 w_2 \in V_\#$ and

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$$\Phi(c_1w_1+c_2w_2) = \left((c_1w_1+c_2w_2)(z_0), (c_1w_1+c_2w_2)'(z_0)\right)$$
$$= c_1(w_1(z_0), w_1'(z_0))$$
$$+ c_2(w_2(z_0), w_2'(z_0))$$
$$= c_1\Phi(w_1) + c_2\Phi(w_2).$$

Φ is onto because for any $\alpha, \beta \in \mathbb{C}$ there is a solution $w \in V_\#$ so that
$w(z_0) = \alpha$ and $w'(z_0) = \beta$.
i.e., $\Phi(w) = (\alpha, \beta)$.

Φ is 1-1 because the solution w satisfying $w(z_0) = \alpha$, $w'(z_0) = \beta$, is unique.

Let $\pi: \tilde{D} \to D$ be the universal covering space, where $D \subseteq \mathbb{C}$ a connected domain. Now we can state the above theorem for $\tilde{D}$, since $\tilde{D}$ is simply connected.

Theorem: Let $\tilde{D} \xrightarrow{\pi} D \subseteq \mathbb{C}$ be as above and P and Q be holomorphic functions on $\tilde{D}$. Let $\tilde{p}_0 \in \tilde{D}$ and $\alpha, \beta \in \mathbb{C}$ arbitrary complex numbers. Then there is a unique holomorphic function $w(\tilde{p}, \tilde{p}_0, \alpha, \beta)$ that satisfies both

$$(\tilde{\#}) \quad \frac{d^2w}{dz^2}(\tilde{p}) + P(\tilde{p})\frac{dw}{dz}(\tilde{p}) + Q(\tilde{p})\, w(\tilde{p}) = 0$$

and the initial conditions

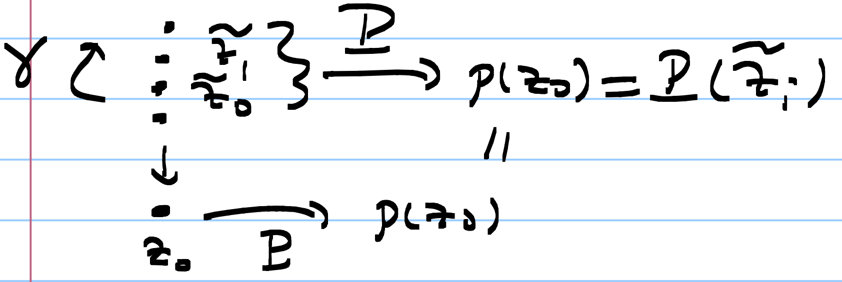
$$(\tilde{*}) \quad w(\tilde{p}_0) = \alpha \text{ and } \frac{dw}{dz}(\tilde{p}_0) = \beta.$$

Moreover, the vector space $\tilde{V}_\#$ of all solutions of $(\#)$ is a two dimensional $\mathbb{C}$ vector space.

Now assume that the functions P and Q are in $\pi^*(\mathcal{O}(D)) \subseteq \mathcal{O}(\tilde{D})^{\Gamma}$, where Γ is the deck transformation group of the universal cover $\pi : \tilde{D} \to D$.

$$\begin{tikzcd} \tilde{D} \arrow[d, "\pi"'] \\ D \end{tikzcd} \qquad P, Q \qquad \begin{array}{l} P = \pi^*(p) = p \circ \pi \\ Q = \pi^*(q) = q \circ \pi \end{array}$$

$p, q : D \longrightarrow \mathbb{C}$



$\gamma \in \Gamma$ Deck trans

If $w \in \tilde{V}_{\#}$ then we have

$$w''(\tilde{z}) + P(\tilde{z})\, w'(\tilde{z}) + Q(\tilde{z})\, w(\tilde{z}) = 0$$

applying γ^* to the above equation we get

$$\gamma^*(w''(\tilde{z})) + \gamma^*(P(\tilde{z})\, w'(\tilde{z})) + \gamma^*(Q(\tilde{z})\, w(\tilde{z})) = 0$$

$$w''(\gamma(\tilde{z})) + P(\underset{\underset{P(\tilde{z})}{\|}}{\gamma(\tilde{z})})\, w'(\gamma(\tilde{z})) + Q(\underset{\underset{Q(\tilde{z})}{\|}}{\gamma(\tilde{z})})\, w(\gamma(\tilde{z})) \underset{}{=0}$$

Thus $\gamma^* w'(\tilde{z}) = w''(\gamma(\tilde{z}))$ is again a soln of $(\#)$.

Theorem: $\left.\begin{array}{l} \omega \in \widetilde{V}_{\#} \\ \gamma \in \Gamma \end{array}\right\} \Rightarrow \gamma^* \omega \in \widetilde{V}_{\#}.$

Moreover, $\gamma^* : \widetilde{V}_{\#} \longrightarrow \widetilde{V}_{\#}$ is a linear transformation.

Proof If $\omega_1, \omega_2 \in \widetilde{V}_{\#}$ and $\lambda_1, \lambda_2 \in \mathbb{C}$ then

$\lambda_1 \omega_1 + \lambda_2 \omega_2 \in \widetilde{V}_{\#}$ and $\gamma^*(\lambda_1 \omega_1 + \lambda_2 \omega_2)$ is also a solution and

$$\begin{aligned} \gamma^*(\lambda_1 \omega_1 + \lambda_2 \omega_2) &= (\lambda_1 \omega_1 + \lambda_2 \omega_2)(\gamma) \\ &= \lambda_1 \omega_1(\gamma) + \lambda_2 \omega_2(\gamma) \\ &= \lambda_1 \gamma^* \omega_1 + \lambda_2 \gamma^* \omega_2. \end{aligned}$$

Notation: For any $\gamma \in \Gamma$ let $M(\gamma)$ denote the linear map $(\gamma^{-1})^*$. Note that in this case

$$\begin{aligned} M(\gamma_1 \gamma_2)(\omega) &= ((\gamma_1 \gamma_2)^{-1})^* \omega \\ &= (\gamma_2^{-1} \gamma_1^{-1})^* \omega \\ &= \omega(\gamma_2^{-1} \gamma_1^{-1}) \\ &= (\gamma_1^{-1})^* \omega(\gamma_2^{-1}) \\ &= (\gamma_1^{-1})^* ((\gamma_2^{-1})^* \omega) \\ &= M(\gamma_1)(M(\gamma_2) \omega) \\ &= M(\gamma_1) M(\gamma_2)(\omega) \end{aligned}$$

$$\Rightarrow M(\gamma_1 \gamma_2) = M(\gamma_1) M(\gamma_2)$$

So we have a group homomorphism

$M : \Gamma \longrightarrow GL(\widetilde{V}_{\#}), \ \gamma \longmapsto M(\gamma).$

Recall that $\tilde{V}_{\#} \simeq \mathbb{C}^2$ as a $\mathbb{C}$-vector space and thus $GL(\tilde{V}_{\#}) = GL(2,\mathbb{C})$, the group of invertible 2×2-complex matrices, once we choose a basis $[w_1, w_2]$ for $\tilde{V}_{\#}$.

The map $\mathcal{M} : \Pi \longrightarrow GL(\tilde{V}_{\#}) = GL(2,\mathbb{C})$ is called the monodromy representation of the $(\#)$.

<u>Definition</u>: Let $\mathcal{M}$ be the monodromy representation of the differential equation $(\#)$. If we can find a matrix representation M corresponding to $\mathcal{M}$ of the form

$$\gamma \longmapsto \mathcal{M}(\gamma) \longmapsto M(\gamma) = \begin{pmatrix} a(\gamma) & b(\gamma) \\ 0 & c(\gamma) \end{pmatrix}$$

by choosing an appropiate basis $[w_1, w_2]$, then we call $\mathcal{M}$ a triangulable representation.

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The Seventeenth Week: Elementary methods of solving Differential Equations

Let's consider a set Σ of "known functions". Constant functions are regarded as known. The following operations produce new functions from old ones:

i) The four arithmetic operations:

$$F_1, F_2 \longrightarrow F_1+F_2,\ F_1-F_2,\ F_1\cdot F_2,\ F_1/F_2$$

Linear combinations $F_1, F_2 \longrightarrow \lambda_1 F_1 + \lambda_2 F_2$

ii) Differentiation: $F \longrightarrow \frac{dF}{dz}$

iii) Integration: $F \longrightarrow \int F(z)\,dz$

iv) Exponentiation: $F=F(z) \longrightarrow e^{F(z)}$

Process of type L_0: Starting with functions $F_1, F_2, \dots, F_n \in \Sigma$ apply procedures (i), (ii), (iii) or (iv) finitely many times to produce a new function.

Example: $F_1, F_2, F_3 \in \Sigma$, then

$$\frac{(F_1+F_2)\int F_2\,dz}{e^{\int F_3\,dz}}$$ is a process of type L_0.

We'll say that a process is of type L if it is a finite sequence of operations of types (i)–(iv) and (v), where (v) is defined as follows:

v) Solving algebraic equations:

$$F \longmapsto \sqrt[n]{F}, \text{ or more generally,}$$

$F_1, \dots, F_n \in \Sigma \longmapsto$ A root of Ψ of the equation

$$\Psi^n + F_1 \Psi^{n-1} + F_2 \Psi^{n-2} + \dots + F_n = 0.$$

A function obtained from Σ by a process of type L will be called a function of type L on Σ. The set of all such functions will be denoted by $L(\Sigma)$. We have obvious inclusion $L_0(\Sigma) \subseteq L(\Sigma)$.

Example: If $F_1, F_2, F_3 \in \Sigma$ then

$$\sqrt[7]{e^{-\int F_1 dz} \int F_2 e^{\int F_1 dz} + \sqrt[5]{F_3} + \sqrt{\frac{dF_2}{dz}}} \in L(\Sigma).$$

Here L is used in honor of Liouville.

Definition: Suppose that the coefficient functions $P(z)$ and $Q(z)$ of the equation

$$\frac{d^2w}{dz^2} + P(z)\frac{dw}{dz} + Q(z)w = 0$$

belong to Σ, the set of known functions. If all solutions of this differential equation are of type L_0 on Σ, we say that the differential equation is of type L_0 on Σ. If all the solutions are of type L on Σ, the differential equation is of type L on Σ.

Remark: $\Sigma \longrightarrow L_0(\Sigma) \longrightarrow L(\Sigma)$

Solutions of algebraic equations with coefficients in $L(\Sigma)$ are also solutions of some other equations with coefficients in Σ. This is a consequence of basic theory of field extension.

Preparation Theorem 17.1 Suppose that one non-trivial solution of the differential equation

$$\frac{d^2w}{dz^2} + P(z)\frac{dw}{dz} + Q(z)w = 0$$

is of type L_0 on Σ. Then all the solutions of this equation are of type L_0 on Σ. Similar statement also holds for L.

Proof: Let w_1 be a non-trivial solution of the equation and w_1 is of type L_0 on Σ. Let w be another solution of the equation. must show: w is of type L_0 on Σ.

Write $w = w_1 u$ for some u. Then
$w(z) = w_1(z)u(z)$.
$w' = w_1'u + w_1u'$ and
$w'' = w_1''u + 2w_1'u' + w_1u''$.
Since w is a solution we have

$$\begin{aligned} 0 &= w'' + Pw' + Qw. \\ &= (\underline{w_1''u} + 2w_1'u' + w_1u'') \\ &\quad + P(\underline{w_1'u} + w_1u') \\ &\quad + Q\underline{w_1u} \end{aligned}$$

$$\Rightarrow \quad 0 = \underbrace{(w_1'' + Pw_1' + Qw_1)}_{0=} + 2w_1'u' + w_1u'' + Pw_1u'$$

So, we have $0 = w_1 u'' + (2w_1' + P w_1) u'$

$u = ?$

Let $v \doteq u'$. Then $v' = u''$ and thus the equation becomes

$$0 = w_1 v' + (2w_1' + P w_1) v$$

$$w_1 \frac{dv}{dz} = -(2w_1' + P w_1) v$$

$$\Rightarrow \frac{dv}{v} = -\frac{(2w_1' + P w_1)}{w_1} dz$$

$$\ln|v| = -\int \left(\frac{2w_1'}{w_1} + P\right) dz$$

$$v = C_1 e^{-\int \left(\frac{2w_1'}{w_1} + P\right) dz}$$

$$= C_1 e^{-2\ln|w_1| - \int P dz}$$

$$= C_1 w_1^{-2} \cdot e^{-\int P(z) dz}$$

$$u' = v = c_1 w_1^{-2} e^{-\int P(z) dz}$$

$$u = c_1 \int w_1^{-2} e^{-\int P(z) dz} + c_2$$

So, $w = u w_1 = c_1 w_1 \int w_1^{-2} e^{-\int p(z) dz} + c_2 w_1$

and thus w is type L_0 on Σ. ∎

Let $D \subseteq \mathbb{C}$ be a domain (connected open subset) and set $\Sigma = K(D)$ the set of all meromorphic functions on D. In particular, Σ contains all holomorphic

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functions on D. Hence, all single-valued holomorphic functions on D are considered as "known" functions.

