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METU MATH. DEPT. SUMMER 2021
MATH 420 - TOPOLOGY

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<u>Textbook</u>: James Munkres, Topology 2nd Ed.

CHAPTER 2: Topological Spaces and Continuous Functions:

One may regard topological spaces as a generalization of Euclidean or more generally metric spaces. It turns out that many mathematical concepts can be defined or studied by means of open/closed sets even in the absence of a metric. There are important topological spaces which do not admit a compatible metric and therefore one has to consider topological spaces which do not admit compatible metric.

Definition: A topology on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

1) ϕ and X are in $\mathcal{T}$.

2) The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$:
$\{U_\alpha\}_{\alpha\in\Lambda}$, $U_\alpha \in \mathcal{T} \Rightarrow \bigcup_{\alpha\in\Lambda} U_\alpha \in \mathcal{T}$.

3) The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$:
$U_1, \dots, U_n \in \mathcal{T} \Rightarrow U_1 \cap \dots \cap U_n \in \mathcal{T}$.

In this case, $\mathcal{T}$ is called a topology on X and the pair $(X, \mathcal{T})$ is called a topological space.

Examples: 1) X any set. Let $\mathcal{T} = \{\phi, X\}$. Clearly $\mathcal{T}$ is a topology on X, called the trivial or smallest (weakest) topology on X.

2) X any set. Let $\mathcal{T} = \mathcal{P}(X)$ the collection

of all subsets of X. Again clearly $\mathcal{P}(X)$ is a topology on X, called the largest or strongest topology on X.

If $\mathcal{J}$ is a topology on X then the elements of $\mathcal{J}$ are called open subsets of $\mathcal{J}$. A subset C of X will be called closed if $X \setminus C$ is open.

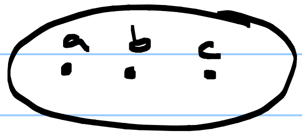
<u>Examples:</u> 1) If (X,d) is metric space then the collection of open subsets $\mathcal{J}$ of (X,d) then clearly $\mathcal{J}$ is a topology on X.

2) Let X be any set. Suppose that $\mathcal{J}$ is a topology on X s.t. it contains all one point subsets: $\forall x \in X$, $\{x\} \in \mathcal{J}$. Then any subset U of X can be written as

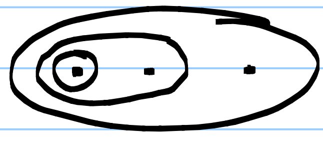
$$U = \bigcup_{x \in U} \{x\}$$ and by assumption U is open: $U \in \mathcal{J}$.

Hence, $\mathcal{J} = \mathcal{P}(X)$. This topology is also called the discrete topology on X.

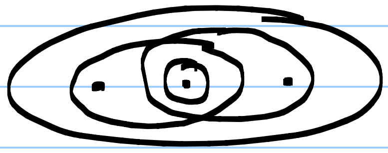
3) $X = \{a, b, c\}$



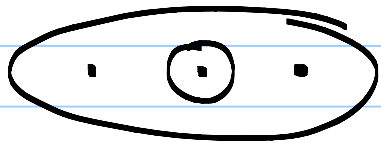
$\mathcal{J}_1 = \{X, \phi\}$



$\mathcal{J}_2 = \{\{a\}, \{a,b\}, \{a,b,c\}, \phi\}$



$\mathcal{J}_3 = \{\phi, \{a,b\}, \{b\}, \{b,c\}, \{a,b,c\}\}$



$\mathcal{J}_4 = \{\phi, X, \{b\}\}$

4) Finite complement topology: X any set.
Let $\mathcal{T}_f$ be the collection of all subsets of X whose complements are finite:
$U \subseteq X$ is open in $\mathcal{T}_f$ if and only if $X \setminus U$ is finite or X.

$\mathcal{T}_f$ is a topology on X.

1) $\phi \in \mathcal{T}_f$ because $X \setminus \phi = X$.
$X \in \mathcal{T}_f$ because $X \setminus X = \phi$, which is finite.

2) Let $\{U_\alpha\}_{\alpha \in \Lambda}$ be a collection of open subsets.

Then each $X \setminus U_\alpha$ is finite. Hence

$$X \setminus \left(\bigcup_{\alpha \in \Lambda} U_\alpha \right) = \bigcap_{\alpha \in \Lambda} (X \setminus U_\alpha),$$ which is clearly finite.

Hence, $\bigcup_{\alpha \in \Lambda} U_\alpha$ is open.

3) Let $U_1, U_2, \ldots, U_n$ be open sets. Then each $X \setminus U_i$ is finite. Hence,
$$X \setminus (U_1 \cap \cdots \cap U_n) = \bigcup_{i=1}^{n} (X \setminus U_i),$$ which is clearly finite.

Hence, $U_1 \cap \ldots \cap U_n$ is still open.

Thus, $\mathcal{T}_f$ is a topology on X, called the finite complement topology.
Note that the closed subsets of X are just the finite subsets of X and X itself.

3') $X = \mathbb{R}$ or $\mathbb{C}$ consider $\mathcal{T}_f$ on X. $\mathcal{T}_f$ is called also called the Zariski topology on X.

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4) (Exercise) Let X be any set. Let $\mathcal{T}_c$ be the collection of subsets of X, whose complements are countable or X. Then $\mathcal{T}_c$ is also a topology on X, called the countable complement topology.

Definition: Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a set X. If $\mathcal{T}' \supset \mathcal{T}$, then we say that $\mathcal{T}'$ is finer (or stronger) than $\mathcal{T}$. Equivalently, we say that $\mathcal{T}$ is coarser (weaker) than $\mathcal{T}'$.

Thus the topology $\mathcal{T} = \mathcal{P}(X)$ is really the strongest topology on X.

Basis for a Topology: If X is a set, a basis for a topology on X is a collection $\mathcal{B}$ of subsets of X (called basis elements) such that

1) For each $x \in X$, there is at least one basis element $B \in \mathcal{B}$ containing x: $x \in B$.

2) If x belongs to the intersection of two basis elements B_1 and B_2, then there is a third basis element B_3 so that $x \in B_3 \subseteq B_1 \cap B_2$.

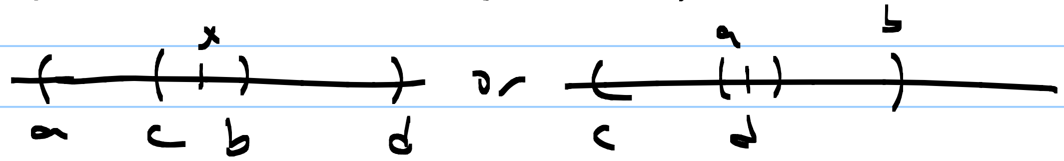
If $\mathcal{B}$ is a basis for a topology, then we define the topology $\mathcal{T}$ generated by $\mathcal{B}$ as follows:
A subset U of X is said to be open in X (i.e. $U \in \mathcal{T}$) if for each $x \in U$ there is some $B \in \mathcal{B}$ so that $x \in B \subseteq U$. Note that each basis element is itself an element of $\mathcal{T}$.

Exercise: Check that $\mathcal{T}$ is indeed a topology on X.

Examples: 1) $X=\mathbb{R}$, $\mathcal{B}=\{(a,b) \mid a<b\}$.

Claim: $\mathcal{B}$ is a basis for a topology.

1) Let $x\in\mathbb{R}$ then $x\in(x-1,x+1)\in\mathcal{B}$.
2) If $x\in\mathbb{R}$ and $x\in(a,b)\cap(c,d)$ then



Let $e=\max\{a,c\}$ and $f=\min\{b,d\}$.
Note that $x\in(a,b)\cap(c,d)$ and hence

$$\begin{matrix} a<x<b \\ c<x<d \end{matrix} \Rightarrow e=\max\{a,c\}<x<\min\{b,d\}=f.$$

$x\in(e,f)\subseteq(a,b)\cap(c,d)$.

Hence, $\mathcal{B}$ is a basis for a topology on X. If U is an open generated by $\mathcal{B}$ then for any $x\in U$ there is some $(a,b)\in\mathcal{B}$ so that $x\in(a,b)\subseteq U$. However this is just the standard definition of being open set in $\mathbb{R}$. Thus, this topology is also called the standard topology on $\mathbb{R}$, which we denote as $\mathbb{R}_{std}$.

Example: Recall the standard topology on $\mathbb{R}^2$, considered as a metric space with the Euclidean metric: $U\subseteq\mathbb{R}^2$ is called open if and only if for any $p\in U$ there is some $r>0$ so that

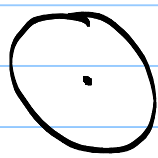
$$p\in B(p,r)\subseteq U.$$

U

Two bases for this topology are as follows:

$$\mathcal{B}_1 = \{ B_{d_2}(p,r) \mid p \in \mathbb{R}^2, r > 0 \}$$

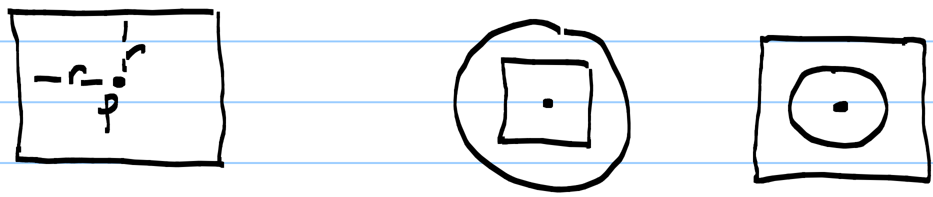
$$B_{d_2}(p,r) = \{ q \in \mathbb{R}^2 \mid d_2(p,q) < r \}.$$



$$d_2((x_1,y_1),(x_2,y_2)) = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

$$\mathcal{B}_2 = \{ B_{d_1}(p,r) \mid p \in \mathbb{R}^2, r > 0 \}$$

$$d_1((x_1,y_1),(x_2,y_2)) = \max\{ |x_1-x_2|, |y_1-y_2| \}.$$



Note that $\mathcal{B}_1$ and $\mathcal{B}_2$ generate the same topology on $\mathbb{R}^2$, the standard topology.

<u>Lemma:</u> Let X be a set, let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathcal{B}$.

<u>Proof:</u> Let $U \subseteq X$ be an open subset of $\mathcal{T}$. Then for any $x \in U$ there is some $B_x \in \mathcal{B}$ so that $x \in B_x \subseteq U$. Then

$$U = \bigcup_{x \in U} \{x\} \subseteq \bigcup_{x \in U} B_x \subseteq U.$$

Hence, $U = \bigcup_{x \in U} B_x$.

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On the other hand, by the definition each basis element is open in $\mathcal{T}$. Hence, their arbitrary unions are also open in X. ∎

Lemma: Let X be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open U of X and each $x \in U$, there is an element C of $\mathcal{C}$ such that $x \in C \subseteq U$. Then $\mathcal{C}$ is a basis for the topology of X.

Proof: First let us show that $\mathcal{C}$ is a basis for a topology on X.

1) Let $x \in X$. Since $U = X$ is an open set and $x \in X$ by the hypothesis there is some $C \in \mathcal{C}$ so that $x \in C \subseteq U$.

2) Let C_1 and C_2 be in $\mathcal{C}$ and $x \in C_1 \cap C_2$. Since C_1 and C_2 are open so is $C_1 \cap C_2$. Again by the assumption there is some $C_3 \in \mathcal{C}$ so that $x \in C_3 \subseteq C_1 \cap C_2$.

Hence, the collection $\mathcal{C}$ is a basis for a topology. Let $\mathcal{T}'$ be the topology generated by $\mathcal{C}$. If $\tilde{\mathcal{T}}$ is the topology of X then $\mathcal{T}'$ is contained in $\tilde{\mathcal{T}}$, because $\mathcal{C}$ consists of elements of $\tilde{\mathcal{T}}$ and elements of $\mathcal{T}'$ are arbitrary union of elements of $\mathcal{C}$, which are in $(X, \tilde{\mathcal{T}})$.

So we have $\mathcal{T}' \subseteq \tilde{\mathcal{T}}$.

For the converse direction let $U \in \tilde{\mathcal{T}}$. By the assumption for any $x \in U$ there is some $C_x \in \mathcal{C}$ so that $x \in C_x \subseteq U$. Then

$$U = \bigcup_{x \in U} \{x\} \subseteq \bigcup_{x \in U} C_x \subseteq U \text{ and hence } U = \bigcup_{x \in U} C_x.$$

Hence, $U \in \mathcal{T}'$. Therefore, $\mathcal{T} \subseteq \mathcal{T}'$ and thus $\mathcal{T} = \mathcal{T}'$, in particular, $\mathcal{C}$ is a basis for $\mathcal{T}$ also. ∎

Lemma: Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively, on a set X. Then the followings are equivalent:

1) $\mathcal{T}'$ is finer than $\mathcal{T}$.
2) For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.

Proof: $(1 \Rightarrow 2)$ Let $x \in X$ and $B \in \mathcal{B}$ a basis element with $x \in B$. Since $\mathcal{B}$ is a basis for $\mathcal{T}$, B is open in $\mathcal{T}$ and thus in $\mathcal{T}'$ since $\mathcal{T} \subseteq \mathcal{T}'$. Now since $\mathcal{B}'$ is a basis for $\mathcal{T}'$ there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.
This finishes the proof.

$(2 \Rightarrow 1)$ Let $U \in \mathcal{T}$. must show: $U \in \mathcal{T}'$.
Let $x \in U$. Since $\mathcal{B}$ is a basis for $\mathcal{T}$ there is some $B \in \mathcal{B}$ s.t. $x \in B_x \subseteq U$. By by (2) there is some $B_x' \in \mathcal{B}$ so that $x \in B_x' \subseteq B_x$.
In particular,

$$U = \bigcup_{x \in U} \{x\} \subseteq \bigcup_{x \in U} B_x' \subseteq U \text{ and hence,}$$

$$U = \bigcup_{x \in U} B_x'.$$ Since each $B_x' \in \mathcal{B}'$, $U \in \mathcal{T}'$.

Hence, $\mathcal{T} \subseteq \mathcal{T}'$. ∎

Definition: If $\mathcal{B}$ is the collection of all open intervals in the real line,

$$(a,b) = \{x \in \mathbb{R} \mid a < x < b\},$$

the topology generated by $\mathcal{B}$ is called the standard topology on the line.

Unless stated otherwise, we will assume that $\mathbb{R}$ is equipped with the standard topology.

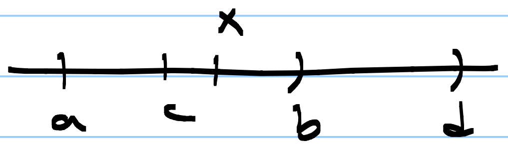
Consider the following basis $\mathcal{B}'$ consisting of all half-open intervals of the form

$$[a, b) = \{x \mid a \leq x < b\}.$$

Claim: $\mathcal{B}'$ is a basis for a topology on $\mathbb{R}$.

Proof: 1) Let $x \in \mathbb{R}$ then $x \in [x, x+1) \in \mathcal{B}'$.

2) Let $x \in [a,b) \cap [c,d)$. So $x \in [e,f)$ where



$e = \max\{a,c\}$ and
$f = \min\{b,d\}$.

Moreover, $x \in [e, f) \subseteq [a,b) \cap [c,d)$, because

$$\max\{a,c\} = e \leq x < f = \min\{b,d\}.$$

Hence, $\mathcal{B}'$ is a basis for a topology on $\mathbb{R}$. The topology generated by $\mathcal{B}'$ is called the lower limit topology on $\mathbb{R}$, and will be denoted $\mathbb{R}_\ell$.

Let K denote the set of all numbers of the form $1/n$, $n \in \mathbb{Z}^+$ and let $\mathcal{B}''$ be the collection of all open intervals (a,b), along with all sets

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of the form $(a,b) \setminus K$. The topology generated by $\mathcal{B}''$ will be called the K-topology on $\mathbb{R}$.

<u>Claim:</u> $\mathcal{B}''$ is a basis for a topology on $\mathbb{R}$.

Proof is left as an exercise.

The topology generated by $\mathcal{B}''$ is denoted as $\mathbb{R}_K$.

<u>Lemma:</u> The topologies $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are strictly finer than the standard topology on $\mathbb{R}$, but they are <u>not</u> comparable with one another.

<u>Proof:</u> Let $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}''$ denote the topologies of $\mathbb{R}_{std}$, $\mathbb{R}_\ell$ and $\mathbb{R}_K$.

<u>$\mathbb{R}_\ell > \mathbb{R}_{std}$:</u> Let (a,b) be an open interval. Then

$$(a,b) = \bigcup_{n=1}^{\infty} [a+\tfrac{1}{n}, b),$$

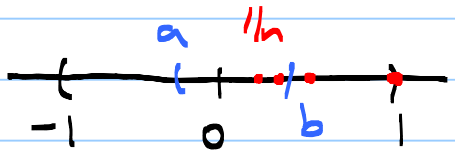
because if $x \in (a,b)$ then $x > a$. Since $\lim (a+1/n) = a$ and $(a+1/n)$ is decreasing there is some n_0 so that $a + 1/n_0 < x$ and then $x \in [a+1/n_0, b)$.

Since $[a+1/n, b)$ are open in $\mathbb{R}_\ell$ so is their union (a,b). Finally, since open intervals (a,b) form a basis for $\mathbb{R}_{std}$ we see that $\mathbb{R}_\ell$ is stronger than $\mathbb{R}_{std}$.

On the other hand, $[0,1)$ is open in $\mathbb{R}_\ell$ but <u>not</u> open in $\mathbb{R}_{std}$. Hence, $\mathbb{R}_\ell$ is strictly stronger than $\mathbb{R}_{std}$.

$\mathbb{R}_K > \mathbb{R}_{std}$: Since any open interval (a,b) is in $\mathcal{B}''$, $\mathbb{R}_K$ is stronger than $\mathbb{R}_{std}$.
Note that $B = (-1,1)\setminus K$ is a basis element for $\mathbb{R}_K$, so that it is open in $\mathbb{R}_K$.

Claim: $B = (-1,1)\setminus K$ is not open in $\mathbb{R}_{std}$.
Proof:



$K = \{1, 1/2, 1/3, \ldots\}$

Note that $0 \in B$ since $0 \notin K$. Assume on the contrary that B is open in $\mathbb{R}_{std}$. Then there must be a basis element (a,b) of $\mathbb{R}_{std}$ so that
$0 \in (a,b) \subseteq B = (-1,1)\setminus K$. Then there is some $n \in \mathbb{Z}^+$ so that $0 < 1/n < b$ and hence

$1/n \in (a,b) \subseteq B = (-1,1)\setminus K$, which is a contradiction since $1/n \in K$. Hence, B is not open in $\mathbb{R}_{std}$ so that $\mathbb{R}_K$ is strictly finer than $\mathbb{R}_{std}$.

Showing that $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are not comparable is left as an exercise! ∎

Definition: A subbasis $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X. The topology generated by the subbasis $\mathcal{S}$ is defined to be the collection $\mathcal{T}$ of all unions of finite intersections of elements of $\mathcal{S}$.

must check that $\mathcal{T}$ is indeed a topology:
1) $\emptyset$, X
2) If $\{U_\alpha\}_{\alpha\in A}$ a collection of open sets in $\mathcal{T}$, then

each U_α is a union of finite intersections of elements of $\mathcal{S}$. Then $\bigcup U_\alpha$ is also a union of finite intersections of elements of $\mathcal{S}$.