If $\tilde{D} \to D$ is the universal cover then the functions on $\tilde{D}$ are considered as "unknown" functions.

Let $P, Q \in \mathcal{O}(D)$ and consider the equation

$$(\#) = (\tilde{\#}) : \frac{d^2 w}{dz^2} + P(z)\frac{dw}{dw} + Q(z)\, w = 0$$

$$\begin{tikzcd} \tilde{D} \arrow[d, "\pi"'] \\ D \end{tikzcd} \qquad \begin{array}{l} \pi^*(\underline{\underline{\mathcal{O}(D)}}) \subseteq \mathcal{O}(\tilde{D}) \\ P, Q \in \mathcal{O}(D) \Rightarrow \pi^* P = P \circ \pi,\ \pi^* Q = Q \circ \pi \in \mathcal{O}(\tilde{D}) \end{array}$$

Theorem: The equation $(\#) = (\tilde{\#})$ is of type L_0 on $\Sigma = K(D)$ if and only if the monodromy representation $\mathcal{M}$ of $(\tilde{\#})$ is triangulable.

Proof: The "only if" part is omitted here and left as an exercise.
For the "if" part assume that $\mathcal{M}$ is triangulable. We'll show that $(\tilde{\#})$ is of type L_0 on Σ.

Since $\mathcal{M}$ is triangulable there is a basis $\{w_1, w_2\}$ of $V_{\tilde{\#}}$ s.t. that for any $\gamma \in \Gamma$

$$\begin{aligned} ((\gamma^{-1})^* w_1, (\gamma^{-1})^* w_2) &= (\mathcal{M}(\gamma) w_1, \mathcal{M}(\gamma) w_2) \\ &= (w_1, w_2) \begin{pmatrix} a(\gamma) & b(\gamma) \\ 0 & d(\gamma) \end{pmatrix} \end{aligned}$$

(1) So, $(\gamma^{-1})^* \omega_1 = \omega_1 \circ \gamma^{-1} = a(\gamma)\, \omega_1$ and

(2) $(\gamma^{-1})^* \omega_2 = \omega_2 \circ \gamma^{-1} = b(\gamma)\, \omega_1 + d(\gamma)\, \omega_2$

Take derivative of (1) to get

(3) $\frac{d\omega_1}{dz} a(\gamma) = \frac{d((\gamma^{-1})^* \omega)}{dz} = (\gamma^{-1})^* \frac{d\omega_1}{dz}$

Take the quotient of (3) by (1) to get

$$\frac{\frac{d\omega_1}{dz} a(\gamma)}{\omega_1\, a(\gamma)} = \frac{(\gamma^{-1})^* \frac{d\omega_1}{dz}}{(\gamma^{-1})^* \omega_1}$$

(4) $\Rightarrow \frac{d\omega_1/dz}{\omega_1} = (\gamma^{-1})^* \left(\frac{d\omega_1/dz}{\omega_1} \right)$

Let $A(z) = \frac{d\omega_1/dz}{\omega_1}$.

<u>want to show:</u> The equation (#) is of type Lo.

We must show that all solutions are of type Lo. Thus $A \in K(\tilde{D})$. So by (4) we have

$(\gamma^{-1})^* (A) = A$ for any element $\gamma \in \Gamma$.

Thus A is in $K(\tilde{D})^{\Gamma} = K(D)$. Therefore A is a known function

Since $A = \frac{d\omega_1/dz}{\omega_1}$ we have $\int A\, dz + C = \ln \omega_1$

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$\Rightarrow \omega_1 = e^{\int A dx + C}$. Since $A \in K(D)$ is a known function $\omega_1 \in L_0$. Hence $V_{\tilde{\Pi}}$ contains a function of type L_0 on Σ. Now by the Preparation theorem all solutions (all elements of $V_{\tilde{\Pi}}$) of type L_0 on Σ.

∎

The Eighteenth Week: Regular Singularities

Recall from the differential equation course that an equation of type
$y''+P(x)y'+Q(x)y=0$ is said have regular singularity at some $x_0\in\mathbb{R}$ if

$\lim\limits_{x\to x_0}(x-x_0)P(x)=\alpha$ and $\lim\limits_{x\to x_0}(x-x_0)^2Q(x)=\beta$ both exist (assuming x_0 is a singular point for P or Q)

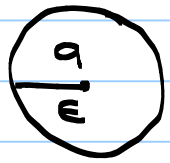
Ex $y''+y'+\frac{1}{(x-2)^2}y=0$

$x_0=2$ is a singular point for $Q(x)=1/(x-2)^2$.
This is a regular singular point since
$\lim\limits_{x\to 2}(x-2)P(x)=\lim\limits_{x\to 2}(x-2)\cdot 1=0=\alpha$ and

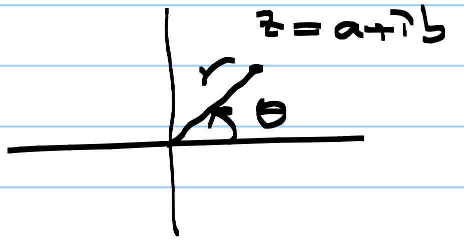
$\lim\limits_{x\to 2}(x-2)^2Q(x)=\lim\limits_{x\to 2}(x-2)^2\frac{1}{(x-2)^2}=1=\beta$.

Universal cover of punctured disc:

$U\doteq U(a,\epsilon)=\{z\in\mathbb{C}\mid |z-a|<\epsilon\}$



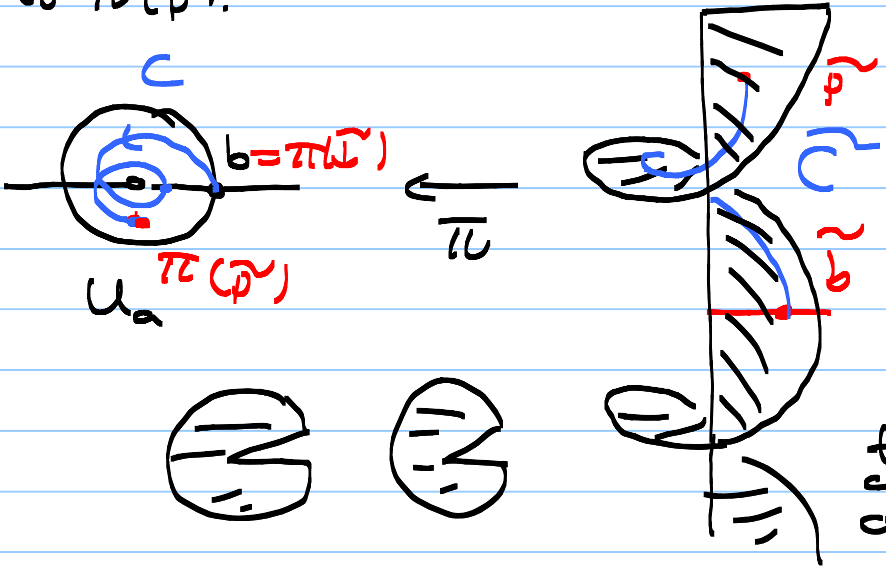
$U_a\doteq U\setminus\{a\}=\{z\in\mathbb{C}\mid 0<|z-a|<\epsilon\}$.
(punctured disc)



$r=|z|=\sqrt{a^2+b^2}$
$\tan\theta=b/a$

$\theta=\arg(z)$

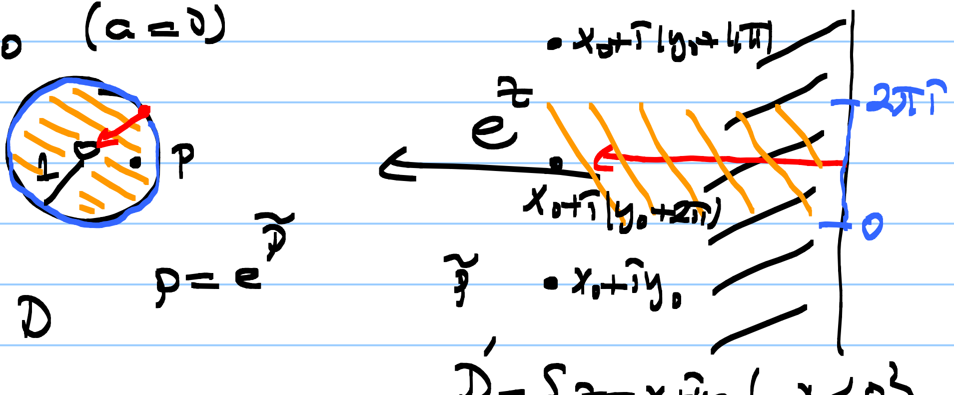
How to define $\arg(p)$ on the universal cover? Let $\tilde{C}$ be a curve in $\tilde{U}_a$ connecting b to $\tilde{p}$, where b is a fixed point in U_a with $\operatorname{Arg}(b)=0$. Let C denote the image of $\tilde{C}$ in U_a, starting at b and ending at $\pi(\tilde{p})$. Then $\arg(\tilde{p})$ is defined to be the total angle about the center a swept out by a point which moves along the curve C from b to $\pi(\tilde{p})$.



Remark. Note that the curve $\tilde{C}$ and thus C is unique up to homotopy since the universal cover is simply connected.

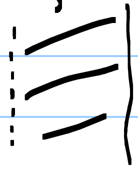
Note that $\arg(\tilde{p})$ is single valued even though $\arg(p)$ on U_a is multi valued.

Another model for the universal cover of U_0 $(a=0)$



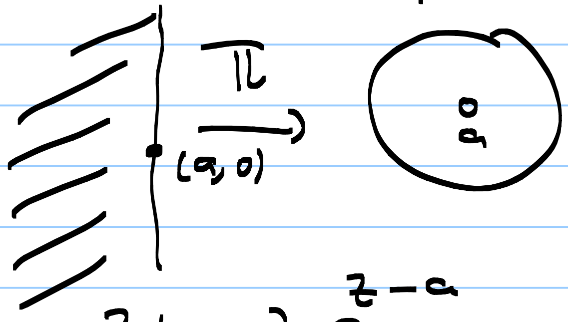
$$D'=\{z=x+iy \mid x<0\}$$

$$e^z=e^{x+iy}=e^x(\cos y+i\sin y)$$



$\operatorname{Arg}(\tilde{D})$ is a single valued function on D!

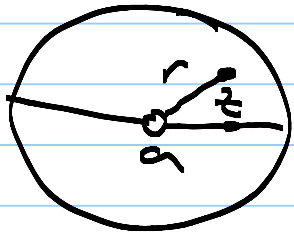
$\tilde{U}_a \longrightarrow U_a = \{z \in \mathbb{C} \mid 0 < |z-a| < 1\}$



$\tilde{U}_a = \{z \in \mathbb{C} \mid \operatorname{Re} z < a\}$

$z \longmapsto e^{z-a} + a$

Local inverse of π is given by logarithms

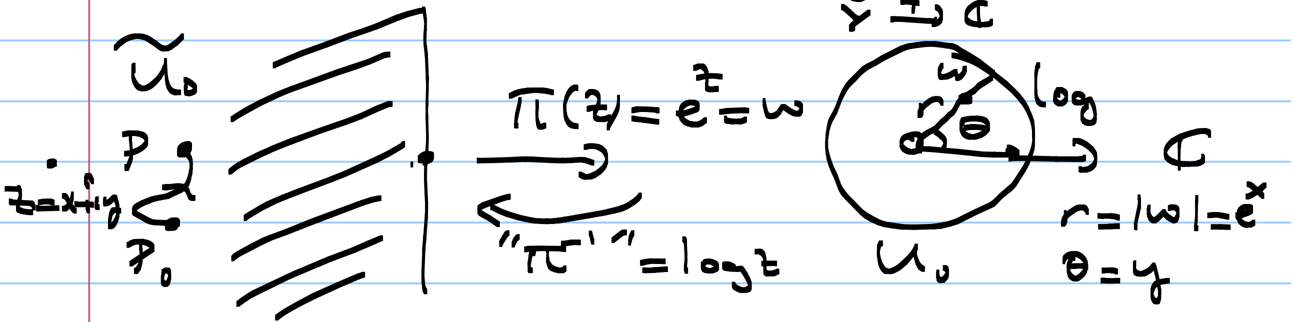


$z - a = r + i\arg(z-a)$
$\arg(z-a)$ is <u>not</u> well defined on U_a.

$\operatorname{Arg}(z-a)$ is well defined on $\tilde{U}_a$:

$\log : \tilde{U}_0 \longrightarrow \mathbb{C}$,

$$\begin{tikzcd} X \arrow[d, "Z"'] \arrow[dr, "f\circ p = p^* f"] & \\ Y \arrow[r, "f"] & \mathbb{C} \end{tikzcd}$$



$$z = x+iy \longmapsto e^z = e^x e^{iy} = e^x\cos y + i e^x \sin y = w \longmapsto \log w$$

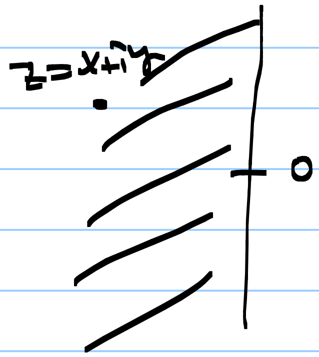
$$\log w = \ln|w| + i\operatorname{Arg}(w)$$
$$= x + iy = z$$

Hence, $\log : \tilde{U}_0 \longrightarrow \mathbb{C}$, $z \longmapsto z$.

Once the logarithm function on $\tilde{U}_a$ is well defined we may define exponential function on $\tilde{U}_a$.

Remark: Note that $z^{1/2}$ is not a single valued function on $\mathbb{C}$. However, on the universal cover $\tilde{U}_a$ they become single valued.

$$(z-a)^{\alpha} = e^{\alpha \log(z-a)}$$

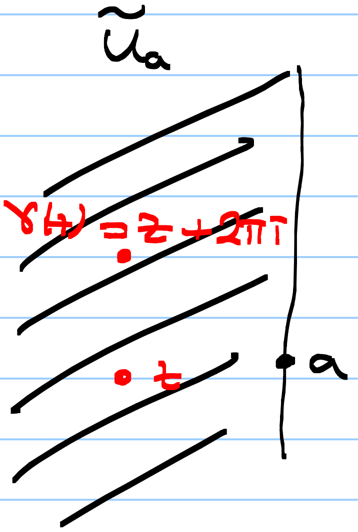


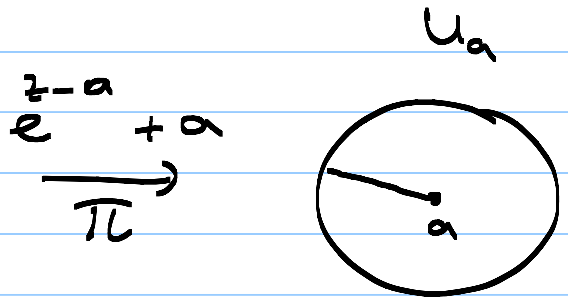
Example: $z^{1/2} = e^{1/2 \log z}$

$$z = x+iy \longmapsto \log z = z$$

So, $z^{1/2} = e^{1/2 \log z} = e^{z/2}$

$$z = x+iy \longmapsto z^{1/2} = e^{x/2 + iy/2} = e^{x/2}\left(\cos\frac{y}{2} + i \sin\frac{y}{2}\right).$$





$\gamma(z) = z+2\pi i \qquad \pi(z) = \pi(\gamma(z))$

$\Gamma = \langle \gamma \rangle$ Deck transformation group

$$\begin{aligned}\gamma^*(\log(z-a)) &= \log(\gamma(z-a)) \\ &= \log(z-a+2\pi i) \\ &= |z-a| + i(\operatorname{Arg}(z-a)+2\pi) \\ &= |z-a| + i\operatorname{Arg}(z-a) + 2\pi i \\ &= \log(z-a) + 2\pi i\end{aligned}$$

Similarly, $\gamma^*((z-a)^\alpha) = \gamma^*(e^{\alpha\log(z-a)})$

$$= e^{\alpha\gamma^*\log(z-a)}$$

$$= e^{\alpha(\log(z-a)+2\pi i)}$$

$$= e^{\alpha\log(z-a)} e^{\alpha 2\pi i}$$

$$= e^{2\pi i\alpha}(z-a)^\alpha$$

So we have obtained the following formulas:

$$\boxed{\begin{aligned} \gamma^*(\log(z-a)) &= \log(z-a)+2\pi i \\ \gamma^*((z-a)^\alpha) &= e^{2\pi i\alpha}(z-a)^\alpha \end{aligned}}$$

<u>Definition</u>: Let F be a function on $\tilde{U}_a$ given by the expansion

$$F(z) = \sum_{i=1}^{m_0} (z-a)^{\alpha_i} A_i(z-a)$$

$$+ \log(z-a)\sum_{i=1}^{m_1} (z-a)^{\beta_i} B_i(z-a)$$

$$+ (\log(z-a))^2 \sum_{i=1}^{m_2} (z-a)^{\gamma_i} C_i(z-a)$$

$$\vdots$$

$$+ (\log(z-a))^\nu \sum_{i=1}^{m_\nu} (z-a)^{\omega_i} W_i(z-a),$$

where ν, m_i are positive integers, $\alpha_i, \beta_i, \dots, \omega_i$ are complex numbers and $A_i, B_i, \dots, C_i$ are convergent power series (analytic functions near a).

In this case, we say that a is a regular singular point of F.

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<u>Remark</u>: Note that if $\alpha_i \in \mathbb{Z}$ and $B_i = C_i = \ = W_i =$
then F is convergent powers at $z=a$.

<u>Preparation Thm (18.1)</u>: If $F, G \in \mathcal{O}(\tilde{U}_a)$ have regular singular points at a, then
$F+G$, $F-G$, $\lambda F + \mu G$, $F \cdot G$ and $\frac{dF}{dz}$
also have regular singular points at a.

<u>Proof</u>: Exercise.

<u>Preparation Theorem 18.2</u>: If $F, G \in \mathcal{O}(\tilde{U}_a)$ have regular singular points at a and $F/G \in K(\tilde{U}_a)$ lies in $K(U_a)$ indeed, then F/G (as a function on U_a) has a pole at a. (That is, a is <u>not</u> an essential singularity).

To prove this we need two lemmas.

<u>Lemma 18.3.</u> Let $\alpha_1, \dots, \alpha_n$ be complex numbers such that $\alpha_i - \alpha_j$ is <u>not</u> an integer for $i \neq j$. Then, the following $(N+1)n$ sequences are linearly independent over $\mathbb{C}$:

$$\left(m^k e^{2\pi\sqrt{-1}\alpha_i m}\right)_{m=1}^{\infty} \qquad (i=1,\dots,n,\ k=0,\dots,N).$$

In other words, if the equation

$$\sum_{i=1}^{n} \sum_{k=0}^{N} c_{ik}\, m^k e^{2\pi\sqrt{-1}\alpha_i m} = 0 \qquad (m=1,2,\dots)$$

holds for some complex numbers c_{ik}, then $c_{ik}=0$ for all i,k.

__Proof__ $(m^k e^{2\pi\sqrt{-1}\alpha_j m})_{m=1}^{\infty}$ $k=0,\ldots,N$, $j=1,\ldots,n$.

$\alpha_i - \alpha_j \notin \mathbb{Z}, i \neq j, \quad e^{2\pi\sqrt{-1}\alpha_i} \neq e^{2\pi\sqrt{-1}\alpha_j}$

Let $\beta_i = e^{2\pi\sqrt{-1}\alpha_i}$ the our sequence becomes

$(m^k \beta_i^m)_{m=1}^{\infty}$ $\quad \beta_i \neq \beta_j$, if $i \neq j$.

$k=0$
$$(\beta_1, \beta_1^2, \beta_1^3, \ldots, \beta_1^m, \ldots)$$
$$(\beta_2, \beta_2^2, \beta_2^3, \ldots, \beta_2^m, \ldots)$$
$$\vdots$$
$$(\beta_n, \beta_n^2, \beta_n^3, \ldots, \beta_n^m, \ldots)$$

$k=1$
$$(\beta_1, 2\beta_1^2, 3\beta_1^3, \ldots, m\beta_1^m, \ldots)$$
$$(\beta_2, 2\beta_2^2, 3\beta_2^3, \ldots, m\beta_2^m, \ldots)$$
$$\vdots$$
$$(\beta_n, 2\beta_n^2, 3\beta_n^3, \ldots, m\beta_n^m, \ldots)$$

$k=2$
$$(\beta_1, 2^2\beta_1^2, 3^2\beta_1^3, \ldots, m^2\beta_1^m, \ldots)$$
$$(\beta_2, 2^2\beta_2^2, 3^3\beta_3^3, \ldots, m^2\beta_2^m, \ldots)$$
$$\vdots$$
$$(\beta_n, 2^2\beta_n^2, 3^2\beta_n^3, \ldots, m^2\beta_n^m, \ldots)$$

$\vdots$

$k=N$
$$(\beta_1, 2^N\beta_1^2, 3^N\beta_1^3, \ldots, m^N\beta_1^m, \ldots)$$
$$(\beta_2, 2^N\beta_2^2, 3^N\beta_2^3, \ldots, m^N\beta_2^m, \ldots)$$
$$\vdots$$
$$(\beta_n, 2^N\beta_n^2, 3^N\beta_n^3, \ldots, m^N\beta_n^m, \ldots)$$

We have $\underline{n(N+1)}$ lines.

__Ex__: $n=2, N=1$
$$\begin{vmatrix} \beta_1 & \beta_1^2 & \beta_1^3 & \beta_1^4 \\ \beta_2 & \beta_2^2 & \beta_2^3 & \beta_2^4 \\ \beta_1 & 2\beta_1^2 & 3\beta_1^3 & 4\beta_1^4 \\ \beta_2 & 2\beta_2^2 & 3\beta_2^3 & 4\beta_2^4 \end{vmatrix} = \beta_1^3\beta_2^3(\beta_2-\beta_1)^4$$

$\beta_i = e^{2\pi\sqrt{-1}\,\alpha_i} \neq 0$, $\beta_i - \beta_j \neq 0$, if $i \neq j$

Hence, the determinant is nonzero.

Exercise: Prove the general case.

Hint:

$$\underbrace{\begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-1} \\ & & \vdots & & \\ 1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^{n-1} \end{vmatrix}_{n\times n}}_{\substack{\text{hom. polynomial in } \lambda_1, \dots, \lambda_n \\ \text{of degree } 0+1+2+\cdots+n-1 = \frac{n(n-1)}{2}}} = \mp \prod_{i \neq j} (\lambda_i - \lambda_j)$$

$F(\lambda_1, \dots, \lambda_n) = 0$ if $\lambda_i = \lambda_j \Rightarrow \lambda_i - \lambda_j \mid F$ if $i \neq j$.

$\frac{n(n-1)}{2}$ $\quad \lambda_i - \lambda_j \quad \lambda_j - \lambda_i$

$\prod_{i\neq j} (\lambda_i - \lambda_j) \mid F$ and $\deg F = \deg \prod_{i \neq j} (\lambda_i - \lambda_j)$

So $F = c \prod_{i\neq j} (\lambda_i - \lambda_j)$ $(c = 1$ ✓$)$.