3) Let U_1 and U_2 be open in $\mathcal{T}$. Then we have

$$U_1 = \bigcup_\alpha (B_1^\alpha \cap \dots \cap B_{n_\alpha}^\alpha), \quad B_i^\alpha \in \mathcal{S}.$$

$$U_2 = \bigcup_\lambda (B_1^\lambda \cap \dots \cap B_{n_\lambda}^\lambda), \quad B_i^\lambda \in \mathcal{S}.$$

Then $U_1 \cap U_2 = \bigcup_{\alpha,\lambda} (B_1^\alpha \cap \dots \cap B_{n_\alpha}^\alpha \cap B_1^\lambda \cap \dots \cap B_{n_\lambda}^\lambda)$

so that $U_1 \cap U_2$ is also a union of finite intersections of elements of $\mathcal{S}$. Hence, $U_1 \cap U_2$ is in $\mathcal{T}$. By induction it follows that $U_1 \cap \dots \cap U_n \in \mathcal{T}$ if each $U_i \in \mathcal{T}$.

Remark: If $\mathcal{S}$ is a basis for a topology and $\mathcal{B}$ is the collection consisting of finite intersections of elements of $\mathcal{S}$ then $\mathcal{B}$ is a basis for a topology. Moreover, the topologies generated by $\mathcal{S}$ and $\mathcal{B}$ are the same.

§ 14. The Order Topology:

Let X be a simply ordered set with order "$<$". (i.e., if $x, y \in X$ then we have either $x<y$, $y<x$ or $x=y$). Let's first define intervals:

$(a,b) = \{x \in X \mid a<x<b\}$ open interval

$(a,b] = \{x \in X \mid a<x\leq b\}$ half-open interval

$[a,b) = \{x \in X \mid a \leq x < b\}$ half-open interval
$[a,b] = \{x \in X \mid a \leq x \leq b\}$ closed interval

<u>Definition</u>: Let X be a set with simple order $<$. Assume that X has at least two points. Let $\mathcal{B}$ be collection of all sets of the following types:

1) All open intervals (a,b) in X,
2) All intervals of the form $(a, b_0]$, where b_0 is the largest element (if any) of X.
3) All intervals of the form $[a_0, b)$, where a_0 is the smallest element (if any) of X.

<u>Claim</u>: $\mathcal{B}$ is a basis for a topology on X.

<u>Proof</u>: We have to show two things:

1) Let $x \in X$.
<u>Case 1</u>. $x = b_0$ the largest element of X
Since X has at least two elements there is some $y \in X$ and $y < b_0 = x$. Then $x \in (y, b_0]$.
<u>Case 2</u>. $x = a_0$ the smallest element of X.
Since X has at least two elements there is one $y \in X$ with $y > a_0 = x$. Then $x \in [a_0, y)$.
<u>Case 3</u>. x is neither the smallest nor the largest element. Then there are elements $y, z \in X$ so that $y < x < z$. Hence, $x \in (y,z)$.

Hence, every point of X is contained in some element of $\mathcal{B}$.

2) Let I, J be two elements of $\mathcal{B}$ and $x \in I \cap J$.
<u>must show</u>: There is some $K \in \mathcal{B}$ so that

$x \in K \subseteq R \cap J$.

The rest is left as an exercise!

The topology generated by $\mathcal{B}$ is called the order topology on ordered set $(X, <)$.

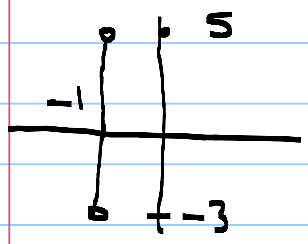
<u>Example:</u> If $<$ is the usual order on $\mathbb{R}$ the order topology on $(\mathbb{R}, <)$ is just the standard topology.

<u>Example:</u> Let $X = \mathbb{R} \times \mathbb{R}$ and $<$ be the dictionary order on X: If $(x_1, y_1), (x_2, y_2) \in \mathbb{R} \times \mathbb{R}$ then $(x_1, y_1) < (x_2, y_2)$ if and only if

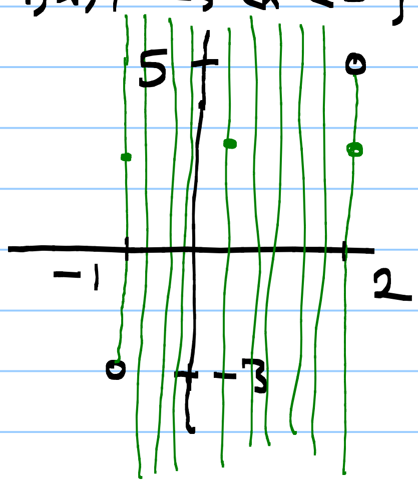
$$x_1 < x_2 \quad \text{or} \quad (x_1 = x_2 \text{ and } y_1 < y_2).$$

Note that X has no smallest or largest element.

a) $((-1,-3), (-1,5)) = \{(-1, x) \mid -3 < x < 5\}$



b) $((-1,-3), (2,5))$



3) Let $X = (0,1]$ with its usual order. Then X has a largest element. Then $\mathcal{B}$ has the following types of intervals:

1) (a,b), $0 < a < b < 1$.

2) $(a,1]$ $0 < a < 1$

§15. The Product Topology on $X \times Y$.

Let X and Y be topological spaces, say with topologies $\mathcal{T}_X$ and $\mathcal{T}_Y$.

Let $\mathcal{B}$ be the collection of all subsets $U \times V$ of $X \times Y$, where U and V are open in X and Y, respectively. In other words, $U \in \mathcal{T}_X$ and $V \in \mathcal{T}_Y$.

Claim: $\mathcal{B}$ is a basis for a topology on $X \times Y$, and the topology generated by $\mathcal{B}$ is called the product topology on $X \times Y$.

Proof: 1) Let $(x,y) \in X \times Y$. Since $X \in \mathcal{T}_X$ and $Y \in \mathcal{T}_Y$, $X \times Y \in \mathcal{B}$ and thus we are done.

2) Let $U_1 \times V_1$ and $U_2 \times V_2$ be elements of $\mathcal{B}$ and let $(x,y) \in (U_1 \times V_1) \cap (U_2 \times V_2)$. Then $x \in U_1 \cap U_2$ and $y \in V_1 \cap V_2$. Clearly, $U_1 \cap U_2$ is open in X and $V_1 \cap V_2$ is open in Y, so that $(U_1 \cap U_2) \times (V_1 \cap V_2) \in \mathcal{B}$.
Finally, since $(x,y) \in (U_1 \cap U_2) \times (V_1 \cap V_2)$ and $(\underline{U_1} \cap \underline{U_2}) \times (\underline{V_1} \cap \underline{V_2}) \subseteq (\underline{U_1} \times \underline{V_1}) \cap (\underline{U_2} \times \underline{V_2})$ so that we are done. ∎

Theorem: If $\mathcal{B}$ is a basis for X and $\mathcal{C}$ is a basis for Y, then the collection

$$\mathcal{D} = \{ B \times C \mid B \in \mathcal{B},\ C \in \mathcal{C} \}$$ is a basis for the product topology on $X \times Y$.

Proof: Since each $B \in \mathcal{B}$ and $C \in \mathcal{C}$ are open in X and Y, respectively, $B \times C$ is open in the product topology $X \times Y$. Let $W \subseteq X \times Y$ be an open in $X \times Y$ and $(x,y) \in W$ a point in W.

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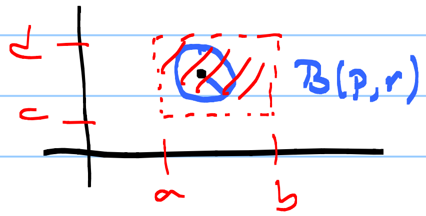
<u>must find</u>: $B\times C \in \mathcal{D}$ so that $(x,y)\in B\times C \subseteq W$.

Since W is an open set and $(x,y)\in W$ there is a basis element of the form $U\times V$, where $(x,y)\in U\times V \subseteq W$. Since $x\in U$ and U is open in X then is a basis element $B\in\mathcal{B}$ with $x\in B\subseteq U$. Similarly, there is a basis element $C\in\mathcal{C}$ with $y\in C\subseteq V$. Hence, $(x,y)\in B\times C\subseteq U\times V\subseteq W$.

This finishes the proof. ∎

<u>Example:</u> Consider the product topology on $\mathbb{R}\times\mathbb{R}$, where each $\mathbb{R}$ has the standard topology. We know that open intervals of the form (a,b) forms a basis for the standard topology on $\mathbb{R}$ and thus by the above theorem the collection below is a basis for the product topology on $\mathbb{R}\times\mathbb{R}$.

$$\mathcal{D}=\{(a,b)\times(c,d) \mid a<b,\ c<d \in\mathbb{R}\}.$$



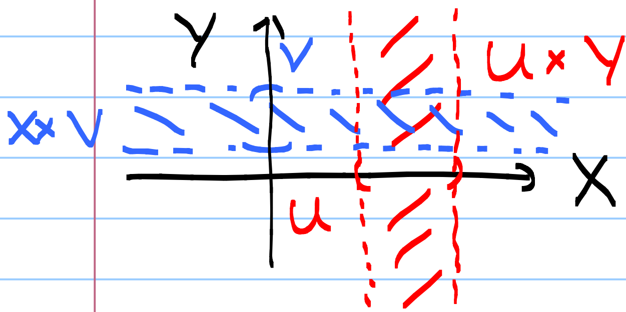
Note that $\mathcal{C}=\{B(p,r)\mid p\in\mathbb{R}^2,\ r>0\}$ is a basis for the product topology on $\mathbb{R}^2$, because for every basis element $(a,b)\times(c,d)$ and a point $(x,y)\in(a,b)\times(c,d)$ we can find some $B(p,r)\in\mathcal{C}$ so that $(x,y)\in B(p,r)\subseteq(a,b)\times(c,d)$.

<u>Definition:</u> Let $\pi_1: X\times Y\to X$ be defined by the equation $\pi_1(x,y)=x$ and $\pi_2: X\times Y\to Y$ be defined by the equation $\pi_2(x,y)=y$, called the projection maps onto the first and second factors, respectively.

Clearly, π_1 and π_2 are surjective.

For any open subset $U \subseteq X$ and $V \subseteq Y$ we have

$$\pi_1^{-1}(U) = U \times Y \quad \text{and} \quad \pi_2^{-1}(V) = X \times V.$$



Note that $(X \times V) \cap (U \times Y) = U \times V$ and therefore the collection $\mathcal{S} = \{ U \times Y, X \times V \mid U \subseteq X, V \subseteq Y \text{ open}\}$ forms a subbasis for the product topology on $X \times Y$.

<u>Theorem</u>: The collection

$$\mathcal{S} = \{ \pi_1^{-1}(U) \mid U \text{ is open in } X\} \cup \{\pi_2^{-1}(V) \mid V \text{ is open in } Y\}$$

is a subbasis for the product topology on $X \times Y$.

§16. <u>The Subspace Topology</u>:

Let X be a topological space with topology $\mathcal{T}$. If Y is a subset of X, then the collection

$$\mathcal{T}_Y = \{ U \cap Y \mid U \in \mathcal{T}\}$$ defines a topology on Y,

called the subspace topology.

<u>Exercise</u>: Show that $\mathcal{T}_Y$ is a basis for a topology on Y.

<u>Lemma</u>: If $\mathcal{B}$ is a basis for the topology on X then $\mathcal{B}_Y = \{ B \cap Y \mid B \in \mathcal{B}\}$ is a basis for the

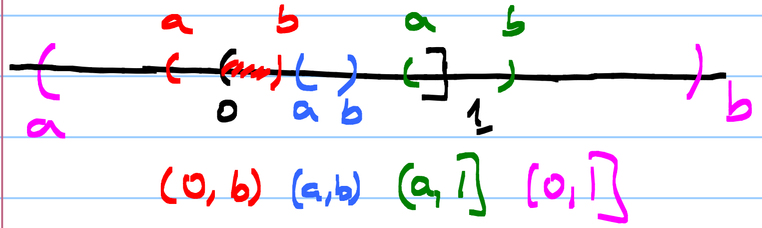
subspace topology on Y.

Proof: must prove: For any open set $U\cap Y$ of Y and $x\in U\cap Y$ there is some element $B\cap Y$ of $\mathcal{B}_Y$ so that $x\in B\cap Y\subseteq U\cap Y$.

Since by definition U is open in X and $x\in U$ there is some basis element $B\in\mathcal{B}$ with $x\in B\subseteq U$. Then $x\in B\cap Y\subseteq U\cap Y$. ∎

Example: Consider $Y=[0,1]$ as a subspace of $\mathbb{R}_{std}$. Then the collection

$$\{(a,b)\cap[0,1] \mid a<b\in\mathbb{R}\}.$$



Lemma: Let Y be a subspace of X. If U is open in Y and Y is open in X, then U is open in X.

Proof: Since U is open in Y, $U=W\cap Y$ where W is open in X. Since both W and Y are open in X so is their intersection $U=W\cap Y$. ∎

Theorem: If A is a subspace of X and B is a subspace of Y, then the product topology on $A\times B$ is the same as the subspace topology on $A\times B$ considered as a subset of the product space $X\times Y$.

Proof: Let $\mathcal{T}$ and $\mathcal{T}'$ denote the product topology on $A \times B$ and the subspace topology on $A \times B$, inherited from $X \times Y$, respectively.

$\mathcal{T} \subseteq \mathcal{T}'$: Let $W \in \mathcal{T}$.

must show $W \in \mathcal{T}'$: Let $(x,y) \in W$. Then there are open sets $U \subseteq X$, $V \subseteq Y$ s.t. $x \in A \cap U$, $y \in B \cap V$ s.t. $(x,y) \in (A \cap U) \times (B \cap V) \subseteq W$. (Then $(x,y) \in U \times V$, where $U \times V$ is open in $X \times Y$.)

So $(x,y) \in (A \times B) \cap (U \times V)$, where $(A \times B) \cap (U \times V)$ is open in the subspace topology on $A \times B$.

Note that $(A \times B) \cap (U \times V)_{(x,y)} \overset{(=)}{\subseteq} (A \cap U) \times (B \cap V) \subseteq W$.

Then $W = \bigcup_{(x,y) \in W} \{(x,y)\} \subseteq \bigcup_{(x,y) \in W} (A \times B) \cap (U \times V)_{(x,y)} \subseteq W$

$\Rightarrow W = \bigcup_{(x,y) \in W} (A \times B) \cap (U \times V)$ so that $W \in \mathcal{T}'$!

Hence, $\mathcal{T} \subseteq \mathcal{T}'$!

Next we must show that any $W \in \mathcal{T}'$ lies in $\mathcal{T}$. I leave this as an exercise! ●

§17. Closed Sets and Limit Points:

A subset A of a topological space X is called closed if $X \setminus A$ is an open subset.

Theorem: Let X be a topological space. Then the following conditions hold:

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1) $\emptyset$ and X are closed.
2) Arbitrary intersections of closed sets is closed.
3) Finite unions of closed sets is closed.

Proof: 1) Since $\emptyset$ and X are open $X \setminus \emptyset = X$ and $X \setminus X = \emptyset$ are both closed.

2) Let $\{A_\alpha\}_{\alpha \in \Lambda}$ be a collection of closed sets in X. Then
$$X \setminus \left(\bigcap_\alpha A_\alpha\right) = \bigcup_\alpha (X \setminus A_\alpha) \text{ is open since each } X \setminus A_\alpha \text{ is open.}$$
Hence, $\bigcap_\alpha A_\alpha$ is closed.

3) If $A_1, \ldots, A_n$ are closed then

$X \setminus (A_1 \cup \cdots \cup A_n) = (X \setminus A_1) \cap \cdots \cap (X \setminus A_n)$ is open since each $X \setminus A_i$ is open.

Definition: Let Y be a subspace of a topological space X. A set A is closed in Y if A is a subset of Y and if A is closed in the subspace topology of Y.

Theorem: Let Y be a subspace of X. Then a set A is closed in Y if and only if it equals the intersection of a closed set in X with Y.

Proof: ($\Leftarrow$) Let $A \subseteq Y$ be a subset and assume that $A = Y \cap C$, for some closed subset C of X.
must show: A is closed in Y.
Let $U = Y \setminus A$. Since C is closed in X, $X \setminus C$ is open in X. So, $(X \setminus C) \cap Y$ is open in Y. Note that $Y \cap (X \setminus C) = Y \setminus C = Y \setminus A$

Since $A = Y \cap C$. So, $Y \setminus A$ is open in Y and thus A is closed in Y.

($\Leftarrow$) Let $A \subseteq Y$ be closed in Y. Then $Y \setminus A$ is open in Y. So there is some open U in X so that $Y \setminus A = U \cap Y$. Let $C = X \setminus U$, which is closed in X. Note that

$$C \cap Y = (X \setminus U) \cap Y = Y \setminus (Y \cap U)$$
$$= Y \setminus (Y \setminus A)$$
$$= A.$$

This finishes the proof. ∎

<u>Theorem</u>: Let Y be subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X.

<u>Proof</u>: Exercise.

<u>Closure and Interior of a Set</u>: Let X be a topological space and A a subset of X. Then the interior of A is defined to be the union of all open subsets of X contained in A:

$$\operatorname{Int} A = \bigcup_{\substack{U \subseteq X \text{ open} \\ U \subseteq A}} U \qquad (\operatorname{Int} A \subseteq A)$$

Note that $\operatorname{Int} A$ is open in X and indeed it is the largest open subset contained in A.

The closure of A is defined to be the intersection of all closed subsets of X containing A:

$$\operatorname{Cl} A = \overline{A} = \bigcap_{\substack{C \subseteq X \text{ closed} \\ A \subseteq C}} C \qquad (A \subseteq \overline{A})$$

Note that $\bar{A}$ is closed in X and indeed it is the smallest closed set containing A.

<u>Remark</u>: Note that a subset A of X is open if and only if $A = \operatorname{Int} A$. Similarly, A is closed if and only if $A = \bar{A}$.

<u>Theorem</u>: Let Y be a subspace of X; let A be a subset of Y; let $\bar{A}$ denote the closure of A in X. Then the closure of A in Y is $\bar{A} \cap Y$.

<u>Proof</u>: Let B denote the closure of A in Y.

Since $\bar{A}$ is closed in X, $\bar{A} \cap Y$ is closed in Y. Clearly, $A \subseteq \bar{A} \cap Y$. In particular, B is contained in $\bar{A} \cap Y$. So $\underline{B \subseteq \bar{A} \cap Y}$.

B is closed in Y and thus $B = Y \cap C$ for some closed subset C of X. So

$A \subseteq B \subseteq C \subseteq X$, where C is closed in X.

In particular, $\bar{A} \subseteq C$. So $\underline{\bar{A} \cap Y} \subseteq Y \cap C = \underline{B}$.

Hence, $B = \bar{A} \cap Y$. ■

<u>Theorem</u>: Let A be a subset of the topological space X.

a) Then $x \in \bar{A}$ if and only if every open set U containing x intersects A.

b) Supposing the topology of X is given by a basis, then $x \in \bar{A}$ if and only if every basis element B containing x intersects A.

<u>Proof</u>: a) Let $x \in \bar{A}$. Let $U \subseteq X$ be an open set with $x \in U$.

<u>must show</u>: $U \cap A \neq \emptyset$.