For simplicity let's assume that the center a of U_a is zero: $a = 0$.

Lemma 18.4. Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be complex numbers such that $\alpha_i - \alpha_j$ is not an integer for $i \neq j$. Then, the $(N+1)n$ functions

$$z^{\alpha_i} (\log z)^k \quad (i = 1, \dots, n;\ k = 0, \dots, N)$$

are linearly independent over $K(U_a)$; namely, if the equation $\sum_{i=1}^{n} \sum_{k=0}^{N} F_{i,k}(z)\, z^{\alpha_i} (\log z)^k \equiv 0$ holds,

for some functions $F_{i,k}$, then $F_{i,k} \equiv 0$.

Proof: Apply $(\gamma^m)^*$ to both sides of the above equation, for every m to get

$$\sum_{i=1}^{n} \sum_{k=0}^{N} \underline{F_{i,k}(z)}\, z^{\alpha_i}\, e^{2\pi\sqrt{-1}\,\alpha_i m} (\log z + \underline{\underline{2\pi\sqrt{-1}\, m}})^k \equiv 0.$$

Compare the coefficients of $m^h e^{2\pi\sqrt{-1}\alpha_i m}$ and use the above lemma to deduce

$$\sum_{k=h}^{N} F_{i,k}(z) (\log z)^{k-h} \binom{k}{h} \equiv 0, \text{ for all } i=1,\ldots,n.$$

Apply $(\gamma^*)^m$ again to get

$$\sum_{k=h}^{N} F_{i,k}(z) (\log z + 2\pi\sqrt{-1}\,m)^{k-h} \binom{k}{h} \equiv 0, \quad \begin{matrix} i=1,\ldots,n \\ m=1,\ldots\infty \end{matrix}$$

Similar arguments proves that $F_{ik}(z) = 0\ \forall z$ or $F_{ik} \equiv 0$, using the fact that

$$\det \begin{vmatrix} 1 & c_1 \cdots & c_1^{n-1} \\ 1 & c_2 \cdots & c_2^{n-1} \\ \vdots & & \\ 1 & c_n \cdots & c_n^{n-1} \end{vmatrix} = \prod (c_i - c_j) \neq 0 \text{ if } c_i \neq c_j.$$

This finishes the proof of the lemma. ∎

Remark: For a proof of Lemma 18.3 one may consider constant coefficients linear differential equations or recursion sequences:

Ch. Poly $\Delta = (\lambda - \lambda_1)^{r_1} \cdots (\lambda - \lambda_n)^{r_n}$

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Proof of Preparation Theorem 18.2: We may assume that F and G have the following forms:

$$F = \sum_{i=0}^{r} \sum_{k=0}^{\infty} z^{\alpha_i} (\log z)^k \underline{\underline{P_{ik}(z)}}$$

$$G = \sum_{i=0}^{r} \sum_{k=0}^{\infty} z^{\alpha_i} (\log z)^k Q_{ik}(z)$$

We may further assume that $\alpha_i - \alpha_j \notin \mathbb{Z}$ if $i \neq j$, because if $\alpha_j = \alpha_i + m$ $(m \in \mathbb{Z})$ then

$$z^{\alpha_j} = z^{\alpha_i} z^{m}$$ and we can rewrite $P_{jk}(z)$ to include z^m.

Let $f = F/G$, then $\underline{\underline{F}} = f\underline{\underline{G}}$ or $F - fG = 0$. This gives us

$$\sum_i \sum_k z^{\alpha_i} (\log z)^k (P_{ik} - fQ_{ik}) \equiv 0.$$

Since $f \in K(U_a)$, we can assume that $P_{ik} - fQ_{ik} \in K(U_a)$, by choosing a smaller U_a if necessary. Now by Lemma 18.4. $P_{ik} - fQ_{ik} = 0$ for all i, k. Since $G \not\equiv 0$ there is some $Q_{ik} \not\equiv 0$. Hence, $f = P_{ik}/Q_{ik}$, where P_{ik} and Q_{ik} are convergent power series. Hence, f has only poles as singularities. So f has meromorphic at $z=0$. ■

Now consider the differential equation below

$$(\#) \quad \frac{d^2 w}{dz^2} + P(z)\frac{dw}{dz} + Q(z)w = 0,$$ where

$P(z)$ and $Q(z)$ are holomorphic functions in U_a.

The two-dimensional vector space of solutions $V_\#$ is contained in $\mathcal{O}(\widetilde{U}_a)$. The differential equation (#) is of Fuchsian type at the point a if every solution of (#) has a regular singular point at a.

Theorem 18.5 The differential equation

$$\frac{d^2w}{dz^2} + P(z)\frac{dw}{dz} + Q(z)w = 0, \text{ where } P, Q \in \mathcal{O}(U_a)$$

is Fuchsian type at the point a if and only if $P(z)$ and $Q(z)$ have Laurent expansions at a of the form

$$P(z) = \frac{\alpha_0}{z-a} + \alpha_1 + \alpha_2(z-a) + \alpha_3(z-a)^2 + \cdots$$

$$Q(z) = \frac{\beta_0}{(z-a)^2} + \frac{\beta_1}{(z-a)} + \beta_2 + \beta_3(z-a) + \cdots$$

Remark: Note that that in this case

$$\lim_{z\to a}(z-a)P(z) = \alpha_0 \text{ and } \lim_{z\to a}(z-a)^2Q(z) = \beta_0.$$

Conversely, if these limits exist then P and Q have the forms stated in the theorem.

First we state a

Lemma 18.6. Let φ and ψ be linearly independent solutions of (#). Then we have

$$\varphi\psi' - \varphi'\psi = \begin{vmatrix} \varphi & \psi \\ \varphi' & \psi' \end{vmatrix} = Ce^{-\int P(z)dz}, \text{ where}$$

$$P(z) = -(\varphi\psi' - \varphi'\psi)'/(\varphi\psi' - \varphi'\psi).$$

Video 3P

Proof $W = \begin{vmatrix} \varphi & \psi \\ \varphi' & \psi' \end{vmatrix} = \varphi\psi' - \varphi'\psi$ and thus

$$W' = \cancel{\varphi'\psi'} + \varphi\psi'' - \varphi''\psi - \cancel{\varphi'\psi'}, \text{ where}$$
$$= \varphi\psi'' - \varphi''\psi$$

$\varphi'' = -P\varphi' - Q\varphi$ and $\psi'' = -P\psi' - Q\psi$

$$\Rightarrow W' = \varphi(-P\psi' - \cancel{Q\psi}) + (P\varphi' - \cancel{Q\varphi})\psi$$
$$= P(\psi\varphi' - \varphi\psi')$$
$$= -PW$$

Hence, $\dfrac{W'}{W} = -P \Rightarrow \dfrac{dW}{W} = -P(z)\,dz$

$\Rightarrow W = c\, e^{-\int P(z)dz}$, where $P = -W'/W$.

Since the covering transformation group $\Gamma = \Gamma(\tilde{U}_a \xrightarrow{\pi} U_a) \simeq \pi_1(U_a)$ is the infinite cyclic group generated by say γ, the monodromy representation of Γ defined by (#) is determined by the action $w \longmapsto \gamma^*(w)$:

$$\begin{tikzcd} V_{\#} \arrow[r] \arrow[d, phantom, "\text{is}"] & V_{\#} \arrow[d, phantom, "\text{is}"] \\ \mathbb{C}^2 & \mathbb{C}^2 \end{tikzcd}, \quad w \longmapsto \gamma^*(w)$$

From linear algebra we know that there is a basis $\mathcal{B} = \{w_1, w_2\}$ of $V_{\#}$ in which γ^* has the form

$$[\gamma^*]_{\mathcal{B}} = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \text{ or } [\gamma^*]_{\mathcal{B}} = \begin{pmatrix} c & 1 \\ 0 & c \end{pmatrix}.$$

First we prove the "only if" direction.

Case 1: $[\gamma^*]_\beta = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$

$$\gamma^*\omega_1 = c\,\omega_1 \quad \text{and} \quad \gamma^*\omega_2 = d\,\omega_2.$$

Choose some $\lambda \in \mathbb{C}$ s.t. $e^{2\pi i\lambda} = c$ and consider the function $\omega_1/(z-a)^\lambda$.

Then
$$\gamma^*\left(\frac{\omega_1}{(z-a)^\lambda}\right) = \frac{\gamma^*\omega_1}{\gamma^*((z-a)^\lambda)}$$
$$= \frac{\cancel{c}\,\omega_1}{(z-a)^\lambda \cdot \cancel{e^{2\pi i\lambda}}}$$
$$= \frac{\omega_1}{(z-a)^\lambda}.$$

Since $\Gamma = \langle \gamma^* \rangle$ we have $\frac{\omega_1}{(z-a)^\lambda} \in K(\tilde{U}_a)^\Gamma = K(U_a)$. Let $\frac{\omega_1}{(z-a)^\lambda} = \sum_{n=-\infty}^{\infty} c_n(z-a)^n$ be Laurent expansion. By assumption (#) is of Fuchsian type at ω and thus ω has a regular singular point at a. Now by the Preparation Theorem 18.2. this function is meromorphic. So
$$\frac{\omega_1}{(z-a)^\lambda} = \sum_{n \geq -n_0} c_n(z-a)^n.$$

Let $\lambda_1 = \lambda - n_0$ and rename c_{n-n_0} by c_n. Then we have
$$\omega_1 = (z-a)^{\lambda_1} \sum_{n=0}^{\infty} c_n(z-a)^n \quad (c_0 \neq 0).$$

Note that we still have $e^{2\pi i\lambda_1} = e^{2\pi i\lambda} = c$.

Similarly, $w_2 = (z-a)^{\lambda_2} \underbrace{\sum_{n=0}^{\infty} d_n (z-a)^n}_{R_2(t)}$ $(d_0 \neq 0)$,
when $e^{2\pi i \lambda_2} = d$.

Let's compute the Wronskian $W = \begin{vmatrix} w_1 & w_2 \\ w_1' & w_2' \end{vmatrix}$.
Let $t = z-a$ for simplicity.
Then $w_j = t^{\lambda_j} R_j(t)$.

$$W = t^{\lambda_1+\lambda_2-1}\{(\lambda_2-\lambda_1)R_1R_2 + t(R_1R_2' - R_1'R_2)\}$$

$$= t^{\nu} R(t), \text{ where } \underline{\underline{R(0) \neq 0}}.$$

By the Lemma, $P = -\frac{W'}{W} = -\frac{\nu t^{\nu-1} R + t^{\nu} R'}{t^{\nu} R}$

$\Rightarrow P = -\frac{\nu}{t} + \frac{R'}{R} = -\nu/t + \text{c.p.s.}$, because $R(0) \neq 0$ and thus $1/R$ is also a c.p.s. (convergent power series).

$\Rightarrow P = \frac{-\nu}{z-a} + \text{c.p.s.}$ and thus P is as stated in the theorem.

For Q note that we have $0 = w_1'' + P w_1' + Q w_1$, and thus

$$Q = -\frac{w_1'' + P w_1'}{w_1} = \frac{t^{\lambda_1 - 2}\,\text{c.p.s.} + \left(\frac{\nu}{t} + \text{c.p.s.}\right) t^{\lambda_1 - 1}\,\text{c.p.s.}}{t^{\lambda_1} \cdot R_1(t)}$$

$$= t^{-2}\,\text{c.p.s}$$

$$= \frac{1}{(z-a)^2}\boxed{\text{c.p.s.}} = \beta + \delta(z-a) + \underline{\xi(z-a)^2 + \cdots}$$

$$= \frac{\beta}{(z-a)^2} + \frac{\delta}{(z-a)} + \text{c.p.s.}$$

This completes the proof of "only if" part of Theorem 8.5 in Case 1.

<u>Case 2</u> : $[\gamma^*]_\beta = \begin{pmatrix} c & 1 \\ 0 & c \end{pmatrix}$

$\gamma^* w_1 = c w_1$ and $\gamma^* w_2 = w_1 + c w_2$.