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Assume on the contrary that $U \cap A = \emptyset$. Then $A \subseteq X \setminus U$, where $X \setminus U$ is closed in X. Hence $\overline{A} \subseteq X \setminus U$. However, this is a contradiction since $x \in U \cap \overline{A} \subseteq U \cap (X \setminus U) = \emptyset$.

Now let $x \in X$ so that every open set U containing x intersects with A.

Must show: $x \in \overline{A}$.

Assume on the contrary that $x \notin \overline{A}$. Then $x \in X \setminus \overline{A} =: U$, which is an open set. Now by the assumption $U \cap A \neq \emptyset$, which is a contradiction since

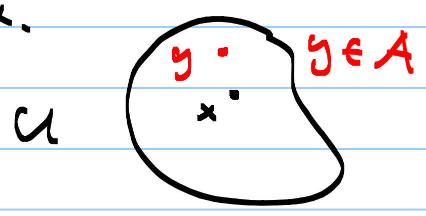
$$\emptyset \neq U \cap A = (X \setminus \overline{A}) \cap A = \emptyset \quad \text{since } A \subseteq \overline{A}.$$

Hence, the first statement is proved.

Part (b) is left as an exercise. ∎

Convention: Usually an open set containing a point x is called a neighborhood of the point x.

Limit Points: Let A be a subset of a space X. A point $x \in X$ is called a limit point (cluster point or accumulation point) of A if every neighborhood U of x contains a point from A different than x.



Equivalently, a point $x \in X$ is a limit point of A if and only if x belongs to the closure of $A \setminus \{x\}$.

Theorem: Let $A \subseteq X$ be subspace of a space X, let A' be the set of all limit points of A. Then

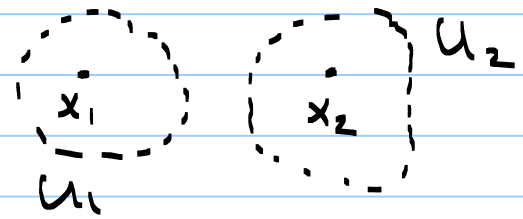
$$\bar{A} = A \cup A'.$$

Proof: Note that every limit point x is in the closure of $A \setminus \{x\}$.
So if $x \in A'$ then $x \in \overline{A \setminus \{x\}} \subseteq \bar{A}$. Hence $A' \subseteq \bar{A}$.

Clearly, $A \subseteq \bar{A}$ and thus $A \cup A' \subseteq \bar{A}$. For the reverse inclusion let $x \in \bar{A}$. Assume that $x \notin A$. Let U be a neighborhood of x. Since $x \in \bar{A}$, $U \cap A$ cannot be empty. However, $x \notin A$ and thus $U \cap A$ contains a point different than x. Hence, x is a limit point of A. So, $x \in A' \subseteq A \cup A'$. Therefore, $\bar{A} \subseteq A \cup A'$ and hence $\bar{A} = A \cup A'$. ∎

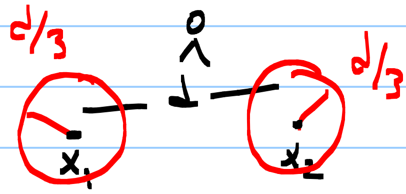
Corollary: A subset of a topological space is closed if and only if it contains all of its limit points.

Hausdorff Spaces: A space X is called Hausdorff if for every pair of distinct points x_1 and x_2 then there are disjoint open subsets U_1 and U_2 so that $x_1 \in U_1$ and $x_2 \in U_2$.



$U_1 \cap U_2 = \emptyset.$

Example: $\mathbb{R}^n_{std}$ is Hausdorff



Example: 

$X=\{a,b\}$
$\mathcal{T}=\{\phi,\{a\},\{a,b\}\}$

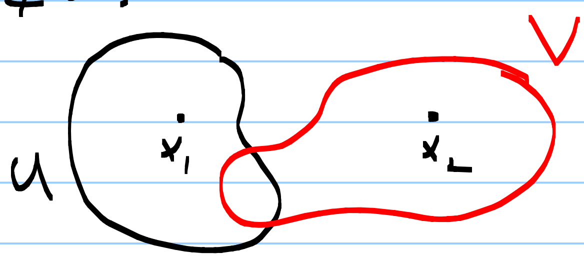
Note $a\neq b$ and the only open set containing b is X, itself. Hence, $(X,\mathcal{T})$ is not Hausdorff.

Theorem: Every finite point set in a Hausdorff space is closed.

Proof: For any $x\in X$ the subset $X\setminus\{x\}$ is open. This is because if $y\in X\setminus\{x\}$ then $y\neq x$ and thus there are open subsets U,V of X with $x\in U$, $y\in V$ and $U\cap V=\phi$.
In particular, $y\in V\subset X\setminus\{x\}$.

So $\{x\}$ is closed in X. Finally, if $A=\{x_1,\dots,x_n\}$ is a finite set then $A=\{x_1\}\cup\{x_2\}\cup\dots\cup\{x_n\}$, which is closed, because it is a finite union of closed subsets. ∎

Definition: A space X is said satisfy the T_1-axiom if for any two distinct points x_1 and x_2 of X there is an open set U in X with $x_1\in U$ and $x_2\notin U$.



Note that every Hausdorff space X is T_1.
We'll see later that there are non Hausdorff T_1-spaces (e.g. the real line with double origin).

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Theorem: Let X be a space satisfying T_1-axiom. Let A be a subset of X. Then a point $x \in X$ is a limit point of A if and only if every neighborhood of x contains infinitely many points of A.

Proof: Let $x \in X$ be a limit point of A. Let U be a neighborhood of x.

must show: $A \cap U$ is infinite.

Note that every one point (and hence finite) subset of a T_1-space is closed.

Now if $A \cap U$ is finite then

$V = U \setminus (A \cap U \setminus \{x\})$ is open in X. Clearly, $x \in V$ since $x \in U$. However, $V \cap A$ is either empty or equal $\{x\}$. So x cannot a limit point of A, which is a contradiction.

Hence, $A \cap U$ must be infinite.

Conversely, if every neighborhood of x contains infinitely many points from A then it clearly contains points from A that are different from x. This finishes the proof. ∎

Theorem: If X is a Hausdorff space, then a sequence of points of X converges to at most one point of X.

Definition: Let (x_n) be a sequence in a space X. We say that (x_n) converges to a point $a \in X$ if for any neighborhood U of a there is some index n_0 so that $n \geq n_0$ implies $x_n \in U$.

Proof of the Theorem: Assume on the contrary that the sequence (x_n) converges to different points say $a \in X$ and $b \in X$. Since $a \neq b$ and our space X is Hausdorff there are open sets U and V so that $a \in U$, $b \in V$ and $U \cap V = \emptyset$.

Since (x_n) converges to a there is some $n_1 \in \mathbb{N}$ so that $n \geq n_1 \Rightarrow x_n \in U$. Similarly, there is some $n_2 \in \mathbb{N}$ so that $n \geq n_2 \Rightarrow x_n \in V$.
Let $n_0 = \max\{n_1, n_2\}$. Then $n_0 \geq n_1$ and $n_0 \geq n_2$. Hence, $x_{n_0} \in U \cap V = \emptyset$, which is a contradiction. Thus, (x_n) may converge to at most one point. ∎

Example: Let X be any set with the indiscrete topology $\mathcal{T} = \{\emptyset, X\}$. Note that if (x_n) is any sequence in X then (x_n) converges to any element y of X.

Theorem: Every simply ordered set is a Hausdorff space in the order topology. The product of two Hausdorff spaces is Hausdorff. A subspace of a Hausdorff space is Hausdorff.

Proof: Let $(X, <)$ be a simply ordered space and $x, y \in X$ with $x \neq y$. Then without loss of generality we may assume that $x < y$. If there is a point $z \in X$ with $x < z < y$, then $x \in (-\infty, z)$, $y \in (z, \infty)$ and $(-\infty, z) \cap (z, \infty) = \emptyset$. Now suppose there is no such $z \in X$. In other words assume $(x, y) = \emptyset$. Then again $x \in (-\infty, y)$, $y \in (x, \infty)$ so that

$$(-\infty, y) \cap (x, \infty) = (x, y) = \emptyset.$$ Hence, X is Hausdorff.

Product of two Hausdorff spaces is Hausdorff: Let X, Y be Hausdorff spaces and let $(x_1, y_1) \neq (x_2, y_2)$ be two points in $X \times Y$. Without loss of generality assume that $x_1 \neq x_2$. Since X is Hausdorff there are open subsets U and V of X with $x_1 \in U$, $x_2 \in V$ and $U \cap V = \emptyset$. Then $(x_1, y_1) \in U \times Y$, $(x_2, y_2) \in V \times Y$, which are both open in $X \times Y$ and

$$(U \times Y) \cap (V \times Y) = (U \cap V) \times Y = \emptyset \times Y = \emptyset.$$

Hence, $X \times Y$ is Hausdorff.

Let $Y \subseteq X$ be a subspace of a Hausdorff space X. If $y_1 \neq y_2$ are two points in Y, then since X is Hausdorff there are open subsets U and V in X with $y_1 \in U$, $y_2 \in V$ and $U \cap V = \emptyset$. Then $y_1 \in U \cap Y$, $y_2 \in V \cap Y$ and $(U \cap Y) \cap (V \cap Y) = (U \cap V) \cap Y = \emptyset \cap Y = \emptyset$. Finally, since $U \cap Y$ and $V \cap Y$ are open in Y, we are done.

§.18 Continuous Functions:

Let X and Y be topological spaces and $f: X \to Y$ any function. f is called continuous if

$f^{-1}(U) \doteq \{x \in X \mid f(x) \in U\}$, the inverse image of

any open subset U of Y is open in X.

Theorem: Let X and Y be topological spaces; let $f: X \to Y$ be a function. Then the following conditions are equivalent:

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1) f is continuous
2) For every subset A of X, one has $f(\overline{A}) \subset \overline{f(A)}$.
3) For every closed subset B of Y, the set $f^{-1}(B)$ is closed in X.
4) For each $x \in X$ and each neighborhood V of $f(x)$ there is a neighborhood U of x so that $f(U) \subseteq V$.

Proof: (1) $\Rightarrow$ (3) Assume that f is continuous. Let $B \subseteq Y$ be a closed subset. Then $U = Y \setminus B$ is open in Y. Since f is continuous $f^{-1}(U)$ is open in X. Thus

$$f^{-1}(B) = f^{-1}(\underline{\underline{Y \setminus U}}) = f^{-1}(Y) \setminus f^{-1}(U)$$

$\Rightarrow f^{-1}(B) = X \setminus f^{-1}(U)$, which is closed since $f^{-1}(U)$ is open.

(3) $\Rightarrow$ (2) So we assume that inverse image of any closed set is closed. Let $A \subseteq X$ be any subset. Then $A \subseteq f^{-1}(\overline{f(A)})$, where $f^{-1}(\overline{f(A)})$ is a closed subset of X.

Since $A \subseteq f^{-1}(\overline{f(A)})$ and $f^{-1}(\overline{f(A)})$ is closed $\overline{A} \subseteq f^{-1}(\overline{f(A)})$. Hence, $f(\overline{A}) \subseteq \overline{f(A)}$.

(2) $\Rightarrow$ (1): Let $U \subseteq Y$ be an open subset.

must show: $f^{-1}(U)$ is open in X.

Let $B = Y \setminus U$, which is a closed subset of Y.
$Y = U \cup (Y \setminus U)$. Then

$X = f^{-1}(U) \cup f^{-1}(Y \setminus U)$, which is a disjoint union.

By (2), $f(\overline{f^{-1}(Y\setminus U)}) \subseteq \overline{f(f^{-1}(Y\setminus U))}$
$$\subseteq \overline{Y\setminus U} = Y\setminus U,$$
since U is open in Y.

In particular, $\overline{f^{-1}(Y\setminus U)} \subseteq f^{-1}(Y\setminus U)$.

Hence, $f^{-1}(Y\setminus U) = \overline{f^{-1}(Y\setminus U)}$ so that $f^{-1}(Y\setminus U)$ is closed in X. Thus, $f^{-1}(U)$ is open in X.

So far we've shown (1) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1).

(1) $\Rightarrow$ (4) $\Rightarrow$ (1) is left as an exercise. ∎

<u>Definition</u>: Let X and Y be topological spaces. A continuous bijection $f: X \to Y$ is called a homeomorphism if $f^{-1}: Y \to X$ is also continuous.

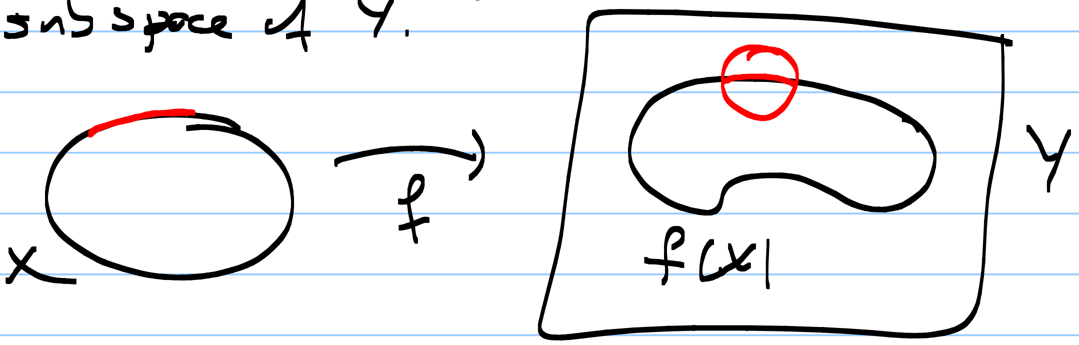
<u>Example</u>: Let $f: (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ be the identity map, $f(x) = x$, $\forall x \in X$, where $X = \{a, b\}$ and

$\mathcal{T}_1 = \{\emptyset, \underline{\{a\}}, \{a,b\}\}$, $\mathcal{T}_2 = \{\emptyset, \{a,b\}\}$.

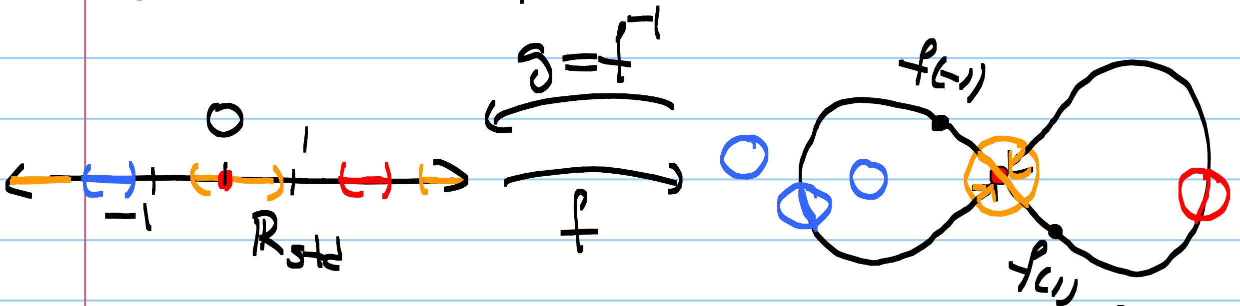
Then the bijection f is clearly continuous. However, $g = f^{-1}: (X, \mathcal{T}_2) \to (X, \mathcal{T}_1)$ is not continuous, because,

$g^{-1}(\{a\}) = f(\{a\}) = \{a\}$, which is not open in $\mathcal{T}_2$. Hence, f is <u>not</u> a homeomorphism.

Definition: A function $f: X \to Y$ is called an *embedding* if f is a homeomorphism onto its image $f(X)$ considered as a subspace of Y.



Example: Consider the map $f: \mathbb{R}_{std} \to \mathbb{R}^2_{std}$ as described below:



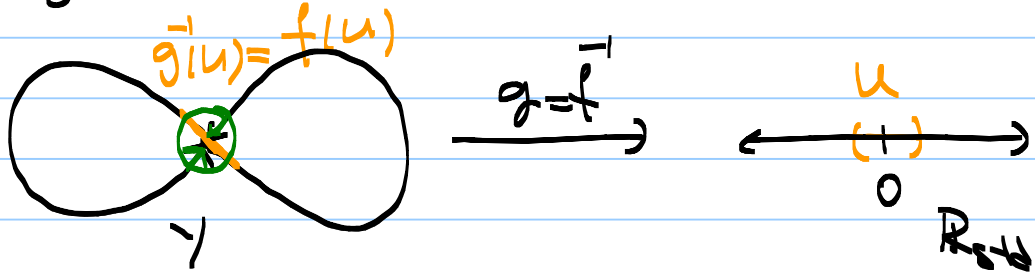
f is clearly 1-1 and onto its image. $f(\mathbb{R}_{std}) = Y$
f is also continuous.

Question: Is f an embedding?

$$f: \mathbb{R}_{std} \longrightarrow f(\mathbb{R}_{std}) = Y \subseteq \mathbb{R}^2_{std}$$

Answer: No! Indeed, f^{-1} is not continuous.

Let $g = f^{-1}: Y \longrightarrow \mathbb{R}_{std}$.



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Claim: $g^{-1}(U) = f(U)$ is not open in the subspace Y. Hence, $g = f^{-1}$ is not continuous.

Remark: Note that the following collection $\mathcal{B}$ is a basis for a topology on the set $\mathbb{R}$:

$$\mathcal{B} = \{(a,b) \mid a,b<0\} \cup \{(a,b) \mid a,b>0\}$$
$$\cup \{(a,b) \cup (c,\infty) \cup (-\infty, d) \mid a<0,\ b>0,\ c,d \in \mathbb{R}\}.$$

If $\mathcal{T}$ is the topology generated by $\mathcal{B}$ then

$f : (\mathbb{R}, \mathcal{T}) \longrightarrow Y \subseteq \mathbb{R}^2_{std}$ is a homeomorphism.

Clearly $\mathcal{T}$ is weaker than $\mathbb{R}_{std}$.

Constructing Continuous Functions:

Theorem: Let X, Y and Z be topological spaces.

a) (Constant Function) If $f: X \to Y$ maps all the points into the single point $y_0 \in Y$, then f is continuous.

b) (Inclusion map) If A is a subspace of X, the inclusion map $j: A \to X$, $j(a) = a$, for all $a \in A$, is continuous.

c) (Composition) If $f: X \to Y$ and $g: Y \to Z$ are continuous functions then so is $g \circ f: X \to Z$.

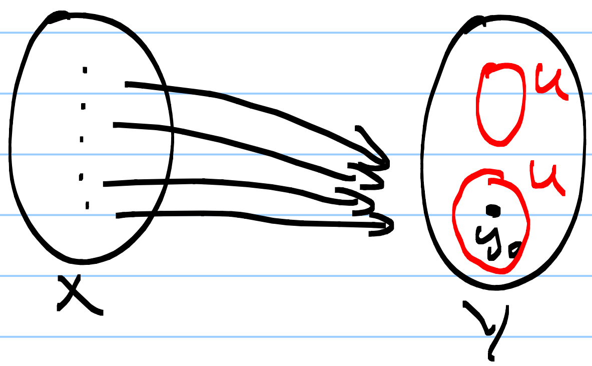
d) (Restricting the Domain) If $f: X \to Y$ is continuous and $A \subseteq X$ is a subspace of X, then the restricted function $f|_A : A \to Y$, $f|_A(x) = f(x)$, for all $x \in A$, then $f|_A : A \to Y$ is continuous.

e) (Restricting or expanding the range) Let $f: X \to Y$ be continuous. If Z is a subspace of Y containing $f(X)$, then the function $g: X \to Z$, given by $g(x) = f(x)$, obtained by restricting the range of f to Z is continuous. Similarly, if Z is a space containing Y as a subspace then $h: X \to Z$ given by $h(x) = f(x)$, $\forall x \in X$, is also continuous.

f) (Local Formulation of Continuity) A map $f: X \to Y$ is continuous if X can be written as the union of open sets U_α such that $f|_{U_\alpha} : U_\alpha \to Y$ is continuous.

__Proof:__ a) $f: X \to Y$, $f(x) = y_0$, $\forall x \in X$.

If $U \subseteq Y$ is open in Y, then $f^{-1}(U) = X$ if $y_0 \in U$ and $f^{-1}(U) = \emptyset$ if $y_0 \notin U$. Since both X and $\emptyset$ are open in X f is continuous.



b) $A \subseteq X$, $J: A \to X$, $J(x) = x$, for all $x \in A$.

Let $U \subseteq X$ be an open subset. Then $J^{-1}(U) = U \cap A$, which is clearly open by the definition of subspace. Hence, J is continuous.

c) Let $f: X \to Y$ and $g: Y \to Z$ be continuous.

Then for any open set U in Z the preimage under $g\circ f : X \to Z$ is

$(g\circ f)^{-1}(U) = \{x \in X \mid g(f(x)) \in U\}$

$= \{x \in X \mid f(x) \in g^{-1}(U)\}$.

$= f^{-1}(g^{-1}(U))$, where $g^{-1}(U)$ is open since g is continuous. Hence, $f^{-1}(g^{-1}(U))$ is open since f is also continuous.
Thus, $g\circ f : X \to Z$ is continuous.

d) $f : X \to Y$ continuous, $A \subseteq X$ subspace.

$g = f|_A : A \to Y$, $g(x) = f(x)$, for all $x \in A$.

Then for any open set U of Y, we have

$g^{-1}(U) = \{x \in A \mid g(x) = f(x) \in U\}$

$= f^{-1}(U) \cap A$.

However, $f^{-1}(U)$ is open in X since f is continuous and then $g^{-1}(U) = f^{-1}(U) \cap A$ is open in A.

e) Let $f : X \to Y$ continuous.

i) $f(X) \subseteq Z \subseteq Y$ subspaces.

$g : X \to Z$, $g(x) = f(x)$, for all $x \in X$.

Let $U \subseteq Z$ be an open subset. Then $U = Z \cap V$ for some open subset V of Y.

Then $g^{-1}(U) = \{x \in X \mid g(x) = f(x) \in U\}$

$\Rightarrow g^{-1}(U) = \{x \in X \mid f(x) \in V \cap Z\}$

$= \{x \in X \mid f(x) \in V\}$ since $f(X) \subseteq Z$.

$= f^{-1}(V)$, which is open in X since $f: X \to Y$ is continuous.

f) $f: X \to Y$ is continuous and $Y \subseteq Z$ is a subspace.

$h: X \to Z$, $h(x) = f(x)$, $\forall x \in X$.

Let $U \subseteq Z$ be an open subset. Then

$h^{-1}(U) = \{x \in X \mid h(x) = f(x) \in U\}$

$= \{x \in X \mid f(x) \in U \cap Y\}$

$= f^{-1}(U \cap Y)$, which is open since

$f: X \to Y$ is continuous and $U \cap Y$ is an open subset of Y.

f) $X = \bigcup_\alpha U_\alpha$, $f: X \to Y$ so that $f|_{U_\alpha}: U_\alpha \to Y$ is continuous for all α.

Let $V \subseteq Y$ be an open subset of Y. Then

$f^{-1}(V) = f^{-1}(V) \cap X$

$= f^{-1}(V) \cap \left(\bigcup_\alpha U_\alpha\right)$ $\qquad f: X \to Y$

$= \bigcup_\alpha \left(f^{-1}(V) \cap U_\alpha\right)$ $\qquad f_\alpha = f|_{U_\alpha}: U_\alpha \to Y$

$= \bigcup_\alpha f_\alpha^{-1}(V)$

Since $f_\alpha: U_\alpha \to Y$ is continuous $f_\alpha^{-1}(V)$ is open in U_α. However, U_α is open in X and thus $f_\alpha^{-1}(V)$ is open in X. Finally

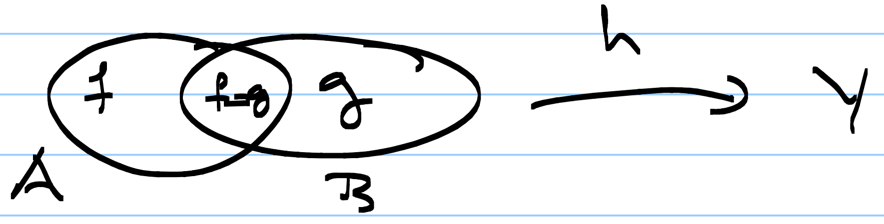
$f^{-1}(V) = \bigcup_\alpha f_\alpha^{-1}(V)$, which is a union of open subsets of X, is also open. Thus, f is continuous. ∎

Local formulation of continuity does not work if we replace open subsets U_α with closed subsets, for infinite union. However, it works for finite union as the lemma below shows:

Lemma (The Pasting Lemma)

Let $X = A \cup B$, where A and B are closed subsets of X. Let $f: A \to Y$ and $g: B \to Y$ be two continuous functions, when A and B are endowed with subspace topology. Assume that $f(x) = g(x)$ for all $x \in A \cap B$. Then the function $h: X \to Y$ defined by

$$h(x) = \begin{cases} f(x) & \text{if } x \in A, \\ g(x) & \text{if } x \in B, \end{cases}$$ is continuous.



Proof: Let $K \subseteq Y$ be a closed subset. Then

$h^{-1}(K) = \{x \in X \mid h(x) \in K\}$

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$= \{x \in A \mid f(x) \in K\} \cup \{x \in B \mid g(x) \in K\}.$

$= f^{-1}(K) \cup g^{-1}(K)$

Since $f: A \to Y$ and $g: B \to Y$ are continuous $f^{-1}(K)$ is closed in A and $g^{-1}(K)$ is closed in B. Since A and B are closed in X, $f^{-1}(K)$ and $g^{-1}(K)$ are both closed in X. Hence,

$h^{-1}(K) = f^{-1}(K) \cup g^{-1}(K)$ is closed in X, so that $h: X \to Y$ is continuous. ∎

Remark: In the proof of the above Pasting Lemma the step where we take unions of closed sets may not for infinite unions. Hence, the Pasting Lemma may not hold for infinite unions.

Theorem (Maps Into Products)
Let $f: A \to X \times Y$ be given by the equation

$f(a) = (f_1(a), f_2(a))$, for some functions $f_1: A \to X$ and $f_2: A \to Y$. Then f is continuous if and only if each f_i is continuous.

Proof: Consider the projection maps $\pi_1: X \times Y \to X$ and $\pi_2: X \times Y \to Y$ given by $\pi_1(x,y) = x$ and $\pi_2(x,y) = y$, for all $(x,y) \in X$.

Note that if $U \subseteq X$ is open then $\pi_1^{-1}(U) = U \times Y$, which is open. Hence, π_1 is continuous. Similarly, π_2 is also continuous.

Also note that since $f(a) = (f_1(a), f_2(a))$

and thus $f_1(a) = \pi_1(f(a))$ and $f_2(a) = \pi_2(f(a))$, for all $a \in A$. Thus $f_1 = \pi_1 \circ f$ and $f_2 = \pi_2 \circ f$.

So, if f is continuous then f_1 and f_2 are continuous, since they are compositions of continuous functions.

For the converse statement, let f_1 and f_2 be continuous. Consider a basis element $U \times V$, where $U \subseteq X$ and $V \subseteq Y$ are basis elements.

Then

$$f^{-1}(U \times V) = \{a \in A \mid f(a) = (f_1(a), f_2(a)) \in U \times V\}.$$

$$= \{a \in A \mid f_1(a) \in U,\ f_2(a) \in V\}.$$

$$= \{a \in A \mid f_1(a) \in U\} \cap \{a \in A \mid f_2(a) \in V\}.$$

$$= f_1^{-1}(U) \cap f_2^{-1}(V).$$

Since both f_1 and f_2 are continuous we see that $f^{-1}(U \times V)$ is open. ■

<u>Remark</u>: This proof does <u>not</u> fit infinite products, since intersection of infinitely many open subsets may <u>not</u> be open.

<u>Remark</u>: Let $f: X \to Y$ be any map and $\mathcal{B}$ a basis for the topology on Y. Then f is continuous if and only if $f^{-1}(B)$ is open in X, for all $B \in \mathcal{B}$.

<u>Proof</u>: If f is continuous then $f^{-1}(B)$ is open because $B \subseteq Y$ is open since it belongs to a basis.

For the other direction assume that $f^{-1}(B)$ is open for all $B \in \mathcal{B}$. Then for any open subset U of Y we have $U = \bigcup_{\substack{B \subseteq U \\ B \in \mathcal{B}}} B$.

Hence, $f^{-1}(U) = f^{-1}\left(\bigcup_{\substack{B\subseteq U \\ B\in\mathcal{B}}} B\right) = \bigcup_{\substack{B\subseteq U \\ B\in\mathcal{B}}} f^{-1}(B)$, which is union of open sets in X.

Hence, $f^{-1}(U)$ is open. ∎

§19. The Product Topology: We already studied the product $X\times Y$ for two spaces. In this section we'll consider arbitrary products.

Definition: Let J be an index set. Given a set X we define a J-tuple of elements of X to be a function $x: J \to X$. If $\alpha \in J$, then we often denote the value $x(\alpha)$ by x_α, rather than $x(\alpha)$, called the α^{th} coordinate of x. Moreover, we denote the function x itself by the symbol

$$(x_\alpha)_{\alpha\in J}.$$

Example: 1) $J = \{1,2,3\}$. So J-tuple in a set X is a function $x: J \to X$.

$(x(1), x(2), x(3)) \longleftrightarrow (x_1, x_2, x_3)$ triples in X.

2) $J = \mathbb{N} = \{0,1,2,3,\ldots\}$. Then a J-tuple in a set X is a function $x: \mathbb{N} \to X$, which is just a sequence in X.

$x: \mathbb{N} \to X$, $\quad x \longleftrightarrow x(0), x(1), x(2), \ldots, x(n)$

$\qquad\qquad\qquad\qquad\quad x_0, x_1, x_2, \ldots, x_n$

$x = (x_n)_{n\in\mathbb{N}}$.

3) $J = \mathbb{R}$. Then a J-tuple in $X = \mathbb{R}$ is

a function $x: \mathbb{R} \to \mathbb{R}$. $x = x(t)$

$$x = (x_t)_{t \in \mathbb{R}}.$$

<u>Definition:</u> Let $\{A_\alpha\}_{\alpha \in J}$ be an indexed family of sets, let $X = \bigcup_{\alpha \in J} A_\alpha$. The Cartesian product of this indexed family of sets, denoted by

$$\prod_{\alpha \in J} A_\alpha$$

is defined to be the set of all J-tuples $(x_\alpha)_{\alpha \in J}$ of elements of X such that $x_\alpha \in A_\alpha$, for any $\alpha \in J$.

In other words,

$$\prod_{\alpha \in J} A_\alpha = \{ x : J \to \bigcup_{\alpha \in J} A_\alpha \mid \underbrace{x(\alpha)}_{= x_\alpha} \in A_\alpha \}$$

<u>Example:</u> $X_1 \times X_2 \times X_3 = \{ \underbrace{(a_1, a_2, a_3)}_{= (a_i)_{i \in J}} \mid a_i \in X_i, i = 1, 2, 3. \}$ $\quad J = \{1, 2, 3\}$

$a: J = \{1, 2, 3\} \to X_1 \cup X_2 \cup X_3$ so that
$a_1 = a(1) \in X_1$, $a_2 = a(2) \in X_2$, $a_3 = a(3) \in X_3$.

<u>Product Topology</u> Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of topological spaces.

Consider the product set

$$\prod_{\alpha \in J} X_\alpha = \{ x : J \to \bigcup_{\alpha \in J} X_\alpha \mid x(\alpha) \in X_\alpha, \forall \alpha \in J \}$$

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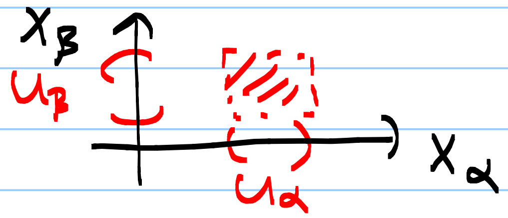
$$= \{ (x_\alpha)_{\alpha \in J} \mid x_\alpha \in X_\alpha, \forall \alpha \in J \}.$$

One may define several topologies on the product $\prod_\alpha X_\alpha$, some of which will be described below:

<u>Box Topology</u>: Consider the collection of subsets

$$\prod_{\alpha \in J} U_\alpha, \text{ where each } U_\alpha \subseteq X_\alpha \text{ is open.}$$

This collection, say $\mathcal{B}$, is a basis for a topology called the Box topology on the product $\prod_{\alpha \in J} X_\alpha$.



<u>Claim</u>: $\mathcal{B}$ is a basis for a topology on $\prod_\alpha X_\alpha$.

<u>Proof</u>: 1) If $x = (x_\alpha) \in \prod_\alpha X_\alpha$, then since $\prod_\alpha X_\alpha \in \mathcal{B}$ we are done with 1^{st} condition for $\mathcal{B}$ to be a base for a topology.

2) Let $x = (x_\alpha) \in (\prod_\alpha U_\alpha) \cap (\prod_\alpha V_\alpha)$. Then let

$$W = (\prod_\alpha U_\alpha) \cap (\prod_\alpha V_\alpha) = \prod_\alpha (U_\alpha \cap V_\alpha) \in \mathcal{B},$$

since $U_\alpha \cap V_\alpha$ is open in X_α. So

$$x \in W \subseteq (\prod_\alpha U_\alpha) \cap (\prod_\alpha V_\alpha). \qquad \blacksquare$$

Example: $J = \mathbb{N}$, $X_n = \mathbb{R}_{std}$, $n \in J = \mathbb{N} = \{1, 2, \dots\}$

Consider $\prod_{n \in \mathbb{N}} X_n = \prod_{n \in \mathbb{N}} \mathbb{R}_{std}$, with box topology.

The the function $f: \mathbb{R}_{std} \to \prod_{n \in \mathbb{N}} \mathbb{R}_{std}$, given by $f(t) = (t, t, t, \dots, t, \dots)$ is not continuous, even though each coordinate of $f(t)$ is continuous.

Let $U = \prod_{n=1}^{\infty} (-1/n, 1/n)$, which is open in the Box Topology.

$$\begin{aligned}
f^{-1}(U) &= \{t \in \mathbb{R}_{std} \mid f(t) \in U\} \\
&= \{t \in \mathbb{R}_{std} \mid (t, t, \dots, t, \dots) \in (-1, 1) \times (-1/2, 1/2) \times \dots \times (-1/n, \tfrac{1}{n}) \times \dots\} \\
&= \{t \in \mathbb{R}_{std} \mid t \in (-1, 1) \cap (-1/2, 1/2) \cap \dots \cap (-1/n, 1/n) \cap \dots\} \\
&= \{t \in \mathbb{R}_{std} \mid |t| < \tfrac{1}{n}, n = 1, 2, \dots\} \\
&= \{0\} \text{ is not open in } \mathbb{R}_{std}.
\end{aligned}$$

To obtain a reasonable topology on the product $\prod_\alpha X_\alpha$ we need to a coarser topology, avoiding arbitrary products of open sets.

Recall that the product topology on $X \times Y$ had a subbasis consisting of elements of the form $\pi_1^{-1}(U) = U \times Y$ and $\pi_2^{-1}(V) = X \times V$.

Definition: Let $\prod_{\alpha \in J} X_\alpha$ be a product of sets where each X_α is a topological space. Let $\pi_\beta : \prod_{\alpha\in J} X_\alpha \longrightarrow X_\beta$ denote the projection map given by $\pi_\beta((x_\alpha)) = x_\beta$. The topology generated by the subbasis $\pi_\alpha^{-1}(U_\alpha)$, where $U_\alpha \subseteq X_\alpha$ is open for each $\alpha\in J$, is called the product topology on $\prod_{\alpha\in J} X_\alpha$.

__Remark:__ $S = \{ \pi_\alpha^{-1}(U_\alpha) \mid \alpha\in J,\ U_\alpha \subseteq X_\alpha \text{ open} \}$

$$\pi_\beta^{-1}(U_\beta) = \prod_{\alpha\in J} \underline{\underline{V_\alpha}}, \qquad V_\beta = U_\beta \text{ and } V_\alpha = X_\alpha,\ \forall\ \alpha\neq\beta$$

__Example:__ $J = \underline{\mathbb{N}} = \{1,2,3,\dots\}$, $X_n = \mathbb{R}_{std}$.

$$\pi_m : \prod_{n=1}^{\infty} \mathbb{R}_{std} \longrightarrow \mathbb{R}_{std}, \qquad \pi_m((x_n)) = x_m,\ m\in\mathbb{N}.$$

$$\pi_m^{-1}((a,b)) = \mathbb{R}\times\mathbb{R}\times\dots\times\mathbb{R}\times \overset{m\text{th coordinate}}{\downarrow}\!(a,b)\times\mathbb{R}\times\dots\times\mathbb{R}\times\dots .$$

__Basis elements:__ Finite intersections of subbasis elements.

$$\mathbb{R}\times\dots\times\mathbb{R}\times\underline{(a_1,b_1)}\times\mathbb{R}\times\dots\times\underline{(a_2,b_2)}\times\dots\times\mathbb{R}\times\dots\times\underline{(a_n,b_n)}\times\mathbb{R}\times\dots\times\mathbb{R}\times\dots$$

__Remark:__ Clearly, each projection function $\pi_\beta : \prod_\alpha X_\alpha \longrightarrow X_\beta$ is continuous.

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Indeed, the product topology on $\prod_\alpha X_\alpha$ is the weakest topology on $\prod_\alpha X_\alpha$ so that each π_β is continuous.

Remark: If J is an infinite index set then the box topology may be stronger than the product topology. For instance if for any $\alpha \in J$ there is some open set $U_\alpha \neq X_\alpha$ then the product $\prod_\alpha U_\alpha$ is open in the box topology but it is not open in the product topology.

Convention: Unless stated otherwise a product of topological spaces is considered with its product topology.

Theorem: Let A_α be a subspace of X_α, for each $\alpha \in J$. Then $\prod_{\alpha \in J} A_\alpha$ is a subspace of $\prod X_\alpha$ if both products are given the box topology or product topology.

Proof: Product Topology. Both products $\prod_\alpha X_\alpha$ and $\prod_\alpha A_\alpha$ are considered with product topology.

must show $\prod_\alpha A_\alpha$ is a subspace of $\prod_\alpha X_\alpha$.