By the considerations in Case 1, since $\gamma^* w_1 = c w_1$ we have

$$w_1 = (z-a)^{\lambda_1} \sum_{n=0}^{\infty} c_n (z-a)^n \quad (c_0 \neq 0),$$

$e^{2\pi i \lambda_1} = c$.

Let $w_3 = \frac{1}{2\pi i c} (\log(z-a))\, w_1$.

$$\gamma^* w_3 = \frac{1}{2\pi i c} \gamma^*(\log(z-a))\, \gamma^* w_1$$
$$= \frac{1}{2\pi i c} (\log(z-a) + 2\pi i)\, c w_1$$
$$= w_1 + c w_3.$$

Also let $w_4 = w_2 - w_3$. Then

$$\gamma^* w_4 = \gamma^* w_2 - \gamma^* w_3$$
$$= (\cancel{w_1} + c w_2) - (\cancel{w_1} + c w_3)$$
$$= c(w_2 - w_3)$$
$$= c w_4.$$

Thus by the above arguments $w_4 = (z-a)^{\lambda} R_4(z-a)$.

Hence, $w_2 = w_4 + w_3$
$$= (z-a)^{\lambda} R_4(z-a) + \frac{1}{2\pi i c}(\log(z-a))\, w_1$$

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$$\Rightarrow w_2 = (z-a)^{\lambda_2} \sum b_n (z-a)^n + (z-a)^{\lambda_1} \log(z-a) \sum d_n (z-a)^n$$

where $e^{2\pi i \lambda_2} = e^{2\pi i \lambda_1} = c$.

Now as in the Case 1, $P = -W'/W$ and $Q = -\frac{w_1'' - P w_1'}{w_1}$, where W is the Wronskian determinant, $W = \begin{vmatrix} w_1 & w_2 \\ w_1' & w_2' \end{vmatrix}$ and it follows that P and Q have the desired form as stated in the theorem.

As a consequence of the above proof we have the following:

Theorem 18.6: Let $\frac{d^2w}{dz^2} + P(z)\frac{dw}{dz} + Q(z)w = 0$ be of Fuchsian type at a, with $P, Q \in O(U_a)$.

1) If the generator of the monodromy group has the Jordan form

$$\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$$

then the solution space $V_\#$ of $(\#)$ is spanned by

$$w_1 = (z-a)^{\lambda} \sum_{n=0}^{\infty} c_n (z-a)^n \quad (c_0 \neq 0), \text{ and}$$

$$w_2 = (z-a)^{\mu} \sum_{n=0}^{\infty} d_n (z-a)^n \quad (d_0 \neq 0), \text{ where}$$

$(\gamma^* w_1, \gamma^* w_2) = (w_1 \; w_2) \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$, where λ and μ satisfy $e^{2\pi i \lambda} = c$, $e^{2\pi i \mu} = d$.

II) If the Jordan form is $\begin{pmatrix} c & 1 \\ 0 & c \end{pmatrix}$, then $V_{\#}$ is spanned by

$$w_1 = (z-a)^{\lambda} \sum_{n=0}^{\infty} c_n (z-a)^n, \quad (c_0 \neq 0), \text{ and}$$

$$w_2 = (z-a)^{\mu} \sum_{n=0}^{\infty} d_n (z-a)^n + \frac{1}{2\pi i c} \log(z-a) \cdot w_1 .$$

Moreover we have $(\gamma^* w_1, \gamma^* w_2) = (w_1\ w_2)\begin{pmatrix} c & 1 \\ 0 & c \end{pmatrix}$, and λ and μ satisfy

$e^{2\pi i \lambda} = e^{2\pi i \mu} = c$ (therefore $\lambda - \mu$ is an integer).

Now let's prove the "if" part of the theorem. Let $P(z)$ and $Q(z)$ have the Laurent expansions as stated in the theorem. In particular,

$A(z) = (z-a) P(z)$ and $B(z) = (z-a)^2 Q(z)$ are holomorphic at $z=a$. So we have

$$A(z) = \sum_{n=0}^{\infty} \alpha_n (z-a)^n \text{ and } B(z) = \sum_{n=0}^{\infty} \beta_n (z-a)^n .$$

Then the equation (#) becomes

$$(z-a)^2 w'' + (z-a) A(z) w' + B(z) w = 0.$$

Let's look for a solution of the equation of the form

$$w(z) = (z-a)^{\lambda} \sum_{n=0}^{\infty} c_n (z-a)^n, \quad c_0 \neq 0$$

for some $c_n \in \mathbb{C}$. Plug this candidate into the equation and for simplicity replace $(z-a)$ by t.

$w = t^{\lambda} \sum_{n=0}^{\infty} c_n t^n = \sum_{n=0}^{\infty} c_n t^{n+\lambda}$. Then

$w' = \sum_{n=0}^{\infty} (\lambda+n) c_n t^{n+\lambda-1}$ and $w'' = \sum_{n=0}^{\infty} (\lambda+n)(\lambda+n-1) c_n t^{n+\lambda-2}$.

$$\Rightarrow 0 = t^2 w'' + t A(t) w' + B(t) w$$
$$= \sum_{n=0}^{\infty} \Big[(\lambda+n)(\lambda+n-1) c_n + \sum_{k=0}^{n} (\lambda+k) c_k \alpha_{n-k} + \sum_{k=0}^{n} c_k \beta_{n-k} \Big] t^{\lambda+n}.$$

Hence, we have
$$(\lambda+n)(\lambda+n-1) c_n + \sum_{k=0}^{n} (\lambda+k) c_k \alpha_{n-k} + \sum_{k=0}^{n} c_k \beta_{n-k} = 0$$
for all n.

So we get
$$\big[(\lambda+n)(\lambda+n-1) + \alpha_0 (\lambda+n) + \beta_0 \big] c_n + \sum_{k=0}^{n-1} \big[(\lambda+k) \alpha_{n-k} + \beta_{n-k} \big] c_k = 0.$$

For $n=0$ since $c_0 \neq 0$ we have

$(\lambda+n)(\lambda+n-1) + \alpha_0 (\lambda+n) + \beta_0 = 0$ so that $\lambda \in \mathbb{C}$ must be a root of the quadratic equation

(3) $X(X-1) + \alpha_0 X + \beta_0 = 0$, which will be called the Indicial Equation of the differential equation.

Let $F(X)$ be the above polynomial: $F(X) = X(X-1) + \alpha_0 X + \beta_0$. Then we have

$$F(\lambda+n) c_n = -\sum_{k=0}^{n-1} \big[\alpha_{n-k} (\lambda+k) + \beta_{n-k} \big] c_k.$$

Assume that the roots of $F(x)=0$ do not differ by an integer. Then since $F(\lambda)=0$ we see that $F(\lambda+n)\neq 0$ for any integer. Hence

$$(4) \quad c_n = \frac{-1}{F(\lambda+n)} \sum_{k=0}^{n-1} \left[\alpha_{n-k}(\lambda+k)+\beta_{n-k}\right]c_k .$$

Conversely, let λ be a root of $F(x)=0$ and let $c_0=1$. Then from (4) we compute $c_1, c_2, \ldots$ and obtain the solution.

Exercise: Show that the series $\sum_{n=0}^{\infty} c_n t^n$ has positive radius of convergence.

Since $F(x)=0$ has two roots say λ and μ, we have two solutions
$w_1(z) = (z-a)^\lambda \sum c_n (z-a)^n$ and
$w_2(z) = (z-a)^\mu \sum c_n' (z-a)^n$, which are clearly linearly independent (assuming $\lambda-\mu\notin\mathbb{Z}$).
Therefore the equation (#) is of Fuchsian type at $z=a$.

Now assume that $\lambda-\mu=m\in\mathbb{Z}$, $m\geq 0$. Since $F(\lambda)=0$ and $\lambda=\mu+m\geq\mu$, we have $F(\lambda+n)\neq 0$, for any $n\neq 0$. So using the recursion formula (4) we compute all c_n's and obtain a solution of the form

$$w_1(z) = (z-a)^\lambda \sum_{n=0}^{\infty} c_n (z-a)^n, \quad c_0\neq 0,$$

where λ is the bigger root of $F(x)=0$.

For the second solution try a candidate

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Let the $w = w_1 \eta$. Then

$$w' = w_1' \eta + w_1 \eta',$$
$$w'' = w_1'' \eta + 2 w_1' \eta' + w_1 \eta''.$$

Plug these into the equation $0 = t^2 w'' + tAw' + Bw$, to get

$$0 = \underbrace{(t^2 w_1'' + tAw_1' + Bw_1)}_{\overset{\shortparallel}{0} \text{ since } w_1 \text{ is a solution}} \eta + t^2 (2 w_1' \eta' + w_1 \eta'') + tA w_1 \eta'$$

$$\Rightarrow t(2 w_1' \eta' + w_1 \eta'') + A w_1 \eta' = 0.$$

$$\Rightarrow \eta'' + \left(2 \frac{w_1'}{w_1} + \frac{A}{t}\right) \eta' = 0.$$

Let $v = \eta'$. Then we have $v' + \left(2 \frac{w_1'}{w_1} + \frac{A}{t}\right) v = 0.$

$$\Rightarrow \frac{dv}{dt} + \left(2 \frac{w_1'}{w_1} + \frac{A}{t}\right) v = 0.$$

$$\Rightarrow \frac{dv}{v} = -\left(\frac{2 w_1'}{w_1} + \frac{A}{t}\right) dt$$

$$\Rightarrow \log|v| = -2 \log w_1 - \int \frac{A}{t}\, dt + C$$

$$v = w_1^{-2} e^{-\int \frac{A}{t} dt}$$

$$\eta' = w_1^{-2} e^{-\int \frac{A}{t} dt} \Rightarrow \eta = \int w_1^{-2} e^{-\int \frac{A}{t} dt}.$$

So $w = w_1 \eta = w_1 \int w_1^{-2} e^{-\int \frac{A}{t} dt}$ is linearly independent from w_1 and they form a basis for $V_{\#}$.

Recall that $w_1 = t^{\lambda} \sum_{n=0}^{\infty} c_n t^n$, $c_0 = 1$, and

$A = \alpha_0 + \alpha_1 t + \cdots, \quad \alpha_0 \neq 0.$

Then $\int \frac{A}{t}\,dt = \alpha_0 \log t + \alpha_1 t + \frac{\alpha_2}{2} t^2 + \cdots$

$$\Rightarrow e^{-\int \frac{A}{t} dt} = t^{-\alpha_0} e^{-(\alpha_1 t + \frac{\alpha_2}{2} t^2 + \cdots)}$$

$$= t^{-\alpha_0}(1 + b_1 t + b_2 t^2 + \cdots)$$

On the other hand,

$$w_1^{-2} = [t^{\lambda}(1 + c_1 t + c_2 t^2 + \cdots)]^{-2} = t^{-2\lambda}(1 + d_1 t + d_2 t^2 + \cdots)$$

$$\Rightarrow w_1^{-2} e^{-\int \frac{A}{t} dt} = t^{-\alpha_0 - 2\lambda}(1 + e_1 t + e_2 t^2 + \cdots)$$

Recall that λ and $\mu = \lambda - m$ are the roots of $F(X) = X(X-1) + \alpha_0 X + \beta_0$ so that

$\lambda + \mu = 2\lambda - m = 1 - \alpha_0$ and so $-\alpha_0 - 2\lambda = -1 - m$

$$\Rightarrow w_1^{-2} e^{-\int \frac{A}{t} dt} = t^{-1-m}(1 + e_1 t + e_2 t^2 + \cdots).$$

Hence, $\eta = \int \left(w_1^{-2} e^{-\int \frac{A}{t} dt}\right) dt$

$$= \frac{t^{-m}}{-m} + e_1 \frac{t^{-m+1}}{-m+1} + \cdots + e_m \log t + e_{m+1} t + \frac{e_{m+2}}{2} t^2 + \cdots + C$$

$$\Rightarrow w = w_1 \eta$$

$$= e_m \log t \; w_1 + \left(\sum_{n=0}^{\infty} t^{\lambda} c_n t^n\right)\left(\frac{t^{m}}{-m} + \cdots\right)$$

$$= e_m (\log t)\, w_1 + t^{\mu} \sum_{n=0}^{\infty} c_n' t^n$$

$$\Rightarrow w = e_m (z-a)^{\lambda} \log(z-a) \sum_{n=0}^{\infty} c_n (z-a)^n + (z-a)^{\mu} \sum_{n=0}^{\infty} c_n' (z-a)^n.$$

In particular, w has regular singularity at $z=a$. Thus, (#) is of Fuchsian type.
This finishes the proof of the theorem. ∎

As a consequence of the proof we have

Theorem 18.8 The exponents λ and μ in Theorem 18.7 are roots of the equation $F(X) = X(X-1) + \alpha_0 X + \beta_0 = 0$.