Let $U \subseteq \prod_\alpha A_\alpha$ be an open subset. Then U is a union of basis elements say

$$U = \bigcup_\lambda O_\lambda, \text{ where each } O_\lambda \text{ is a basis element}$$

of the product space. Then O_λ is the intersection of finitely many subbasis elements of the form

$\pi_\beta^{-1}(U_\beta)$ where $\pi_\beta : \prod_\alpha A_\alpha \to A_\beta$ and $U_\beta \subseteq A_\beta$

is an open subset. Since $A_\beta \subseteq X_\beta$ is a subspace
$U_\beta = A_\beta \cap V_\beta$ for some open subset $V_\beta \subseteq X_\beta$.

Then $\underbrace{\pi_{\beta_1}^{-1}(U_{\beta_1}) \cap \cdots \cap \pi_{\beta_k}^{-1}(U_{\beta_k})}_{\text{“}O_\lambda\text{”}} = \pi_{\beta_1}^{-1}(V_{\beta_1}) \cap \cdots \cap \pi_{\beta_k}^{-1}(V_{\beta_k})$
$\cap \prod_\alpha A_\alpha$.

Hence $\pi_{\beta_1}^{-1}(U_{\beta_1}) \cap \cdots \cap \pi_{\beta_k}^{-1}(U_k)$ is open in the subspace topology of $\prod_\alpha A_\alpha$, because $\pi_{\beta_1}^{-1}(V_{\beta_1}) \cap \cdots \cap \pi_{\beta_k}^{-1}(V_{\beta_k})$ is open in $\prod_\alpha X_\alpha$.

Hence the union of $\pi_{\beta_1}^{-1}(U_{\beta_1}) \cap \cdots \cap \pi_{\beta_k}^{-1}(U_{\beta_k})$, which is $U = \bigcup_\lambda O_\lambda$ is open in the subspace topology.

Conversely, let $U \subseteq \prod_\alpha X_\alpha$ be a open subset.

<u>must show:</u> $U \cap \prod_\alpha A_\alpha$ is open in the product

topology. Let $x = (x_\alpha) \in U \cap \prod_\alpha A_\alpha$. Since U is open in $\prod_\alpha X_\alpha$ there are U_α's s.t. $x_\alpha \in U_\alpha \subseteq X_\alpha$ open.

Then $x = (x_\alpha) \in \prod_\alpha U_\alpha \subseteq \prod_\alpha X_\alpha$ and then

$x = (x_\alpha) \in \prod_\alpha U_\alpha \cap \prod_\alpha A_\alpha = \prod_\alpha (U_\alpha \cap A_\alpha)$, where

each $U_\alpha \cap A_\alpha$ is open in the subspace $U_\alpha \cap A_\alpha$.
Since we are in the product topology $U_\alpha = X_\alpha$ for all but finitely many α. Then $U_\alpha \cap A_\alpha = A_\alpha$, for all but finitely many α. Hence, $\prod_\alpha (U_\alpha \cap A_\alpha)$ is a basis element for the product topology.

Thus U is open in the product topology $\prod_\alpha A_\alpha$.

Note that the same argument gives the proof for the box topology also.

Finally, we must show the following:

Let $U \subseteq \prod_\alpha A_\alpha$ be open in the subspace topology when $\prod X_\alpha$ has the box topology. Then we must show: $U \subseteq \prod_\alpha A_\alpha$ is open in the box topology of $\prod_\alpha A_\alpha$.

<u>Exercise:</u> Prove the above statement.

<u>Theorem:</u> If each X_α is a Hausdorff space, then $\prod X_\alpha$ is Hausdorff in both the box and product topology.

<u>Proof</u>: Note that since box topology is stronger than the product topology the open sets, which will separate two points in the product topology, are also open in the box topology. Thus, it is enough to prove the result only for the product topology.

Let $x = (x_\alpha) \neq (y_\alpha) = y$ be two distinct points in $\prod_\alpha X_\alpha$. Then $x_\beta \neq y_\beta$ for some $\beta \in \Lambda$.

Since X_β is Hausdorff there are open subsets U_β and V_β in X_β s.t. $x_\beta \in U_\beta$, $y_\beta \in V_\beta$ and

$U_\beta \cap V_\beta = \phi$. Then $\prod_\alpha A_\alpha$, where $A_\alpha = \begin{cases} X_\alpha & \alpha \neq \beta \\ U_\beta, & \alpha = \beta \end{cases}$

and $\prod_\alpha B_\alpha$, $B_\alpha = \begin{cases} X_\alpha & \alpha \neq \beta \\ V_\beta & \alpha = \beta \end{cases}$ are disjoint

open sets with $x = (x_\alpha) \in \prod_\alpha A_\alpha$ and $y = (y_\alpha) \in \prod_\alpha B_\alpha$.

This finishes the proof. ∎

<u>Theorem</u>: Let $\{X_\alpha\}$ be an indexed family of spaces; let $A_\alpha \subseteq X_\alpha$ for each α. If $\prod_\alpha X_\alpha$ is given either the product or the box topology then

$$\prod_\alpha \overline{A_\alpha} = \overline{\prod A_\alpha}.$$

<u>Proof</u>: $\subseteq$: Let $x = (x_\alpha) \in \prod_\alpha \overline{A_\alpha}$. Choose any

basis element $\prod_\alpha U_\alpha$, where each $U_\alpha \subseteq X_\alpha$ open

and $x_\alpha \in U_\alpha$. Since $x = (x_\alpha) \in \prod_\alpha \overline{A_\alpha}$, $x_\alpha \in \overline{A_\alpha}$, for each $\alpha \in \Lambda$. Hence, $U_\alpha \cap A_\alpha \neq \phi$. Choose some $y_\alpha \in U_\alpha \cap A_\alpha$, $\alpha \in \Lambda$.

Then $y = (y_\alpha) \in (\prod A_\alpha) \cap (\prod U_\alpha)$ and thus

$(\prod A_\alpha) \cap (\prod U_\alpha) \neq \phi$. Hence, $x = (x_\alpha) \in \overline{\prod A_\alpha}$.

<u>$\supseteq$</u>: Take any $x = (x_\alpha) \in \overline{\prod A_\alpha}$.

<u>must show</u>: $x_\alpha \in \overline{A_\alpha}$, for all α, so that $x = (x_\alpha) \in \prod \overline{A_\alpha}$.

Take any open U_α in X_α s.t. $x_\alpha \in U_\alpha$. Then $\prod U_\alpha$ is open in $\prod X_\alpha$. Since, $x = (x_\alpha) \in \overline{\prod A_\alpha}$ the intersection $(\prod_\alpha U_\alpha) \cap (\prod_\alpha A_\alpha) \neq \phi$

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$\Rightarrow U_\alpha \cap A_\alpha \neq \emptyset$ for each $\alpha \in \Lambda$.
Thus, $x_\alpha \in \bar{A}_\alpha$.

Remark: Assume that each $X_\alpha \neq \emptyset$, $\alpha \in \Lambda$.

The Axiom of Choice states that the product set $\prod X_\alpha$ is not empty. In other words, one can pick an element from each X_α.

Theorem: Let $f: A \to \prod_{\alpha \in J} X_\alpha$ be given by the equation

$$f(a) = (f_\alpha(a))_{\alpha \in J},$$

where each $f_\alpha : A \to X_\alpha$ is a function. Let $\prod_\alpha X_\alpha$ have the product topology. Then f is continuous if and only if each f_α is continuous.

Remark: Let $f: \mathbb{R}_{std} \to \prod_{n=1}^{\infty} \mathbb{R}_{std}$ be given by

$f(t) = (t, t, \dots)$, i.e., $f_n : \mathbb{R}_{std} \to \mathbb{R}_{std}$, $t \mapsto t$, $t \in \mathbb{R}_{std}$. Consider the product $U = \prod_{n=1}^{\infty} (-1/n, 1/n)$, which is open in the box topology.

However, $f^{-1}(U) = \{t \in \mathbb{R} \mid f(t) \in U = \prod_{n=1}^{\infty} (-1/n, 1/n)\}$

$$= \{t \in \mathbb{R} \mid t \in (-1/n, 1/n),\ n = 1, 2, \dots\}$$

$$= \bigcap_{n=1}^{\infty} (-1/n, 1/n)$$

$= \{0\}$, which is not open in $\mathbb{R}_{std}$.

Thus, f is not continuous even if each

$f_n : \mathbb{R}_{std} \to \mathbb{R}_{std}$ is the identity.

<u>Proof</u>: Assume that each f_α is continuous.

<u>must show</u> $f = (f_\alpha)$ is continuous.

Enough to show that $f^{-1}(U)$ is open for any basis element U of $\prod_\alpha X_\alpha$. Then we may take

$U = \prod U_\alpha$, where $U_\alpha = X_\alpha$ for all but finitely many $\alpha \in J$.

$$f^{-1}(U) = \{ x \in \mathbb{R} \mid f(x) = (f_\alpha(x)) \in \prod_\alpha U_\alpha = U \}.$$

$$= \{ x \in A \mid f_\alpha(x) \in U_\alpha,\ \alpha \in J \}.$$

$$= \underline{\bigcap_{\alpha \in J}} f_\alpha^{-1}(U_\alpha),$$ where each $f_\alpha^{-1}(U_\alpha)$ is open since $f_\alpha : A \to X_\alpha$ is continuous.

If $U_\alpha = X_\alpha$, then $f_\alpha^{-1}(U_\alpha) = f_\alpha^{-1}(X_\alpha) = A$ so that the above intersection is indeed a finite intersection. Thus it is open.

Conversely, assume that $f : A \to \prod X_\alpha$ is continuous. Then $f_\alpha = \pi_\alpha \circ f$, where $\pi_\beta : \prod_\alpha X_\alpha \to X_\beta$ is the projection. Since

f and each π_α are continuous, so is their composition f_α. ∎

§ 22. The Quotient Topology:

Definition: Let X and Y be topological spaces, let $p: X \to Y$ be a surjective map. The map p is said to be a quotient map provided that a subset U of Y is open if and only if $p^{-1}(U)$ is open in X.

Construction: Let X be any set and $\sim$ an equivalence relation on X. Let Y be the set of equivalence classes of $\sim$.

$$Y = X/\sim \;= \{ [x] \mid x \in X \}$$

$$[x] = \{ x' \in X \mid x \sim x' \}.$$

Then $p: X \to X/\sim \,= Y$, $p(x) = [x]$ is a surjection.

Conversely, if $p: X \to Y$ is a surjection of two sets X and Y, then we can define an equivalence relation of X as follows:

$x_1, x_2 \in X$, $x_1 \sim x_2$ if and only $p(x_1) = p(x_2)$.

Note that if one starts with an equivalence relation and then obtain a surjection as described above. Then the equivalence relation obtained from this surjection is the equivalence relation we started with. Similarly the same is true if we had started with a surjection.

Thus for a given set X, the collection of equivalence relations on X is the same as the collection of surjections from X.

Let $f: X \to Y$ be a surjection and $g: X \to Z$ be any map of sets.

$$
\begin{tikzcd}
x' \sim x \arrow[d, mapsto] \\
{[x]}
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "f"'] & Z \\
Y = X/\sim \arrow[ur, dotted, "h"] &
\end{tikzcd}
$$

Question: Is there a function $h: Y \to Z$ so that the diagram commutes: $g = h \circ f$?

If such $h: Y = X/\sim \to Z$ exists then

$$(h \circ f)(x) = g(x) \Rightarrow h([x]) = g(x)$$

$$h([x']) = g(x')$$

$\Rightarrow g$ is constant on each equivalence class.

Conversely, if g is constant on each equivalence class then one can define $h: Y = X/\sim \to Z$ as $h([x]) = g(x)$.

$$[x] = f^{-1}([x])$$

Example X = set of quiz papers
$Z = \{0, 1, 2, 3, \ldots, 10\}$ set of scores.

$x, x' \in X$, $x \sim x' \Leftrightarrow x$ and x' are equivalent.

$$
\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "f"'] & Z \\
Y = X/\sim \arrow[ur, "h"'] &
\end{tikzcd}
$$

g: grading function
$g(x)$ = the grade of the quiz paper x.

$Y = X/\sim$ ↑ grading scheme (rubric)

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If X has topology $\mathcal{T}$. Then the strongest topology on $Y = X/\sim$ so that $f: X \to Y = X/\sim$ is continuous is called the quotient topology on $X/\sim = Y$.

$$\begin{array}{c} (X, \mathcal{T}) \\ f \big\downarrow \\ X/\sim = Y \end{array}$$

So a set $U \subseteq Y$ will be open if and only if $f^{-1}(U)$ is open in X.

<u>Theorem</u>: Let $f: X \to Y$ be a quotient map of topological spaces X and Y. Let $g: X \to Z$ be any map from X to a topological Z. Then there is a map $h: Y \to Z$ so that $g = h \circ f$ if and only if g is constant on the fibers of f. Moreover, g is continuous if and only if h is continuous.

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & Z \\ Y = X/\sim \arrow[ur, dashed, "\underline{\underline{h}}"'] & \end{tikzcd} \qquad g = h \circ f$$

<u>Proof</u>: We know that h exists if and only if

g is constant on each equivalence class (i.e., fibers of f).

<u>First assume that g is continuous</u>:

must show: h is continuous.

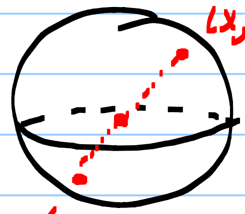
Let <u>$U \subseteq Z$</u> be an <u>open</u> subset. Then

$g^{-1}(U) = (h \circ f)^{-1}(U) = f^{-1}(h^{-1}(U))$. By assumption g is continuous and thus $g^{-1}(U)$ is open in X. Since Y has the quotient topology <u>$h^{-1}(U)$</u> must be open in X. Hence, $h: Y \to Z$ is continuous.

The other direction is proved the same way also. ∎

Example: Let S^2 denote the unit sphere in $\mathbb{R}^3$.

$S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1\} \subseteq \mathbb{R}^3$.



Equip S^2 with the subspace topology inherited from $\mathbb{R}^3$.

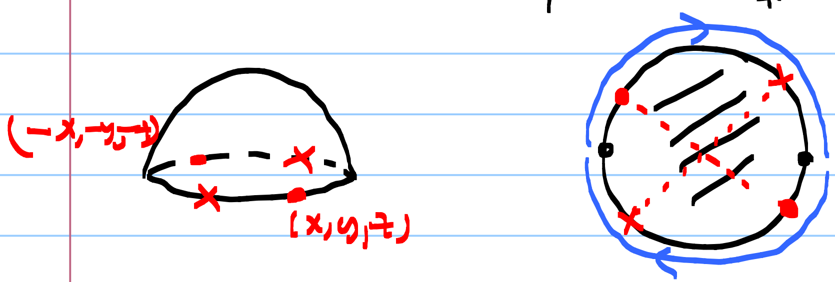
Consider an equivalence relation on S^2 defined as follows:

$$(x_1,y_1,z_1) \sim (x_2,y_2,z_2) \iff (x_2,y_2,z_2) = \pm(x_1,y_1,z_1).$$

The quotient of S^2 by this equivalence relation is called the real projective plane denoted as

$$\mathbb{RP}^2 = S^2/\sim$$

Can we draw a picture of $\mathbb{RP}^2$?



Fact: There is no topological embedding $h: \mathbb{RP}^2 \to \mathbb{R}^3$.

Proof requires "algebraic topology" or "intersection theory".

Fact: There is an embedding $h: \mathbb{RP}^2 \to \mathbb{R}^4$.

Proof: First let's construct an embedding into $\mathbb{R}^5$.

$$\begin{tikzcd}
{[x:y:z]} \in S^2 \arrow[r, "g", "2:1"'] \arrow[d, "f"] & \mathbb{R}^5 \\
{[x:y:z]} \in \mathbb{RP}^2 \arrow[ur, "h", "1:1"'] &
\end{tikzcd}$$

$[x:y:z] = \{(x,y,z), (-x,-y,-z)\}$

$g: S^2 \to \mathbb{R}^5$ should be constant on each equivalence class.

In other words, $g(x,y,z) = g(-x,-y,-z)$, for all $(x,y,z) \in S^2$.

Let $g: S^2 \to \mathbb{R}^5$, $g(x,y,z) = (x^2, y^2, xy, xz, yz)$.

Clearly, $g(-x,-y,-z) = g(x,y,z)$. So, since g is constant on each equivalence class then $\exists$ $h: \mathbb{RP}^2 = S^2/\sim \;\to \mathbb{R}^5$. Moreover, since g is continuous

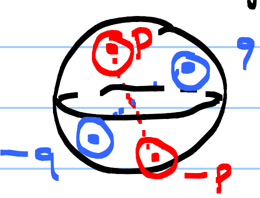
$h: \mathbb{RP}^2 \to \mathbb{R}^5$, $h([x:y:z]) = g(x,y,z)$ is continuous.

Exercise: h is one to one.

We'll see later that since $\mathbb{RP}^2$ is a compact Hausdorff space h is a homeomorphism onto its image. i.e., h is an embedding.

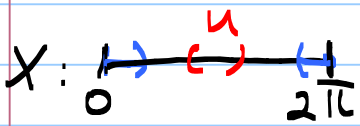
Exercise: $\mathbb{RP}^2$ with its quotient topology is Hausdorff.

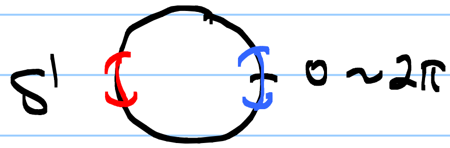
$[p] \neq [q]$ $\quad U \; \odot_p \quad \odot_q \; V$

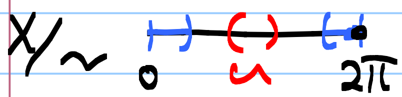


Example: $X = [0, 2\pi] \subseteq \mathbb{R}_{std}$.

Define $\sim$ on X as follows: $x, y \in X$, $x \sim y$ if and only if $x, y \in \{0, 2\pi\}$.







Claim: $X/\sim$ is homeomorphic to the circle S^1 in $\mathbb{R}^2$.

Proof $S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$.

$$\begin{tikzcd} X = [0, 2\pi] \arrow[r, "g"] \arrow[d, "f"'] & S^1 \\ X/\sim \, = [0,2\pi]/_{0 \sim 2\pi} \arrow[ur, "h"'] & \end{tikzcd}$$

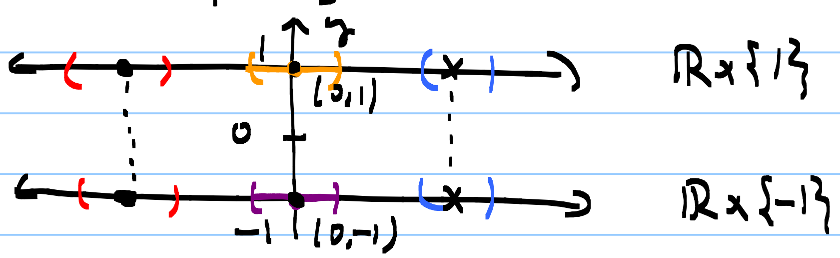
, $g(t) = (\cos t, \sin t)$. continuous and onto.

Since $g(0) = g(2\pi)$, $h : X/\sim \;\to S^1$ exists and it is continuous.

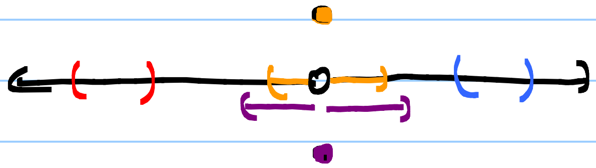
Clearly, h is one. So, h is a 1-1, onto and continuous mapping. (Since $X/\sim$ is compact and Hausdorff) h is a homeomorphism.