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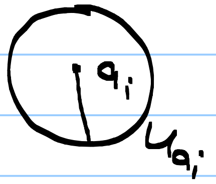
The Nineteenth Week: Fuchsian Differential Equations.

A) $D=\mathbb{C}\setminus\{a_1,\dots,a_n\}$ or $D=\mathbb{CP}^1=S^2\setminus\{a_1,\dots,a_n,\underset{a_{n+1}}{\underset{\shortparallel}{\infty}}\}$

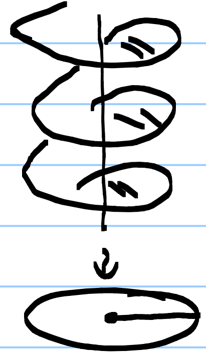
$\mathbb{CP}^1=\mathbb{C}\cup\{\infty\}=\mathbb{C}\,\dot\cup\,\mathbb{C}/z\sim\frac{1}{z},\ z\neq0$

$U_{a_i}=\{z\in\mathbb{C}\mid 0<|z-a_i|<\epsilon\}$



$\tilde{U}_{a_i}\to U_{a_i}$ universal cover.

Spiral staircase covering



For a function $F\in\mathcal{O}(\tilde{D})$, we say that F has regular singular points at $a_1,\dots,a_n$ if $F|_{\tilde{U}_{a_i,j}}$ the restriction of F to each connected component $\tilde{U}_{a_i,j}$ of $\tilde{U}_{a_i}$, has regular singular point at a_i.

We say that $F(z)$ has a regular singular point at $z=\infty$ if $H(t)=F(1/t)$ has a regular singular point at $t=0$.

Lemma: Any meromorphic function $f\in K(\mathbb{CP}^1)$ is rational.

Proof: $\mathbb{CP}^1=\mathbb{C}\,\dot\cup\,\mathbb{C}/z\sim 1/z \quad z\neq0$

Assume on the contrary that $f(z)$ has an infinite power series expansion at some point, say at $z=0$:

$f(z)=\sum_{n=-n_0}^{\infty}a_nz^n$. Then on the coordinate system we have $f(1/z)=\sum_{n=n_0}^{\infty}a_n\,1/z^n$.

Since f is meromorphic we must have

$a_n = 0$ for all $n \geq m$ for some m.
So $f(z) = \sum_{n=-n_0} a_n z^n = a_{-n_0} \frac{1}{z^{n_0}} + \cdots + a_{-1}\frac{1}{z} + a_0$

$+ a_1 z + a_2 z^2 + \dots + a_m z^m$.

Theorem 19.1: If $F, G \in \mathcal{O}(\tilde{D})$, $(G \not\equiv 0)$ have regular singular points at $a_1, a_2, \dots, a_n, a_{n+1} = \infty$, and F/G is in $K(D)$ $(= p^*(K(D)) \subseteq K(\tilde{D}))$, then F/G is a rational function.

(Here $p: \tilde{D} \to D$ is the universal covering.)

Proof: Clearly, F/G is meromorphic on D.

We also know that it is meromorphic at $a_1, a_2, \dots, a_n, \infty$ by the Preparation Theorem 18.2. Therefore, F/G is meromorphic on the entire Riemann sphere. Finally, by the lemma F/G is a rational function. ■

Let $B(\tilde{D})$ be the set of meromorphic functions on $\tilde{D}$ having regular singular points at $a_1, a_2, \dots a_n, a_{n+1} = \infty$. Also let $M(\tilde{D})$ be the field of quotients of $B(\tilde{D})$.

Let's denote by $K = K(R) = \mathbb{C}(z)$ the field of rational functions on $\mathbb{CP}^1$.

So by the previous theorem

$$M(\tilde{D}) \cap K(D) \subset K(R) = \mathbb{C}(z).$$

B) Consider the differential equation

$$(\#) \quad \frac{d^2w}{dz^2} + P(z)\frac{dw}{dz} + Q(z)w = 0.$$

Definition: The differential equation (#) is said to be of Fuchsian Type if the conditions below are satisfied:

a) $P(z)$ and $Q(z)$ are rational function on $\mathbb{CP}^1$, and their poles are contained in the set $\{a_1, a_2, \dots, a_n\}$.

b) Every solution of (#) has regular singular points in the set $\{a_1, a_2, \dots, a_n, \infty\}$.

Theorem 19.2. The differential equation (#) is of Fuchsian type if and only if $P(z)$ $Q(z)$ are rational functions of the following form

$$(*) \quad P(z) = \sum_{i=1}^{n} \frac{\alpha_i}{z-a_i} = \frac{\text{poly. in } z \text{ of degree} \leq n-1}{(z-a_1)\cdots\cdots(z-a_n)}.$$

$$(**) \quad Q(z) = \sum_{i=1}^{n} \left\{ \frac{\beta_i}{(z-a_i)^2} + \frac{\delta_i}{z-a_i} \right\}, \text{ where}$$

$$\sum_{i=1}^{n} \delta_i = 0. \quad \text{So } Q(z) = \frac{\text{poly. in } z \text{ of degree} \leq 2(n-1)}{(z-a_1)^2 \cdots\cdots (z-a_n)^2}$$

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Proof: The "if" part: So we assume that P and Q are rational functions having the forms given in (*) and (**). In particular, $a_1, a_2, \dots, a_n$ are the only poles of $P(z)$ and $Q(z)$. Moreover, by Theorem 19.5 the equation (#) is Fuchsian at any a_i. To finish the proof we just need to show that the equation (#) is Fuchsian at $z = \infty$.

So let $z = 1/t$ and plug this in $P(z)$ and $Q(z)$.

$$\frac{dw}{dz} = \frac{dw}{dt}\frac{dt}{dz} = -\frac{1}{z^2}\frac{dw}{dt} = -t^2\frac{dw}{dt}$$

$$\begin{aligned}\frac{d^2w}{dz^2} &= \frac{2}{z^3}\frac{dw}{dt} - \frac{1}{z^2}\frac{d}{dz}\left(\frac{dw}{dt}\right) \\ &= \frac{2}{z^3}\frac{dw}{dt} - \frac{1}{z^2}\frac{d}{dt}\left(\frac{dw}{dt}\right)\frac{dt}{dz} \\ &= 2t^3\frac{dw}{dt} + t^4\frac{d^2w}{dt^2}\end{aligned}$$

$$\Rightarrow 0 = \frac{d^2w}{dz^2} + P(z)\frac{dw}{dz} + Q(z)w = t^4\frac{d^2w}{dt^2} + 2t^3\frac{dw}{dt} - Pt^2\frac{dw}{dt} + Qw$$

$$\Rightarrow \frac{d^2w}{dt^2} + \left(\frac{2}{t} - \frac{1}{t^2}P(1/t)\right)\frac{dw}{dt} + \frac{1}{t^4}Q(1/t)w = 0.$$

<u>must check</u>: The above equation is Fuchsian at $t=0$. By Theorem 18.5 this reduces to show that the functions $t\left(2/t - 1/t^2\, P(1/t)\right)$ and $t^2\left(1/t^4\, Q(1/t)\right)$ are

holomorphic at $t=0$.

Since $P(z)=\sum_{i=1}^{n}\frac{\alpha_i}{z-a_i}$ and thus

$$
\begin{aligned}
t\left(\frac{2}{t}-\frac{1}{t^2}P(1/t)\right) &= 2-\frac{1}{t}P(1/t)\\
&= 2-\frac{1}{t}\sum_{i=1}^{n}\frac{\alpha_i}{1/t-a_i}\\
&= 2-\sum_{i=1}^{n}\frac{\alpha_i}{1-ta_i}\\
&= 2-\sum_{i=1}^{n}\sum_{k=0}^{\infty}\alpha_i(ta_i)^k\\
&= \left(2-\sum_{i=1}^{n}\alpha_i\right)-\sum_{i=1}^{n}\sum_{k=1}^{\infty}\alpha_i(ta_i)^k,
\end{aligned}
$$

which is clearly a convergent p.s.

Note that $\lim_{t\to 0} t\left(\frac{2}{t}-\frac{1}{t^2}P(1/t)\right)=2-\sum_{i=1}^{n}\alpha_i$.

Notation $\alpha_\infty \doteq 2-\sum_{i=1}\alpha_i$.

Similarly, $t^2\left(\frac{1}{t^4}Q(1/t)\right)=\frac{1}{t^2}Q(1/t)$

$$
\begin{aligned}
&= \frac{1}{t^2}\sum_{i=1}^{n}\left(\frac{\beta_i}{(1/t-a_i)^2}+\frac{\delta_i}{1/t-a_i}\right)\\
&= \sum_{i=1}^{n}\beta_i\frac{1}{(1-ta_i)^2}+\frac{1}{t}\frac{\delta_i}{1-ta_i}
\end{aligned}
$$

$$= \sum_{i=1}^{n}\left(\beta_i \frac{1}{(1-ta_i)^2} + \frac{1}{t}\delta_i \sum_{k=0}^{\infty}(ta_i)^k\right)$$

$$= \sum_{i=1}^{n}\left(\beta_i\left(\sum_{k=0}^{\infty}(ta_i)^k\right)^2 + \frac{\delta_i}{t}\sum_{k=0}^{\infty}(ta_i)^k\right)$$

$$= \sum_{i=1}^{n}\beta_i + \underbrace{\sum_{i=1}^{n}\frac{\delta_i}{t}}_{=0 \text{ by assumption.}} + \sum_{i=1}^{n}\delta_i a_i + \text{c.p.s.} + \text{c.p.s.}$$

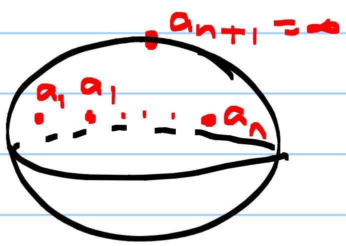
$$= \left(\sum_{i=1}^{n}\beta_i + \delta_i a_i\right) + \text{c.p.s. in } t$$

Hence, the equation is Fuchsian at $t=0$.

Moreover, $\beta_\infty = \lim_{t\to 0} t^2\left(\frac{1}{t^n}Q(1/t)\right) = \sum_{i=1}^{n}\beta_i + \delta_i a_i$.

This finishes the proof of "if part". ■

Remark $\alpha_\infty = 2 - \sum_{i=1}^{n}\alpha_i$, $\beta_\infty = \sum_{i=1}^{n}(\beta_i + \delta_i a_i)$



Let $\lambda_1,\dots,\lambda_n, \lambda_{n+1}=\lambda_\infty$ and $\mu_1,\dots,\mu_n, \mu_{n+1}=\mu_\infty$ are the roots of $x(x-1)+\alpha_i x+\beta_i=0$, $i=1,\dots,n,\overset{\infty}{\overset{\shortparallel}{n+1}}$.

$$\sum_{i=1}^{n}(\lambda_i+\mu_i) + (\lambda_\infty+\mu_\infty) = \sum_{i=1}^{n}(1-\alpha_i) + (1-\alpha_\infty)$$

$$= \sum_{i=1}^{n}(1-\alpha_i) + \left(1-\left(2-\sum_{i=1}^{n}\alpha_i\right)\right)$$

$$= n-1.$$

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c) $P(z)$ and $Q(z)$ polynomials as in (*) and (**) of Theorem 19.2. so that the equation

(#) $\frac{d^2w}{dz^2} + P(z)\frac{dw}{dz} + Q(z)w = 0$ is Fuchsian.