Example: The real line with double origin.

Let $X = \mathbb{R} \times \{-1, 1\} \subseteq \mathbb{R} \times \mathbb{R}$



Define $\sim$ on X as follows: $(x_1,-1) \sim (x_2,1)$ if and only if $x_1 = x_2 \neq 0$.



$X/\sim \; = Y$ is called the real line with double origin.

Note that any open subset containing the two origins must intersect. Hence, $Y = X/\sim$ is <u>not</u> Hausdorff.

Since any subspace of $\mathbb{R}^n_{std}$ is Hausdorff, we see that $Y = X/\sim$ cannot be embedded into $\mathbb{R}^n_{std}$.

§20 <u>The Metric Topology</u>:

<u>Definition</u>: A metric on a set X is a function

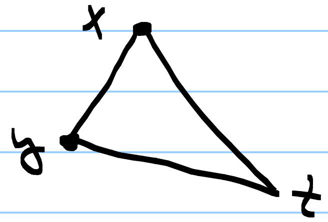
$d: X \times X \longrightarrow \mathbb{R}_{std}$ having the following properties:

1) $d(x,y) \geq 0$, for all $x,y \in X$, and equality holds if and only if $x = y$.

2) $d(x,y) = d(y,x)$, for all $x,y \in X$.

3) (Triangle Inequality) If $x,y,z \in X$ then

$d(x,z) \leq d(x,y) + d(y,z)$.



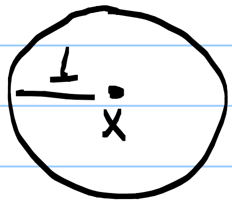
<u>Definition</u>: Let $x \in (X,d)$ and $\epsilon > 0$, then

$B(x,\epsilon) \doteq \{y \in X \mid d(x,y) < \epsilon\}$ is called the open ball with center x and radius $\epsilon > 0$.

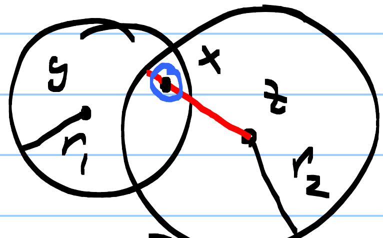
Similarly $B[x,\epsilon] = \{y \in X \mid d(x,y) \leq \epsilon\}$ is called the closed ball with center x and radius ϵ.

Proposition: If (X,d) is a metric space then the collection of all open balls form a basis for a topology on X, called the metric topology induced by the metric d.

Idea:

i) 

$x \in B(x,1)$



ii) $x \in B(y,r_1) \cap B(z,r_2)$

Let $r_3 = \min\left\{ \frac{r_2 - d(x,z)}{2}, \frac{r_1 - d(x,y)}{2} \right\}$.

Then $x \in B(x,r_3) \subseteq B(y,r_1) \cap B(z,r_2)$.

Example: On $\mathbb{R}^n$ the function $d_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$d_2((x_1, \dots, x_n),(y_1, \dots, y_n)) = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{1/2},$$

the Euclidean distance from $(x_1,\dots,x_n)$ to $(y_1,\dots,y_n)$, is a metric on $\mathbb{R}^n$. The topology generated by the metric is the standard topology $\mathbb{R}^n_{std}$.

Example: The discrete metric.

For any set X define $d : X \times X \to \mathbb{R}$ by

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$$

Then d defines a metric on X. Note that the open balls

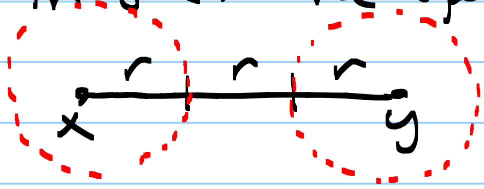
$$B(x,1) = \{ y \in X \mid d(x,y) < 1 \} = \{x\}.$$

Hence, in the metric topology every singleton is an open subset. Hence, the metric topology on X coincides with the discrete topology on X.

Proposition: Every metric topology is Hausdorff.

Proof: If $x \neq y$ in the metric space (X,d)

Let $r = \dfrac{d(x,y)}{3}$.



Then the open balls $B(x,r)$ and $B(y,r)$ are disjoint (if $z \in B(x,r) \cap B(y,r)$ then
$3r = d(x,y) \leq d(x,z) + d(z,y) < r + r = 2r$, which is a contradiction because $r > 0$.) $\blacksquare$

Remark: Hence, a non Hausdorff topological space cannot be metric topology.

Proposition: If the topology of a space X is induced by a metric d on X, then the topologies on every subspace of X is also induced by the same metric.

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Definition: Let X be a metric space with metric d. A subset A of X is said to be bounded if there is some number M so that $d(x,y) \leq M$ for all $x,y \in A$.
For a bounded A in X we define its diameter as $\operatorname{diam} A = \sup\{d(x,y) \mid x,y \in A\}$.

Theorem: Let X be a metric space with metric d. Define $\bar{d} : X \times X \to \mathbb{R}$ as

$$\bar{d}(x,y) = \min\{d(x,y), 1\}.$$

Then $\bar{d}$ is a metric on X that induces the same topology as d.
The metric $\bar{d}$ is called the standard bounded metric corresponding to d.

Proof: First let's prove that $\bar{d}$ is a metric.

M1) $\bar{d}(x,y) = \min\{\underline{d(x,y), 1}\} \geq 0$ since $d(x,y) \geq 0$.

If $\bar{d}(x,y) = 0$ then $d(x,y) = 0$, which implies $x=y$ since d is a metric.

M2) $\bar{d}(x,y) = \bar{d}(y,x)$ is trivially true since $d(x,y) = d(y,x)$.

M3) Let $x,y,z \in X$.
Case 1) $d(x,y) \geq 1$, $d(y,z) \geq 1$. Then $\bar{d}(x,y) = 1 = \bar{d}(y,z)$.

So, $\bar{d}(x,z) \leq 1 \leq 1+1 = \bar{d}(x,y) + \bar{d}(y,z)$.

Case 2) $d(x,y) \geq 1$ and $d(y,z) \leq 1$. Then $\bar{d}(x,y) = 1$ and $\bar{d}(y,z) = d(y,z)$.
$\bar{d}(x,z) \leq 1 \leq 1 + \bar{d}(y,z) = \bar{d}(x,y) + \bar{d}(y,z)$.

Case 3) $d(x,y) \leq 1$ and $d(y,z) \leq 1$. Then

$\bar{d}(x,y) = d(x,y)$ and $\bar{d}(y,z) = d(y,z)$.

So $\bar{d}(x,z) \leq d(x,z) \leq d(x,y) + d(y,z) = \bar{d}(x,y) + \bar{d}(y,z)$.
Hence, $\bar{d}$ is a metric on X.

Note that metric topologies are generated by open balls. Thus it is enough to prove the following statement.

a) Consider a ball $B_d(x,r)$ and a point $y \in B_d(x,r)$.

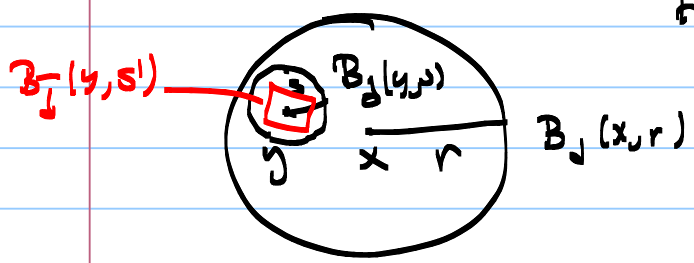
Let $r' = \min\{r, 1/2\}$.

Claim: $B_{\bar{d}}(x, r') \subseteq B_d(x,r)$.

If $z \in B_{\bar{d}}(x,r')$ then $\bar{d}(x,z) < r' \leq 1/2$. Then $d(x,z) = \bar{d}(x,z) < r' \leq r$. So, $z \in B_d(x,r)$.

Hence, $B_{\bar{d}}(x,r') \subseteq B_d(x,r)$.

Now choose first a ball around $y \in B_d(y,s)$ with $B_d(y,s) \subseteq B_d(x,r)$. Then $B_{\bar{d}}(y,s') \subseteq B_d(y,s) \subseteq B_d(x,r)$, where $s' = \min\{s, 1/2\}$.



Hence the topology generated by $\bar{d}$ is stronger than the topology generated by d.

b) Now take any ball $B_{\bar{d}}(x,r)$ and let $y \in B_{\bar{d}}(x,r)$ be any point.
must find some $r' > 0$ so that $B_d(y,r') \subseteq B_{\bar{d}}(x,r)$, which implies that the topology induced by d

is stronger than the topology induced by $\bar{d}$.
The rest is exercise. ■

Remark: Note that a metric topology is generated by ball of radius less than 1. Moreover, if $0<r<1$ then $B_{\bar{d}}(x,r)=B_d(x,r)$ and thus the proof finishes.

Metrics on $\mathbb{R}^n$:

$$d_2(x,y)=\|x-y\|=\left(\sum_{i=1}^{n}|x_i-y_i|^2\right)^{1/2}.$$

$$d_p(x,y)=\left(\sum_{i=1}^{n}|x_i-y_i|^p\right)^{1/p}, \quad p\geq 1.$$

$$d_1(x,y)=\sum_{i=1}^{n}|x_i-y_i|$$

$$d_\infty(x,y)=\max\{|x_i-y_i| \mid i=1,\dots,n\}.$$

Definition: Two metrics d_1 and d_2 on a set X are said to be equivalent if there are positive numbers m, M so that

$$m\, d_1(x,y)\leq d_2(x,y)\leq M\, d_1(x,y).$$

Fact: 1) The equivalence of metrics is indeed an equivalence relation.

2) Equivalent metrics induce the same topology.

Remark: $d(x,y)=|x-y|$ on $\mathbb{R}$, $\bar{d}(x,y)=\min\{|x-y|,1\}$.
Although the topologies induced by d and $\bar{d}$ are the same, the metric are not equivalent.

Video 20

Uniform Metric (Topology)

Let X be a set and consider it as an index set J.

$J = X$. On the product set $\mathbb{R}^J = \mathbb{R}^X = \prod_{x \in X} \mathbb{R}_{std}$

$\mathbb{R}^X = \{ f : X \to \mathbb{R} \mid f \text{ a function} \}$.

Ex 1) $X = \{1,2,3\}$, $\mathbb{R}^X = \mathbb{R}^3 = \{ f : \{1,2,3\} \to \mathbb{R} \}$
$= \{ (f(1), f(2), f(3)) \mid f(i) \in \mathbb{R} \}$
$= \{ (f_1, f_2, f_3) \mid f_i \in \mathbb{R} \}$.

2) $X = \mathbb{N} = \{0,1,2,\ldots\}$, $\mathbb{R}^X = \mathbb{R}^{\mathbb{N}} = \{ a : \mathbb{N} \to \mathbb{R} \mid a \text{ is a function} \}$

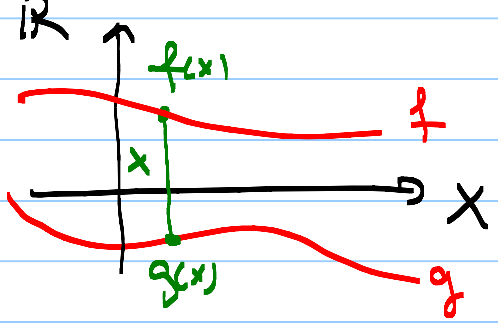
$\mathbb{R}^{\mathbb{N}} = \{ (a_n) \mid a_n \in \mathbb{R} \}$ the set of sequences.

3) $X = \mathbb{R}$, $\mathbb{R}^X = \mathbb{R}^{\mathbb{R}} = \{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ is a function} \}$.

Definition: Let X be any set and denote $B(X)$ the set of all bounded functions $f : X \to \mathbb{R}$, i.e., there is some $M \in \mathbb{R}$ so that $|f(x)| \leq M$ for each $x \in X$.

On $B(X)$ we have a metric called uniform metric d_{sup} defined as follows:

If $f, g \in B(X)$, $d_{sup}(f,g) = \sup_{x \in X} |f(x) - g(x)|$



Fact: d_{sup} is a metric on $B(X)$.

Example: $\mathbb{R}_{std}$, $\bar{d}(x,y)=\min\{|x-y|,1\}$, $x,y\in\mathbb{R}$.

$\mathbb{R}^{\omega}=\mathbb{R}^{\mathbb{N}}=\{(a_n)\mid a:\mathbb{N}\to\mathbb{R} \text{ function}\}$.

Defined d_{sup} on $\mathbb{R}^{\omega}$ as follows

$$d_{sup}((a_n),(b_n))=\sup_{n\in\mathbb{N}}\bar{d}(a_n,b_n).$$

d_{sup} is called the uniform metric on $\mathbb{R}^{\omega}=\mathbb{R}^{\mathbb{N}}$.

Theorem: The uniform topology on $\mathbb{R}^J$ is finer than the product topology and coarser than the box topology. Moreover, they are different provided that J is infinite.
(If J is finite the three topologies coincide)

Proof: Let $x=(x_\alpha)_{\alpha\in J}$ be a point in $\mathbb{R}^J$ and consider a basis element U around x in the product topology.

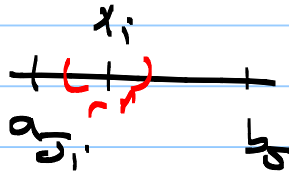
$x\in U=(a_{j_1},b_{j_1})\times\cdots\times(a_{j_n},b_{j_n})\times\mathbb{R}\times\cdots\times\mathbb{R}\times\cdots$

$x_i\in(a_{j_i},b_{j_i})$, $i=1,\dots,n$.

Consider the ball $B(x,r)$ in the uniform topology:

$B(x,r)=\{y\in\mathbb{R}^J\mid d_{sup}(x,y)<r\}$

$x:J\to\mathbb{R}$, $y:J\to\mathbb{R}$, $d_{sup}(x,y)=\sup_{j\in J}\{\bar{d}(x(j),y(j))\}$



Choose $1>r>0$ so that $(x_i-r,x_i+r)\subseteq(a_{j_i},b_{j_i})$, $i=1,\dots,n$.

Then the ball $B_{d_{sup}}(x,r)\subseteq(a_{j_1},b_{j_1})\times\cdots\times(a_{j_n},b_{j_n})\times\mathbb{R}\cdots$

Hence, the uniform topology is stronger than the product topology.

Now consider any ball $B_{\bar{d}_{sup}}(x, r)$. Then

$B_{d_{sup}}(x, r) = \prod_{J \in J} (x(J) - r, x(J) + r)$, which is clearly a basis element for the box topology. Hence, box topology is stronger than the uniform topology.

For the second part assume that J is infinite. Then contains $\mathbb{N} \subseteq J$, and let's work with $\mathbb{N}$ instead of J, for simplicity.

Now, $(-1, 1) \times (-1/2, 1/2) \times \cdots \times (-1/n, 1/n) \times \cdots$ is open in Box topology but it is <u>not</u> open in the uniform topology.

Similarly, the set $(-1, 1) \times (-1, 1) \times \cdots \times (-1, 1) \times \cdots$ is open in the uniform topology, but is not open in the product topology.

Hence, these three topologies are all different.

<u>Theorem</u>: Let $\bar{d}(a, b) = \min\{|a-b|, 1\}$ be the standard bounded metric on $\mathbb{R}$. If x and y are two points of $\mathbb{R}^\omega$, define

$$D(x, y) = \sup_{i \in \mathbb{N}} \left\{ \frac{\bar{d}(x_i, y_i)}{i} \right\}. \qquad (\mathbb{N} = \{1, 2, \ldots\})$$

Then D is a metric on $\mathbb{R}^\omega$ that induces the product topology on $\mathbb{R}^\omega$.

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Proof: First let's see that D is a metric on $\mathbb{R}^\omega$.

M1) $D(x,y) = \sup_i \left\{ \frac{\bar{d}(x_i,y_i)}{i} \right\} \geq 0$, since each $\bar{d}(x_i,y_i) \geq 0$.

If $D(x,y)=0$, then $\bar{d}(x_i,y_i)=0$. Thus $|x_i-y_i|=0$. So, $x_i=y_i$ for all $i\in\mathbb{N}$ and hence $x=y$.

M2) $D(x,y)=D(y,x)$ is clear.

M3) Let $x,y,z\in\mathbb{R}^\omega$. Then, since $\bar{d}$ is a metric on $\mathbb{R}$ we have

$$\frac{\bar{d}(x_i,z_i)}{i} \leq \frac{\bar{d}(x_i,y_i)}{i} + \frac{\bar{d}(y_i,z_i)}{i}, \text{ for all } i\in\mathbb{N}.$$

$$\leq \sup_i\left\{\frac{\bar{d}(x_i,y_i)}{i}\right\} + \sup_i\left\{\frac{\bar{d}(y_i,z_i)}{i}\right\}$$

$$\leq D(x,y) + D(y,z), \text{ for all } i\in\mathbb{N}.$$

Hence, $D(x,z) = \sup_i\left\{\frac{\bar{d}(x_i,z_i)}{i}\right\} \leq D(x,y)+D(y,z)$.

So, D is a metric on $\mathbb{R}^\omega$.

Next we'll show that D induces the product topology on $\mathbb{R}^\omega$.

Let Let U be an open set in the metric topology and let $x\in U$. Choose some $r>0$ so that $B(x,r)\subseteq U$, where

$$B(x=(x_n),r) = \{(y_n)\in\mathbb{R}^\omega \mid D(x,y)<r\}.$$

Choose N large enough st. $1/N < r$.
Let $r' = r/2$.

Let $V=(x_1-r', x_1+r')\times\cdots\times(x_N-r', x_N+r')\times\mathbb{R}\times\mathbb{R}\times\cdots$, which is clearly open in the product topology.

<u>Claim:</u> $(x_n)\in V\subseteq B((x_n), r')\subseteq U$.

<u>Proof</u> If $(y_n)\in V$, then $|x_n-y_n|<r'$, $i=1,\dots,N$.

Then $\frac{\bar d(x_n,y_n)}{n}\le\frac{1}{N}<r'$, for $n\ge N$.

For $n=1,\dots,N$, $\frac{\bar d(x_n,y_n)}{n}<\frac{r'}{n}<r'$.

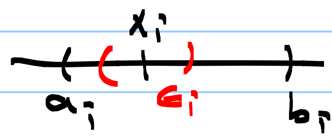
Hence, $D(x,y)=\sup\left\{\frac{\bar d(x_n,y_n)}{n}\right\}\le r'<r$.

So, $V\subseteq B_{d_{\sup}}((x_n), r)\subseteq U$, and thus U is open in the product topology.

Hence, the product topology is stronger than the metric topology induced by D.

For the converse, let $U=\prod_{i\in\mathbb{N}}U_i$ be a basis element of the product topology on $\mathbb{R}^\omega$. So $U_1=(a_1,b_1),\dots,U_n=(a_n,b_n)$, $U_{n+k}=\mathbb{R}$, for all $k\ge1$, where $a_i,b_i\in\mathbb{R}$, $n\in\mathbb{N}$.

Let $x=(x_n)\in U$. So $x_i\in(a_i,b_i)$, $i=1,\dots,n$.



Choose $1>\epsilon_i>0$ so that $(x_i-\epsilon_i, x_i+\epsilon_i)\subseteq(a_i,b_i)$, $i=1,\dots,n$.