Let $D = \mathbb{C}\setminus\{a_1, \dots, a_n\} = \mathbb{CP}^1\setminus\{a_1, \dots, a_n, a_{n+1} = \infty\}$ and set $S_\#$ to be the field obtained by adjoining all the solutions of (#) and their first derivatives to the field of rational functions $K(R) = \mathbb{C}(z)$ $(R = \mathbb{CP}^1)$

$$S_\# = \mathbb{C}(z)\left(\{w, \frac{dw}{dz} \mid w \in V_\#\}\right).$$

Note that since $V_\# = \langle \phi, \psi \rangle$ for any two linearly independent solutions ϕ and ψ of (#) we see that

$$S_\# = \mathbb{C}(z)(\phi, \psi, \phi', \psi').$$

Since the equation is Fuchsian all the solutions in particular, ϕ, ψ and their derivatives ϕ', ψ' have only regular singularities at a_i's.

So by the Preparation Theorem 18.2 we deduce that $S_\# \cap K(D) \subseteq \mathbb{C}(z)$ [just take $F \in S_\# \cap K(D)$ and $G = 1$). On the other hand $\mathbb{C}(z) \subseteq S_\# \cap K(D)$ and thus $S_\# \cap K(D) = \mathbb{C}(z)$.

The next theorem is a generalization of Theorem 17.2.

Theorem 19.3.: For the Fuchsian differential equation (#), every solution of (#) is of type L_0 over $\mathbb{C}(z) = K(R)$ if and only if the monodromy representation of (#) is triangulable.

Proof: We'll prove the "$\Leftarrow$" part only.

So assume that the monodromy representation of $\Gamma = \pi_1(D)$ is triangulable and there is a basis $\{w_1, w_2\}$ of $V_\#$ such that

$$\left((\gamma^{-1})^* w_1, (\gamma^{-1})^* w_2\right) = (w_1 \; w_2)\begin{pmatrix} a(\gamma) & b(\gamma) \\ 0 & c(\gamma) \end{pmatrix}.$$

As in the case of Theorem 17.2. we have

$(\gamma^{-1})^* w_1 = a(\gamma) w_1$ and thus differentiating both sides we get $(\gamma^{-1})^*\left(\frac{dw_1}{dz}\right) = a(\gamma)\frac{dw_1}{dz}$ and thus

$$(\gamma^{-1})^*\left(\frac{dw}{dz}\Big/ w_1\right) = \frac{dw/dz}{w_1}, \; \forall \gamma \in \Gamma.$$

Hence, $dw/dz/w \in K(D)$.

Since $w_1 \in V_\#$ we see that $\frac{dw_1}{dz}/w_1 \in S_\#$.

So $A \doteq \frac{dw_1/dz}{w_1} \in S_\# \cap K(D) = \mathbb{C}(z)$.

It follows that $\frac{dw_1}{w} = A\,dz \Rightarrow w_1 = c\, e^{\int A(z)\,dz}$

and thus w_1 is of type Lo over $\mathbb{C}(z)$.
Finally, by the Preparation Theorem 17.1 every solution of (#) is of type Lo over $\mathbb{C}(z)$. ■

D) Now assume that the equation (#) is of type Lo over $\mathbb{C}(z)$, as in the statement of Theorem 19.3.
We've seen that one solution is given as

$w_1(z) = C e^{\int A(z)\,dz}$. By Theorem H.1 the general solution of (#) is given as

$$w = C e^{\int A(z)\,dz} \int e^{-2\int A(z)\,dz - \int P\,dz} dz + C' e^{\int A(z)\,dz}.$$

Aim: List all solutions of (#).

First let's study $A(z) = \frac{dw_1}{dz} / w_1$.

We know that $w_1 \circ \gamma = a(\gamma) w_1$, for every $\gamma \in \Gamma = \pi_1(D)$. In particular, if $\tilde{p} \in \tilde{D}$ is a zero of w_1 then all of its conjugates $\gamma(\tilde{p})$, $\gamma \in \Gamma = \pi_1(D) = \mathrm{Deck}(\tilde{D} \to D)$ are zeros of w_1.

Let $Z \subseteq \tilde{D}$ be the set of all zeros of w_1.

Claim: If $P : \tilde{D} \to D$ is the universal cover projection then $P(Z)$ is a finite set.

Proof: Suppose that $P(Z)$ is infinite. Since w_1 is analytic on D $P(Z)$ cannot have an accumulation point in D. However, $P(Z)$ must have some accumulation point in the Riemann sphere $R = \mathbb{C}P^1$, since R is compact. So there must be a sequence (z_n) in $P(Z) \subseteq D$ that converges to some point $R \setminus D = \{a_1, \ldots, a_{n+1}\}$, say a_1 without loss of generality. Assume that a_1 is a pole of w_1 of order k. Then $(z - a_1)^k w_1(z)$ is analytic at a_1. However, this is a contradiction since (z_n) is an infinite sequence of zeros of $(z - a_1)^k w_1(z)$ converging to a_1. Thus $P(Z)$ must be finite. ∎

So we see that $Z = \{\tilde{p} \in \tilde{D} \mid P(\tilde{p}) = b_1, \ldots, b_s\}$ for some points $b_1, \ldots, b_s \in D$.

Since w_1 is a solution of a second order linear equation each zero $\tilde{p} \in Z$ of w_1 must have multiplicity one: if $\tilde{p} \in Z$ has multiplicity at least two then $w_1(\tilde{p}) = 0$ and $w_1'(\tilde{p}) = 0$. However, the unique solution of (#) with these initial conditions at $\tilde{p}$ is the zero function thus we must have $w_1 \equiv 0$, a contradiction.

So the multiplicity of each zero of w_1 is one and the Laurent expansion of $A(z) = w_1'/w_1$ at each b_i has the form

$$w_1(z) = \sum_{n=1}^{\infty} a_n (z-b_i)^n = c_1(z-b_i) + c_2(z-b_i)^2 + \cdots \quad (c_1 \neq 0)$$

$$w_1'(z) = c_1 + 2c_2(z-b_i) + \cdots$$

$$A(z) = \frac{w_1'(z)}{w_1(z)} = \frac{1}{(z-b_i)} + (z-b_i) \cdot \text{c.p.s. in } (z-b_i).$$

Let λ_i, μ_i, $i = 1, \ldots, n$ and $\lambda_\infty, \mu_\infty$ be the roots of the indicial equation of (#) at a_i, $i = 1, \ldots, n$, and at $a_{n+1} = \infty$.

Since $w_1 \circ \gamma = a(\gamma) w_1$, $\forall \gamma \in \Gamma$, by the proof of Theorem 18.5 we deduce that

$$w_1 = (z-a_i)^{\lambda_i} \sum_{n=0}^{\infty} c_n (z-a_i)^n \quad (c_0 \neq 0) \quad \text{or}$$

$$w_1 = (z-a_i)^{\mu_i} \sum_{n=0}^{\infty} c_n (z-a_i)^n, \text{ at every } a_i.$$

Here, $a(\gamma_i)$ is either $e^{2\pi\sqrt{-1}\lambda_i}$ or $e^{2\pi\sqrt{-1}\mu_i}$, when γ_i winds once around a_i.

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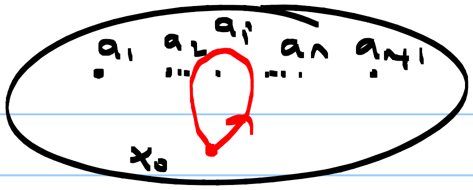

So we see that

$$A(z)=\frac{dw_1/dz}{w_1}=\begin{cases}\frac{\lambda_i}{z-a_i}+(z-a_i).\text{c.p.s. in }(z-a_i)\\ \text{or}\\ \frac{\mu_i}{z-a_i}+(z-a_i).\ \text{c.p.s. in }(z-a_i).\end{cases}$$

Similarly, at $z=\infty$ we have

$$A(z)=\frac{dw_1/dz}{w_1}=-\frac{t^2\,dw_1/dt}{w_1}=\begin{cases}-\lambda_\infty t+\dots\\ -\mu_\infty t+\dots\end{cases}$$

We know that $A(z)$ is a rational function on $\mathbb{R}$. All the poles of $A(z)$ are at the points $a_1,\dots,a_n$, $a_{n+1}=\infty$ and $b_1,\dots,b_s$, where the residues are calculated as above: ρ_i $i=1,\dots,n$, ρ_∞ and $1,\dots,1$, respectively (ρ_i is either λ_i or μ_i).

It follows that the rational function $A(z)$ is given by the

$$A(z)=\sum_{i=1}^{n}\frac{\rho_i}{z-a_i}+\sum_{i=1}^{s}\frac{1}{z-b_i},\text{ where } s+\rho_\infty+\sum_{i=1}^{n}\rho_i=0.$$

On the other hand, since $w_1(z)=c\,e^{\int A(z)dz}$ is a solution of (#) we have $dw_1/dz=A(z)\,w_1$ and hence $\frac{d^2w}{dz^2}=A'(z)w_1+A(z)w_1'=(A'+A^2)w_1$.

So the equation (#) becomes

$(A'+A^2)w_1+P(z)A(z)w_1+Q(z)w_1=0$, which

Implies $A' + A^2 + P(z)A(z) + Q(z) = 0.$ (*)

Let's rewrite $A(z) = \sum_{i=1}^{n} \frac{\rho_i}{z-a_i} + \sum_{i=1}^{s} \frac{1}{z-b_i}$, as

$A(z) = \sum_{i=1}^{m} \frac{\rho_i}{z-a_i}$, where $m = n+s$, $a_{n+i} = b_i$, $\rho_{n+i} = 1$, $i = 1, \dots, s$.

Then $A'(z) = \sum_{i=1}^{m} \frac{-\rho_i}{(z-a_i)^2}$ and

$$A^2(z) = \sum_{i=1}^{m} \frac{\rho_i^2}{(z-a_i)^2} + \sum_{i \neq j} \frac{\rho_i \rho_j}{(z-a_i)(z-a_j)}$$

$$= \sum_{i=1}^{m} \frac{\rho_i^2}{(z-a_i)^2} + \sum_{i \neq j} \left(\frac{1}{z-a_i} - \frac{1}{z-a_j} \right) \frac{\rho_i \rho_j}{a_i a_j}$$

Note that the coefficient of $\frac{1}{z-a_i}$ in the second summation is

$$\sum_{j \neq i} \frac{\rho_i \rho_j}{a_i - a_j} + \sum_{k \neq i} \frac{-\rho_k \rho_i}{a_k - a_i} = 2 \sum_{j \neq i} \frac{\rho_i \rho_j}{a_i - a_j}.$$

Hence, $A^2(z) = \sum_{i=1}^{m} \left\{ \frac{\rho_i^2}{(z-a_i)^2} + \frac{2}{z-a_i} \left(\sum_{j \neq i} \frac{\rho_i \rho_j}{a_i - a_j} \right) \right\}.$

On the other hand,

$P(z) = \sum_{i=1}^{n} \frac{\alpha_i}{z-a_i} = \sum_{i=1}^{m} \frac{\alpha_i}{z-a_i}$, where we define

$\alpha_{n+j} = \beta_{n+j} = \delta_{n+j} = 0$, for $j = 1, \dots, s$, and

$$Q(z) = \sum_{i=1}^{m} \left\{ \frac{\beta_i}{(z-a_i)^2} + \frac{\delta_i}{z-a_i} \right\}.$$

Now, $P(z)A(z)=\left(\sum_{i=1}^{m}\frac{\alpha_i}{z-a_i}\right)\left(\sum_{i=1}^{m}\frac{\rho_i}{z-a_i}\right)$

$$=\sum_{i=1}^{m}\frac{\alpha_i\rho_i}{(z-a_i)^2}+\sum_{i\neq j}\frac{\alpha_i\rho_j}{(z-a_i)(z-a_j)}$$

$$=\sum_{i=1}^{m}\left\{\frac{\alpha_i\rho_i}{(z-a_i)^2}+\sum_{i\neq j}\left(\frac{1}{z-a_i}-\frac{1}{z-a_j}\right)\frac{\alpha_i\rho_j}{a_i-a_j}\right\}$$

$$=\sum_{i=1}^{m}\left\{\frac{\alpha_i\rho_i}{(z-a_i)^2}+\frac{1}{z-a_i}\sum_{j\neq i}\frac{\alpha_i\rho_j+\alpha_j\rho_i}{a_i-a_j}\right\}.$$