Let $\epsilon=\min\left\{\frac{\epsilon_i}{i}\mid i=1,\dots,n\right\}$.

Claim: $x \in B_D(x,\epsilon) \subseteq U$, which implies the metric topology on $\mathbb{R}^\omega$ is stronger than the product topology.

Proof: Let $y \in B_D(x,\epsilon)$. Then for all i

$$D(x,y) < \epsilon \implies \frac{\bar{d}(x_i,y_i)}{i} \leq D(x,y) < \epsilon, \text{ for all } i.$$

If $i = 1, \ldots, n$, then $\epsilon \leq \epsilon_i/i$ so that

$$\bar{d}(x_i,y_i) < \epsilon_i < 1.$$

Hence, $|x_i - y_i| = \bar{d}(x_i,y_i) < \epsilon_i$ and thus
$y_i \in (x_i - \epsilon_i, x_i + \epsilon_i) \subseteq (a_i, b_i)$, $i = 1 \ldots n$.

So $y = (y_n) \in (a_1,b_1) \times \cdots \times (a_n,b_n) \times \mathbb{R} \times \mathbb{R} \times \cdots = U$.

$\implies B_D(x,\epsilon) \subseteq U$. ∎

CHAPTER 3: Connectedness and Compactness

§23. Connected Spaces:

Definition: Let X be a topological space. A separation of X is a pair U, V of disjoint nonempty open subsets of X with $X = U \cup V$.

If X has a separation we say that the topological space X is disconnected. If X admits no separation then we say that X is connected.

Another formulation of connectedness: A space X is connected if and only if the only subsets of X that are both open and closed are the empty set and X itself.

Proof: Suppose X is disconnected. Then X has a separation, say $X = U \cup V$,

1) U, V open
2) $U \cap V = \emptyset$
3) $U \neq \emptyset$, $V \neq \emptyset$.

Then $U = X \setminus V$ is closed since V is open and $U \neq \emptyset$ and $U \neq X$. Hence, U is a subset of X, which is both open and closed so that $\emptyset \neq U \neq X$.

The other direction is left as an exercise.

Lemma: If Y is a subspace of X, a separation of Y is a pair of disjoint nonempty sets A and B whose union is Y, neither of which contains a limit point of the other. The subspace Y is

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connected if there is no separation of Y.

Proof: First suppose that A and B form a separation for Y: $Y = A \cup B$, $A \cap B = \emptyset$ and A, B are open in Y. Then A and B are closed in Y. So $A = \bar{A} \cap Y$ and thus $\emptyset = A \cap B = \bar{A} \cap B$. Hence, B contains no limit point of A. Similarly $\emptyset = A \cap \bar{B}$ and thus A contains no limit point of B.

For the converse assume that A, B are disjoint subsets of Y so that $\bar{A} \cap B = \emptyset = A \cap \bar{B}$, and $A \cup B = Y$. Then $B = (X \setminus \bar{A}) \cap Y$ so that B is open in Y since $X \setminus \bar{A}$ is open in X. Similarly,
$A = (X \setminus \bar{B}) \cap Y$ is open in Y, also.
It follows that A and B form a separation for the subspace Y. ●

Example: 1) $X = \{a, b\}$, $\mathcal{J} = \{\emptyset, X\}$ connected since there is no separation for X.

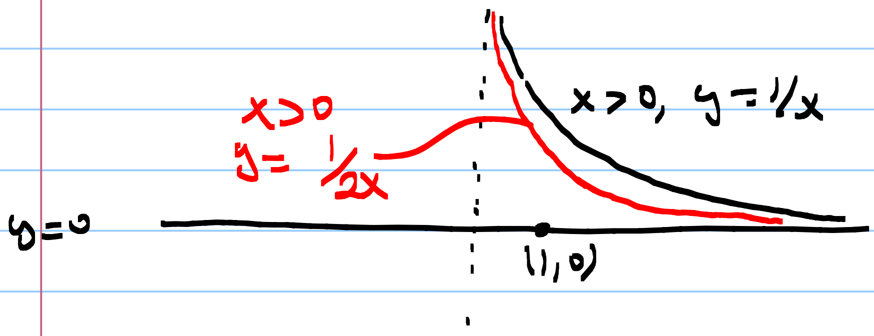
2) $Y = [-1, 0) \cup (0, 1] \subseteq \mathbb{R}_{std}$ disconnected since $[-1, 0)$ and $(0, 1]$ are disjoint subsets and neither one of them contains a limit point of the other: $A = [-1, 0)$, $B = (0, 1]$.
$\bar{A} = [-1, 0] \cap Y = A$, $\bar{B} = [0, 1] \cap Y = (0, 1] = B$.
Since $Y = A \cup B$ and $A \cap B = \emptyset$, they are also both open. Thus, Y is disconnected.

3) $\mathbb{Q} \subseteq \mathbb{R}_{std}$ is disconnected subspace.

$$\mathbb{Q} = \underbrace{\left((-\infty, \sqrt{2}) \cap \mathbb{Q}\right)}_{\text{open in } \mathbb{Q}} \cup \underbrace{\left((\sqrt{2}, \infty) \cap \mathbb{Q}\right)}_{\text{open in } \mathbb{Q}}$$ is a

disjoint union and they are both nonempty. Thus, $\mathbb{Q}$ is disconnected.

4) $X = \{(x,y) \mid y = 0 \text{ or } (x > 0 \text{ and } y = 1/x)\}$
is disconnected.



$U = \{(x,y) \mid x > 0 \text{ and } y > 1/2x\}$, $\overline{U} = \{(x,y) \mid x > 0,\ y \geq 1/2x\}$
$V = \mathbb{R}^2 \setminus \overline{U}$

U and V are open subsets of $\mathbb{R}^2$.

$U \cap X = \{(x,y) \mid x > 0,\ y = 1/x\}$
$V \cap X = \{(x,y) \mid y = 0\}$

<u>Lemma</u>: If the sets C and D form a separation of X, and if Y is a connected subspace then Y lies entirely within either C or D.

<u>Proof</u>: $X = C \cup D$, $C, D \subseteq X$ open

$C \neq \emptyset \neq D$ and $C \cap D = \emptyset$.

If $C \cap Y \neq \emptyset$ and $D \cap Y \neq \emptyset$ then

$Y = (C \cap Y) \cup (D \cap Y)$ would be a separation for the subspace Y. However, Y is connected and thus $C \cap Y = \emptyset$ or $D \cap Y = \emptyset$. Finally, $X = C \cup D$ and thus $C \cap Y = \emptyset \Rightarrow Y \subseteq D$
or $D \cap Y = \emptyset \Rightarrow Y \subseteq C$. ∎

Theorem: The union of a collection of connected subspaces of X that have a point in common is connected.

Proof: $A_\alpha \subseteq X$, $\alpha \in \Lambda$ collection of connected subspaces. By assumption $\bigcap_\alpha A_\alpha \neq \emptyset$.

Let $x_0 \in \bigcap_\alpha A_\alpha$. Let $Y = \bigcup_\alpha A_\alpha$.

must show: Y is connected.

Let $Y = C \cup D$ be a separation for Y. Since $x_0 \in Y = C \cup D$, without loss of generality we may assume that $x_0 \in C$. Since each A_α is connected subspace of Y and $C \cup D$ is a separation for Y, A_α is either contained in C or D. However, since $x_0 \in C \cap A_\alpha$ we see that $A_\alpha \subseteq C$, for any $\alpha \in \Lambda$. Hence, $Y = \bigcup A_\alpha \subseteq C$, a contradiction since $D = \emptyset$. Thus, Y has no separation. ∎

Theorem: Let A be a connected subspace of X. If $A \subseteq B \subseteq \overline{A}$, then B is also connected.

Proof: Let $B = C \cup D$ be a separation for B. Since $A \subseteq B$ is a connected subspace of B, by the above lemma $A \subseteq C$ or $A \subseteq D$. Note that C and D are both closed and open in B. If $A \subseteq C$, then $\overline{A} \subseteq \overline{C}$. Since $\overline{C} \cap D = \emptyset$ and $B \subseteq \overline{A}$ we see that $B \cap D = \emptyset$, a contradiction. ∎

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Theorem: The image a connected space under a continuous map is connected.

Proof: $f: X \to Y$, f continuous, X connected.

must show: $f(X) \subseteq Y$ is a connected subspace.

Suppose $f(X)$ has a separation say
$$f(X) = C \cup D,$$
where C and D are open in $f(X)$. Then $f^{-1}(C)$ and $f^{-1}(D)$ are open subset of X. Moreover,

1) Since $f(X) = C \cup D$, $X = f^{-1}(C \cup D)$
$= f^{-1}(C) \cup f^{-1}(D)$.

2) Since $C \cap D = \emptyset$, $f^{-1}(C) \cap f^{-1}(D) = f^{-1}(C \cap D)$
$= \emptyset$.

3) $C \neq \emptyset$ and $D \neq \emptyset$ implies $f^{-1}(C) \neq \emptyset$ and $f^{-1}(D) \neq \emptyset$ because, $f(X) = C \cup D$.

These imply that $X = f^{-1}(C) \cup f^{-1}(D)$ is a separation for X, a contradiction. Hence, $f(X)$ has no separation, i.e., $f(X)$ is connected.

Theorem: A finite cartesian product of connected spaces is connected.

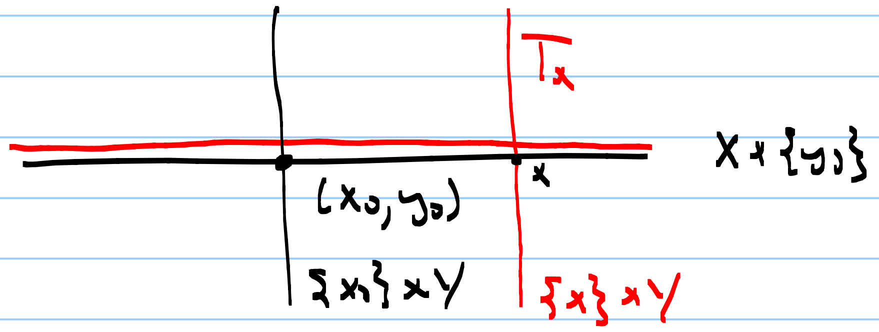
Proof: Let $X_1, \dots, X_n$ be connected spaces

$X_1 \times \cdots \times X_n = X_1 \times (X_2 \times \cdots \times X_n)$ and thus it is enough to prove result for $n = 2$ (by induction).

Now assume that X and Y are connected spaces.
<u>must show:</u> $X \times Y$ is connected.

Let $(x_0, y_0) \in X \times Y$. Then $X \times \{y_0\}$ is the image of the continuous inclusion map

$i_{y_0} : X \to X \times Y$, $i_{y_0}(x) = (x, y_0)$. Since X is connected so is its image $X \times \{y_0\}$ is connected.
Similarly, $\{x_0\} \times Y$ is connected.



Since $X \times \{y_0\}$ and $\{x_0\} \times Y$ are connected subsets and they have a common point we see that
$X \times \{y_0\} \cup \{x_0\} \times Y$ is a connected subspace.

For any $x \in X$, let $T_x \doteq X \times \{y_0\} \cup \{x\} \times Y$

Note that T_x is also connected and $(x_0, y_0) \in T_x$.

Finally, $X \times Y = \bigcup_{x \in X} T_x$ and $\bigcap_{x \in X} T_x \neq \emptyset$ since

$(x_0, y_0) \in T_x$, for all $x \in X$. Hence, $X \times Y$ is also connected.

<u>Remark:</u> We'll see that $\mathbb{R}_{std}$ is connected.
So any finite product of the form

$\mathbb{R}\times\mathbb{R}\times\cdots\times\mathbb{R}=\mathbb{R}^n$ is connected by the above result.

Moreover, $\mathbb{R}^\omega=\mathbb{R}^{\mathbb{N}}$ is connected with its product topology.

However, we'll see that $\mathbb{R}^\omega$ with the box topology is not connected.

$\mathbb{R}^\omega$ with box topology is not connected:

$\mathbb{R}^\omega=\{(x_n)\mid x_n\in\mathbb{R}\}$ real sequences.

$A=$ the set of all bounded sequences in $\mathbb{R}$.
$B=$ the set of all unbounded sequences in $\mathbb{R}$.

Clearly, $A\neq\emptyset$ since $(x_n)=(1,1,\ldots)\in A$ and $B\neq\emptyset$ since $(y_n)=(n)=(1,2,3,\ldots)\in B$.
Moreover, $\mathbb{R}^\omega=A\cup B$ and $A\cap B=\emptyset$.

A is open: Let (x_n) be a bounded sequence. So there is some $M\geq 0$ so that $|x_n|\leq M$ for all $n=1,2,\ldots$.
Then $(x_n)\in U=(x_1-1,x_1+1)\times(x_2-1,x_2+1)\times\cdots\cdots$ and U is open in the box topology of $\mathbb{R}^\omega$.
If $(y_n)\in U$ then $|y_n|\leq M+1$, for all n.
Thus (y_n) is also bounded. In other words, $(y_n)\in A$. So $U\subseteq A$. Hence, A is open in $\mathbb{R}^\omega$.

B is open: Let $(x_n)\in B$. So (x_n) is an unbounded sequence in $\mathbb{R}$. So, for each $N\in\mathbb{N}$ there is some x_{N_0} so that $x_{N_0}\geq N$.

Let $U=(x_1-1,x_1+1)\times(x_2-1,x_2+1)\times\cdots\cdots$, which is open in the box topology of $\mathbb{R}^\omega$.

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If $(y_n) \in U$ then $|y_n - x_n| \le 1$ and thus $|y_{N_0}| \ge |x_{N_0}| - 1 \ge N - 1$. This implies that (y_n) is unbounded. So, $U \subseteq B$ and hence, B is open.

Therefore, A and B form a separation for $\mathbb{R}^\omega$ (in the box topology). Hence, $\mathbb{R}^\omega$ with the box topology is disconnected.

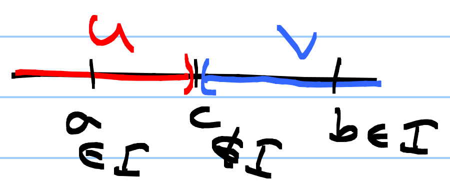
§24. Connected Subspaces of the Real Line.

A subset I of $\mathbb{R}$ is called an interval if whenever $a, b \in I$, for some $a < b$, then $(a,b) \doteq \{x \in \mathbb{R} \mid a < x < b\}$ is in I.

Example $(-\infty, a)$, $(-\infty, a]$, (a,b), $[a,b)$, $(a,b]$, (a, ∞), $[a, \infty)$, $[a,b]$.

Theorem: A subspace I of $\mathbb{R}_{st}$ is connected if and only if I is an interval.

Proof: Suppose I is not an interval. Then there are elements $a, b \in I$ with $a < b$ and $c \in \mathbb{R}$ s.t. $a < c < b$ but $c \notin I$.



Then let $U = (-\infty, c)$ and $V = (c, \infty)$.

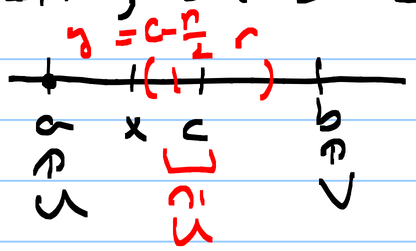
Since $c \notin I$, $I \subseteq \mathbb{R} \setminus \{c\} = U \cup V$. Also, $U \cap V = \emptyset$. Moreover, $a \in U$ and $b \in V$ s.t. $U \cap I \neq \emptyset \neq V \cap I$. So, $I = (U \cap I) \cup (V \cap I)$ is a separation for I. Hence, I is not connected.

Now assume that I is an interval.

must show: I is connected.

Assume on the contrary that I is not connected and $I \subseteq U \cup V$, where U, V open subsets of $\mathbb{R}$, with $U \cap V = \emptyset$ and $I \cap U = \emptyset = I \cap V$. i.e., $I = (I \cap U) \cup (I \cap V)$ is a separation for I.

Let $a \in I \cap U$, let $b \in I \cap V$. Assume that $a < b$.



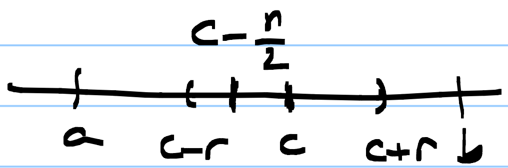
Let $c = \sup\{x \in \mathbb{R} \mid [a,x] \subseteq U\}$. Since $b \in V$ we have $c \leq b$. Since $c \in [a,b]$, $c \in I$.

<u>Case 1</u>: $c \in U$. Since U is open there is some $r > 0$ so that $(c-r, c+r) \subseteq U$. Now $[a,y] \subseteq U$, where $y = c - \frac{r}{2}$. Also $(c-r, c+r) \subseteq U$ and thus $[a, c+\frac{r}{2}] \subseteq U$. Hence, since

$\sup\{x \in \mathbb{R} \mid [a,x] \subseteq U\} \geq c + \frac{r}{2}$ a contradiction.

<u>Case 2</u>: $c \notin U$, then $c \in V$. Since V is also open there is some $r > 0$ so that

$(c-r, c+r) \subseteq V$.



So, $c - \frac{r}{2} \in V$ and thus $c - \frac{r}{2} \notin U$. However, we know that $[a, c-\frac{r}{2}] \subseteq U$ by the choice of c.

This is a contradiction. Hence, I must be connected. ∎

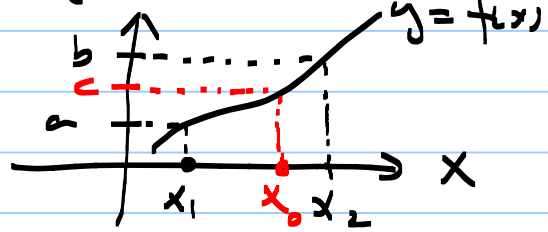
Example: 1) $\mathbb{R} = (-\infty, +\infty)$, (a,b), $[a,b)$, $(-\infty, a]$, ... are all connected subsets.

2) $\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ is connected.

Theorem: (Intermediate Value Theorem)

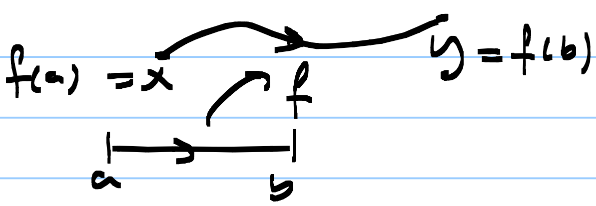
Let $f: X \longrightarrow \mathbb{R}_{std}$ be a continuous map, where X is a connected space. If $a, b \in \mathbb{R}_{std}$ are in the image of f and $c \in (a,b)$ then c is also in the image of f.

Proof:



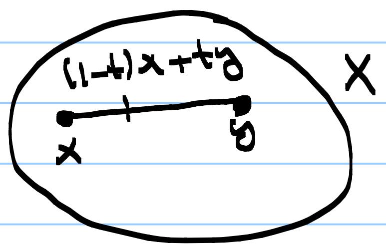
Let x_1 and x_2 be points in X with $f(x_1) = a$ and $f(x_2) = b$. Since X is connected and f is continuous $f(X)$ is a connected subset of $\mathbb{R}_{std}$. Hence, $f(X) = I$ is an interval. Clearly, $a, b \in I$. Since $c \in (a,b)$ and I is an interval, $c \in I$. So there is some $x_0 \in X$ with $f(x_0) = c$. ∎

Definition: Given points x and y of a space X, a path in X from x to y is a continuous map $f: [a,b] \to X$ with $f(a) = x$ and $f(b) = y$.