Plugging these into the equation (*) for $A(z)$ we obtain

$$\sum_{i=1}^{m}\frac{-\rho_i}{(z-a_i)^2}+\sum_{i=1}^{m}\left\{\frac{\rho_i^2}{(z-a_i)^2}+\frac{2}{z-a_i}\left(\sum_{j\neq i}\frac{\rho_i\rho_j}{a_i-a_j}\right)\right\}$$

$$+\sum_{i=1}^{m}\left\{\frac{\alpha_i\rho_i}{(z-a_i)^2}+\frac{1}{z-a_i}\sum_{j\neq i}\frac{\alpha_i\rho_j+\alpha_j\rho_i}{a_i-a_j}\right\}$$

$$+\sum_{i=1}^{m}\left\{\frac{\beta_i}{(z-a_i)^2}+\frac{\delta_i}{(z-a_i)}\right\}=0.$$

$$\Rightarrow 0=\sum_{i=1}^{m}\left[\frac{1}{(z-a_i)^2}\left(-\rho_i+\rho_i^2+\alpha_i\rho_i+\beta_i\right)+\frac{1}{z-a_i}\left\{\sum_{j\neq i}\frac{1}{a_i-a_j}\left(2\rho_i\rho_j+\rho_i\alpha_j+\rho_j\alpha_i\right)+\delta_i\right\}\right].$$

This gives us two sets of equations

a) $\rho_i^2-\rho_i+\alpha_i\rho_i+\beta_i=0$, $i=1,\dots,m$, and

b) $\sum_{j\neq i}\frac{1}{a_i-a_j}\left(2\rho_i\rho_j+\rho_i\alpha_j+\rho_j\alpha_i\right)+\delta_i=0$, $i=1,\dots,m$.

Note that (c) can be written as

$\rho_i(\rho_i-1)+\alpha_i\rho_i+\beta_i=0$ so that ρ_i is a root of the indicial equation for a_i.

On the other hand, the equation (6) enables us to determine $b_k=a_{n+k}$ $(k=1,\dots,s)$ in terms of known coefficients a_i $(i=1,\dots,n)$, ρ_j and α_j.

Finally, solving the above equations for $b_1,\dots,b_s$ we determine

$$A(z)=\sum_{i=1}^{n}\frac{\rho_i}{z-a_i}+\sum_{i=1}^{s}\frac{1}{z-b_i}.$$

<u>Now let's summarize how to determine A(z):</u>

Let $\lambda_1,\mu_1,\lambda_2,\mu_2,\dots,\lambda_\infty,\mu_\infty$ be the roots of the indicial equations of (#) at the regular singular points $a_1,\dots,a_n,a_{n+1}=\infty$, respectively. Let ρ_i to denote either λ_i or μ_i, for each $i=1,\dots,n+1$. So there are 2^{n+1} ways of chosing $\rho_1,\dots,\rho_n,\rho_\infty$.

We know that $A(z)=\sum_{i=1}^{n}\frac{\rho_i}{z-a_i}+\sum_{i=1}^{s}\frac{1}{z-b_i}$,

where $\sum_{i=1}^{n}\rho_i+s=-\rho_\infty$, and thus we use

only those choices satisfying $\sum_{i=1}^{n}\rho_i+\rho_\infty=-s\le 0$ because s is a nonnegative integer.

We know that $a_{n+j}=b_j$, $\rho_{n+j}=1$, $\alpha_{n+j}=\beta_{n+j}=\delta_{n+j}=0$ for all $j=1,\dots,s$.

Moreover, the equation (6)

$$\sum_{j\neq i}\frac{1}{a_i-a_j}(2\rho_i\rho_j+\alpha_i\rho_j+\alpha_j\rho_i)+\delta_i=0\quad(i=1,\dots,m)$$

yields two equations in $x_k = b_k = a_{n+k}$, $k=1,\dots,s$,

c) For $i=1,\dots,n$

$$0 = \delta_i + \sum_{j=1}^{i-1} \frac{1}{a_i-a_j}(2\ell_i\ell_j + \alpha_i\ell_j + \alpha_j\ell_i) + \sum_{j=i+1}^{m} \frac{1}{a_i-a_j}(2\ell_i\ell_j + \alpha_i\ell_j + \alpha_j\ell_i)$$

$$= \delta_i + \sum_{j=1,\, j\neq i}^{n} \frac{1}{a_i-a_j}(2\ell_i\ell_j + \alpha_i\ell_j + \alpha_j\ell_i) + \sum_{j=n+1}^{m} \frac{1}{a_i-a_j}(2\ell_i + \alpha_i)$$

$$\Rightarrow \quad \boxed{0 = \delta_i + \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{a_i-a_j}(2\ell_i\ell_j + \alpha_i\ell_j + \alpha_j\ell_i) + \sum_{k=1}^{s} \frac{1}{a_i-x_k}(2\ell_i + \alpha_i)}$$

(c)

and

(d) For $i=n+1,\dots,m$

$$0 = \sum_{j=1}^{n} \frac{1}{a_i-a_j}(2\ell_i\ell_j + \alpha_i\ell_j + \alpha_j\ell_i) + \sum_{\substack{j=n+1\\ j\neq i}}^{m} \frac{1}{a_i-a_j}(2\ell_i\ell_j + \alpha_i\ell_j + \alpha_j\ell_i)$$

$$= \sum_{j=1}^{n} \frac{1}{x_{i-n}-a_j}(2\ell_j + \alpha_j) + \sum_{\substack{j=n+1\\ j\neq i}}^{n} \frac{2}{x_{i-n}-x_{j-n}}$$

$$\Rightarrow \quad \boxed{0 = \sum_{j=1}^{n} \frac{1}{x_j-a_j}(2\ell_j + \alpha_j) + \sum_{\substack{k=1\\ k\neq l}}^{s} \frac{1}{x_l-x_k} \quad \text{for } l=1,\dots,s.}$$

(d)

If the equations (c) and (d) have a solution

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$(x_1, \dots, x_s) = (b_1, \dots, b_s)$, then

$$A(z) = \sum_{i=1}^{n} \frac{\rho_i}{z-a_i} + \sum_{k=1}^{s} \frac{1}{z-b_k}.$$

$A(z)$ is the function we are looking for to obtain the solutions of (#). If $A(z)$ does not exists the equation (#) is not type L_0 over $\mathbb{C}(z)$.

In the affirmative case the

$$\omega_1(z) = e^{\int A(z)dz} = e^{\sum \rho_i \log(z-a_i) + \sum \log(z-b_k)} = \prod_{i=1}^{n} (z-a_i)^{\rho_i} \prod_{k=1}^{s} (z-b_k).$$

Moreover, the general solution would be

$$\omega(z) = c\,\omega_1 \int \omega_1^{-2} e^{\int P(z)dz} + c'\omega_1$$

$$= c \prod_{i=1}^{n} (z-a_i)^{\rho_i} \prod_{k=1}^{s} (z-b_k) \int \prod_{i=1}^{n} (z-a_i)^{-2\rho_i-\alpha_i} (z-b_k)^{-2}\, dz$$

$$+ c' \prod_{i=1}^{n} (z-a_i)^{\rho_i} \prod_{k=1}^{\infty} (z-b_k).$$

E) **Example:** Solve the equation

$$\frac{d^2w}{dz^2} + \left(\frac{1}{3z} + \frac{1}{6(z-1)}\right)\frac{dw}{dz} + \left(\frac{-1}{3z^2} - \frac{1}{6(z-1)^2} + \frac{1}{2z(z-1)}\right)w = 0.$$

Solution: $P(z) = \dfrac{1}{3z} + \dfrac{1}{6(z-1)}$

$$Q(z) = -\frac{1}{3z^2} - \frac{1}{6(z-1)^2} - \frac{1}{2z} + \frac{1}{2(z-1)}$$

The singular points are 0, 1 and ∞.

$a_1=0$, $a_2=1$, $a_\infty=\infty$. Note that

$\alpha_1=\frac{1}{3}$, $\beta_1=-\frac{1}{3}$, $\delta_1=-\frac{1}{2}$.

$\alpha_2=\frac{1}{6}$, $\beta_2=-\frac{1}{6}$, $\delta_2=1/2$.

$\alpha_\infty=2-\alpha_1-\alpha_2=\frac{3}{2}$, $\beta_\infty=\beta_1+\beta_2+\delta_2=0$.

So we have the following table

Singular pts.	Indicial Equation	Roots λ, μ
$a_1=0$	$x(x-1)+1/3\,x-1/3=0$	$-1/3,\ 1$
$a_2=1$	$x(x-1)+1/6\,x-1/6=0$	$-1/6,\ 1$
$a_\infty=\infty$	$x(x-1)+3/2\,x=0$	$-1/2,\ 0$

$$\underbrace{\lambda_1=-1/3,\ \mu=1}_{\rho_1},\ \underbrace{\lambda_2=-1/6,\ \mu_2=1}_{\rho_2},\ \underbrace{\lambda_\infty=-1/2,\ \mu_\infty=0}_{\rho_\infty}.$$

The only choice of ρ_i's so that $\rho_1+\rho_2+\rho_\infty=-s$ is a negative integer is the following

$\rho_1=\lambda_1=-1/3$, $\rho_2=\lambda_2=-1/6$, $\rho_\infty=\lambda_\infty=-1/2$.

In particular, $s=-(\rho_1+\rho_2+\rho_\infty)=1$. For these choice of ρ_i's the rational function $A(z)$ becomes

$$A(z)=\frac{-1/3}{z}+\frac{-1/6}{z-1}+\frac{1}{z-x_1},$$ where $x_1=b$,

to be determined.

Recall that $A(z)$ must satisfy the equation $A' + A^2 + AP + Q = 0$ or equivalently the equations (c) and (b).

(c) $\tau = 1$, $0 = \delta_1 + \frac{1}{a_1 - a_2}(2e_1 e_2 + \alpha_1 e_2 + \alpha_2 e_1) + \frac{1}{a_1 - b_1}(2e_1 + \alpha_1)$

$$\Rightarrow 0 = -\tfrac{1}{2} + \frac{1}{0-1}\left(2(-1/3)(-1/6) + 1/3(-1/6) + 1/6(-1/3)\right) + \frac{1}{0-b_1}\left(2(-1/3) + 1/3\right)$$

$$\Rightarrow 0 = -\frac{1}{2} - \left(\frac{1}{9} - \frac{1}{18} - \frac{1}{18}\right) - \frac{1}{b_1}\left(-\frac{1}{3}\right)$$

$$\Rightarrow \frac{1}{2} = \frac{1}{3b_1} \Rightarrow b_1 = \frac{2}{3}.$$

Hence, $A(z) = \frac{-1/3}{z} + \frac{-1/6}{z-1} + \frac{1}{z - 2/3}$.

So, $w_1(z) = e^{\int A(z)\,dz} = z^{-1/3}(z-1)^{-1/6}(z - 2/3)$.

A second solution is given by

$$w_2(z) = w_1(z) \int w_1^{-2}(z)\, e^{-\int p(z)\,dz}\, dz$$

$$= z^{-1/3}(z-1)^{-1/6}\left(z - \frac{2}{3}\right) \int z^{2/3}(z-1)^{1/3}\left(z - \frac{2}{3}\right)^{-2} e^{-\int\left(\frac{1}{3z} + \frac{1}{6(z-1)}\right)dz}$$

$$= z^{-1/3}(z-1)^{-1/6}\left(z - \frac{2}{3}\right) \int z^{2/3}(z-1)^{1/3}\left(z - \frac{2}{3}\right)^{-2} z^{-1/3}(z-1)^{-1/6}\, dz$$

$$= z^{-1/3}(z-1)^{-1/6}(z - 2/3) \int z^{1/3}(z-1)^{1/6}\left(z - \frac{2}{3}\right)^{-2} dz.$$

In particular, the general solution of (#) is

$w(z) = c_1 w_1(z) + c_2 w_2(z)$.