A space X is called path connected if any two points of X is connected by a path.

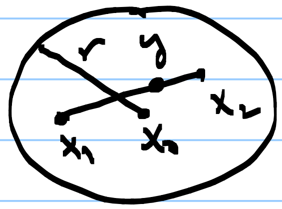
Example: Any convex subset X of $\mathbb{R}^n$ is path connected. Recall that a subset X of $\mathbb{R}^n$ is called convex if whenever $x, y \in X$ then the line segment $t \longmapsto (1-t)x + ty$, $t \in [0,1]$, also belongs to X.



Note that $f(t) = (1-t)x + ty$ is already a continuous path with $f(0) = x$ and $f(1) = y$. Hence, X is path connected.

Example Any Euclidean ball

$$B(x_0, r) = \{ x \in \mathbb{R}^n \mid \|x_0 - x\| < r \} \text{ is path connected.}$$



$$y = (1-t)x_1 + t x_2$$

$$\begin{aligned}
\|y - x_0\| &= \|((1-t)x_1 + t x_2) - (1-t)x_0 - t x_0\| \\
&= \|(1-t)(x_1 - x_0) + t(x_2 - x_0)\| \\
&\le \|(1-t)(x_1 - x_0)\| + \|t(x_2 - x_0)\| \\
&= |1-t| \, \|x_1 - x_0\| + |t| \, \|x_2 - x_0\| \\
&= (1-t)\underbrace{\|x_1 - x_0\|}_{<r} + t\underbrace{\|x_2 - x_0\|}_{<r}
\end{aligned}$$

$0 \le t \le 1$

$$< (1-t) \cdot r + t r = r.$$

Hence, $y \in B(x_0, 1)$.

Proposition: Any path connected space is connected.

Proof: Let X be path connected but not connected. Let $X = U \cup V$ be a separation for X. Choose $x \in U$ and $y \in V$. Then X is path connected

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there is a path $f: [a,b] \to X$ so that $f(a) = x$ and $f(b) = y$. Then $f^{-1}(U)$ and $f^{-1}(V)$ form a separation for $[a,b]$:

1) $x \in U$ and $f(a) = x \Rightarrow a \in f^{-1}(U) \Rightarrow f^{-1}(U) \neq \emptyset$
   $y \in V$ and $f(b) = y \Rightarrow b \in f^{-1}(V) \Rightarrow f^{-1}(V) \neq \emptyset$

2) $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = f^{-1}(\emptyset) = \emptyset$

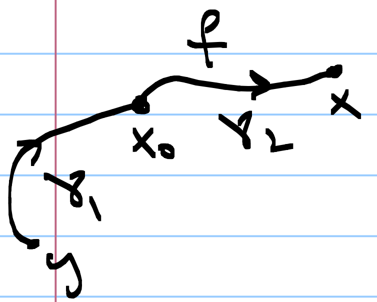
3) $[a,b] = f^{-1}(X) = f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$.

This is a contradiction, since $[a,b]$ is an interval. Hence, X is connected. ∎

<u>Proposition:</u> An open and connected subset of $\mathbb{R}^n_{std}$ is path connected.

<u>Proof:</u> Let $A \subseteq \mathbb{R}^n$ be an open and connected subset of $\mathbb{R}^n$. Choose any $x_0 \in A$. Define the following subset

$$C = \{x \in A \mid \text{there is a path } f:[a,b] \to A \text{ s.t. } f(a) = x_0,\ f(b) = x\}.$$



Clearly, C contains x_0 and thus C is <u>not</u> empty. (Just let $f:[0,1] \to A$, with $f(t) = x_0$ for all $t \in [0,1]$.)

<u>Claim:</u> C is a closed and open subset of A.

Note that since $C \neq \emptyset$ and A is connected we must have $C = A$. Then A is path connected: Let $\gamma_1 : [0,1] \to X$ and $\gamma_2 : [0,1] \to X$

be continuous paths with $\gamma_1(0)=y$, $\gamma_1(1)=x_0$, $\gamma_2(0)=x_0$ and $\gamma_2(1)=x$. Then

$\gamma:[0,1]\to X$ defined by

$$\gamma(t)=\begin{cases}\gamma_1(2t), & 0\le t\le 1/2\\ \gamma_2(2t-1), & 1/2\le t\le 1.\end{cases}$$

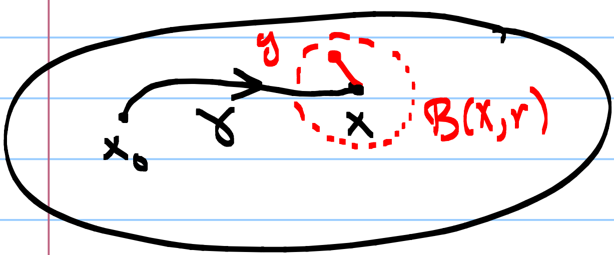
γ is continuous since γ_1 and γ_2 are continuous and

$$\gamma_1(2\cdot\tfrac{1}{2})=\gamma_1(1)=x_0=\gamma_2(0)=\gamma_2(2\cdot\tfrac{1}{2}-1)$$

Hence, γ is continuous by the pasting lemma.

Proof of the claim:

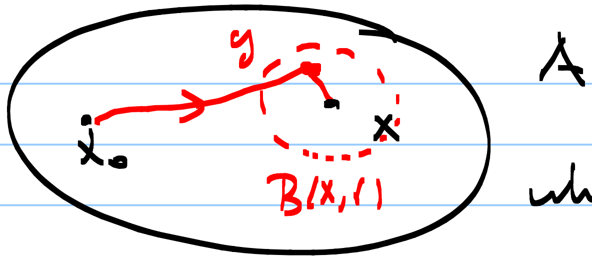
C is open. Let $x\in C$. Then there is a path $\gamma:[0,1]\to A$ with $\gamma(0)=x_0$ and $\gamma(1)=x$. Since A is open there is a ball $B(x,r)$ so that $B(x,r)\subseteq A$.



Now if $y\in B(x,r)$ then the line segment joining x to y is a path. Thus by the above pasting lemma argument we obtain a path joining x_0 to y. Thus $y\in C$. So $B(x,r)\subseteq C$ and hence C is open.

C is closed: Let's show that $A\setminus C$ is open.

Let $x\in A\setminus C$. Since A is open there is a ball $B(x,r)$ with $B(x,r)\subseteq A$. Now any point $y\in B(x,r)$ is connected to x via a line segment and thus there is no path joining y to x_0:

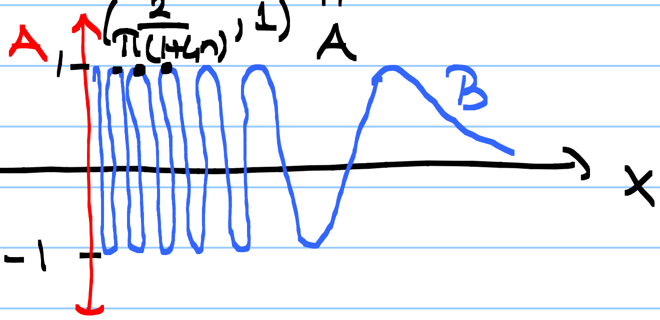


Hence, $B(x,r)\cap C=\emptyset$, which implies $\underline{B(x,r)\subseteq A\setminus C}$.

Thus, $A\setminus C$ is open and hence C is closed. ■

<u>Example</u> (Topologist's Sine Curve)

Let $X=\underbrace{\{(0,y)\mid y\in\mathbb{R}\}}_{A}\cup\underbrace{\{(x,\sin\tfrac{1}{x})\mid x>0\}}_{B}$



<u>Claim</u> X is connected.

A is the image of the map $\mathbb{R}\to\mathbb{R}^2$, $t\longmapsto(0,t)$ and thus A is connected (path connected).
Similarly, B is the image of the map $(0,\infty)\longrightarrow\mathbb{R}^2$, $t\longmapsto(t,\sin 1/t)$ and thus

B is (path) connected.

$p_n=\left(\frac{2}{\pi(1+4n)},1\right)\in A$, for all $n\in\mathbb{N}$.

$\lim p_n=(0,1)\in B$. Since A is connected and $(0,1)\in\bar{A}\cap B$, we see that $A\cup B$ is connected. (Exercise, prove this!)

Hence, X is connected.

Claim. X is not path connected.

Definition. X topological space and let $x \in X$ be any point. Set E_x to be the union of all connected subsets of X containing x:

$$E_x = \bigcup_{\substack{x \in A \\ A \subseteq X \text{ connected}}} A$$

Since all A's in the union contain the point x (i.e. they have a common point) E_x is connected.

Note that E_x is the largest connected subset of X containing x. However, $\overline{E_x}$ is also connected and $E_x \subseteq \overline{E_x}$ and hence $E_x = \overline{E_x}$, so that E_x is closed. E_x is called the connected component of X containing x. If $y \in E_x$ then $E_x = E_y$. In other words, for any $x, y \in X$ we have either $E_x = E_y$ or $E_x \cap E_y = \emptyset$.

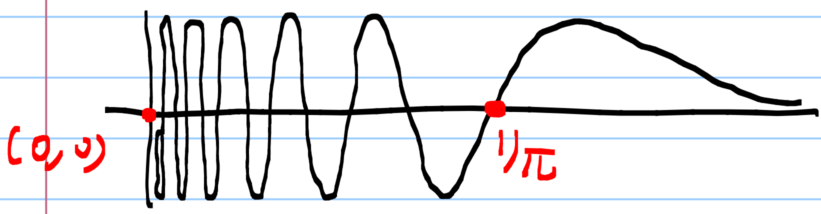
In particular, X is the disjoint union of its connected components:

$$X = \bigcup_{x \in X} E_x = \bigcup_{i \in \Lambda} E_{x_i}$$

Remark. If the number of components of X is finite then $X = E_{x_1} \cup E_{x_2} \cup \dots \cup E_{x_n}$. So

$$E_{x_i} = X \setminus \underbrace{\Big(\bigcup_{j \neq i} E_{x_j} \Big)}_{\text{closed}} \Rightarrow E_{x_i} \text{ is open.}$$

Proof of the claim:



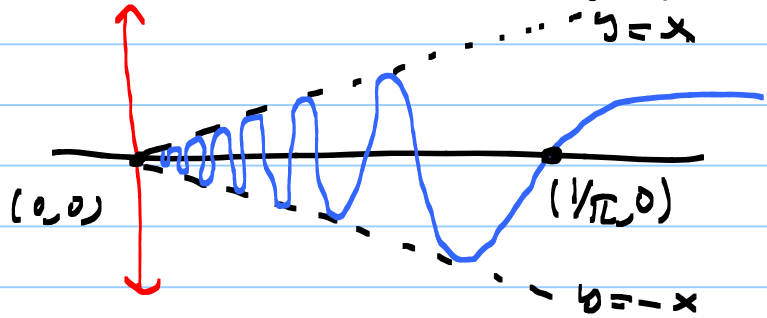
Assume on the contrary that X is path connected. By assumption there is a continuous path $\gamma: [0,1] \longrightarrow X$ so that $\gamma(0) = (0,0)$ and $\gamma(1) = (1/\pi, 0)$.

Let $\gamma(t) = (x(t), y(t))$, where $x(t)$ and $y(t)$ are continuous real valued functions with $x(0) = 0$ and $x(1) = 1/\pi$. The the topological component of the open subset $x^{-1}((0,\infty))$ containing 1 has the form $(a, 1]$, where $1 > a \geq 0$. Note that $x(a) = 0$ because if $x(a) > 0$ the the component $(a,1]$ would be $(b,1]$ where $b < a$. ($x(a) = 0$ since $a \notin x^{-1}((0,\infty))$)
However, for any $t \in (a,1)$ we have $x(t) > 0$ and thus

$$\underline{\underline{y(a)}} = \lim_{t \to a^+} y(t) = \lim_{t \to a^+} \sin 1/x(t) = \lim_{x \to 0} \sin 1/x,$$

where the last limit does not exist. Hence, there is no path joining $(0,0)$ to $(1/\pi, 0)$. In other words, X is not path connected. ■

Example: $X = \{(0,y) \mid y \in \mathbb{R}\} \cup \{(x,y) \mid x > 0, y = x \sin 1/x\}$.



Both components (red and blue) are path connected.

Indeed X is also path connected.

$$\gamma:[0,1/\pi]\longrightarrow X,\quad \gamma(t)=\begin{cases}(0,0) & \text{if } t=0\\ (t, t\sin\frac{1}{t}), & 0<t\leq 1.\end{cases}$$

$\gamma(0)=(0,0)$, $\gamma(1/\pi)=(1/\pi, 0)$.

γ is continuous at $t=0$ since,

$$\gamma(0)=(0,0)=\lim_{t\to 0}(t, t\sin 1/t)=\lim_{t\to 0}\gamma(t).$$

$\blacksquare$

Proposition. $\mathbb{R}^\omega=\mathbb{R}^{\mathbb{N}}$ is connected in the product topology.

Proof: Let $A=\{(x_n)\in\mathbb{R}^\omega \mid x_n=0 \text{ for all } n\geq n_0 \text{ for some } n_0.\}$

A is the set of sequences that are eventually zero.

Note that $A=\bigcup_{m=1}^{\infty}\mathbb{R}^m$, where $\mathbb{R}^m$ is identified with the following subset of A, $\{(x_n)\mid x_{m+k}=0,\ k=1,2,\ldots\}$.

$\mathbb{R}^m$ is connected and they have the common element $(0,0,\ldots,0,\ldots)$. Thus A is connected.

Claim: $\overline{A}=\mathbb{R}^\omega$ and thus $\mathbb{R}^\omega$ is connected.

proof: Take any $(x_n)\in\mathbb{R}^\omega$ and an open subset U containing (x_n). Then there is a basis element of the form $B=(a_1,b_1)\times(a_2,b_2)\times\cdots\times(a_n,b_n)\times\mathbb{R}\times\mathbb{R}\times\cdots$

Note that B contains the element

$$\left(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}, \ldots, \frac{a_n+b_n}{2}, 0, 0, \ldots\right)\in A.$$

Hence, $B\cap A\neq\emptyset \Longrightarrow U\cap A\neq\emptyset$ so that $(x_n)\in\overline{A}$.

So, $\overline{A} = \mathbb{R}^\omega$.

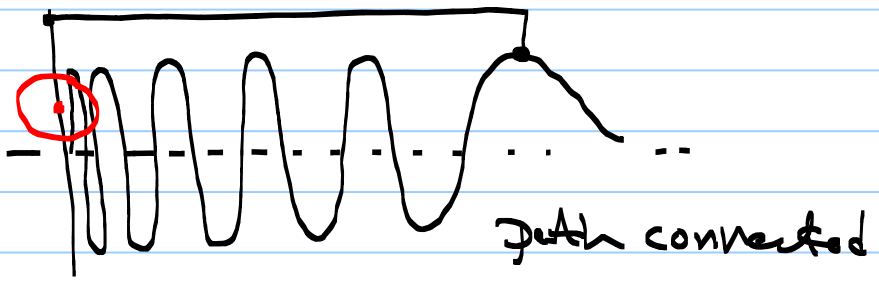
§25. Components and Local Connectedness:

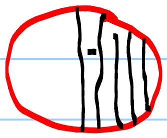
We've already studied connected components.

<u>Definition:</u> A space X is said to be locally connected at x if for every neighborhood U of X, there is a connected neighborhood V of x contained in U.

Similarly, a space X is said to be locally path connected if for every neighborhood U of x, there is path connected neighborhood V of x contained in U.

<u>Example</u>





Hence, X is <u>not</u> locally connected.

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§26. Compact Spaces:

Definition: A collection $\mathcal{A}$ of subsets of a space X is said to cover X, or to be a covering of X, if the union of its elements is equal to X. It is also called an open covering of X if its elements are open subsets of X.

$$X = \bigcup_{A \in \mathcal{A}} A$$

Definition. A space X is called compact if every open covering $\mathcal{A}$ of X contains a finite subcollection that also covers X.

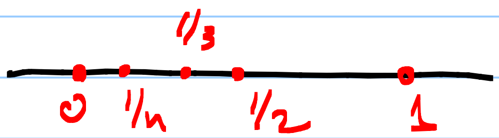
Examples 1) $\mathbb{R}_{std}$ is not compact.

$$\mathbb{R} = \bigcup_{n=1}^{\infty} (-n, n), \qquad A_n = (-n,n), \quad \mathcal{A} = \{A_n\}_{n \in \mathbb{N}}$$

$$A_{n_1} \cup \dots \cup A_{n_k} = A_{n_0}, \text{ where } n_0 = \max\{n_1, \dots, n_k\}.$$

Since the open cover $\mathcal{A}$ has no finite subcover $\mathbb{R}_{std}$ is not compact.

2) $X = \{0\} \cup \{1/n \mid n = 1, 2, \dots\}$ is compact as a subspace of $\mathbb{R}_{std}$.



Let $\mathcal{A} = \{A_\alpha\}$ be an open cover for X.

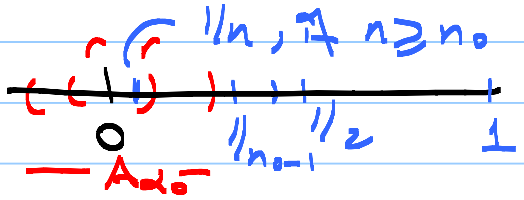
So, $X = \bigcup_{\alpha} A_\alpha$. Let α_0 be so that $0 \in A_{\alpha_0}$.

Since A_{α_0} is an open subset of the subspace X, $A_{\alpha_0} = X \cap U$, where U is open in $\mathbb{R}_{std}$. Since

$0 \in U$ there is some $r > 0$ so that $(-r, r) \subseteq U$. Choose $n_0 \in \mathbb{N}$ so that $n_0 > 1/r$. Then if $n \geq n_0$ we have

$$0 < 1/n \leq 1/n_0 < r \Rightarrow 1/n \in (-r, r) \cap X \subseteq A_{\alpha_0}.$$

Hence, $1/n \in A_{\alpha_0}$ for all $n \geq n_0$.



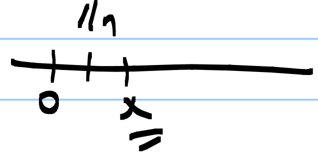
For any $k = 1, \ldots, n_0 - 1$ choose some α_k so that $1/k \in A_{\alpha_k}$. Then

$$X = \{0\} \cup \{1, 1/2, \ldots, 1/(n_0-1), 1/n_0, 1/(n_0+1), \ldots\} \subseteq A_{\alpha_0} \cup A_{\alpha_1} \cup \ldots \cup A_{\alpha_{n_0-1}}$$

In particular, X is covered by finitely many A_α's. Hence, X is compact.

3) $X = (0,1]$ is not compact as a subspace of $\mathbb{R}_{std}$.

$A_n = (1/n, 1]$, $n = 1, 2, \ldots$. Clearly, $X = \bigcup_{n=1}^{\infty} A_n$ so that $\mathcal{A} = \{A_n\}$ is an open cover for X.



On the other hand, any finite union $A_{n_1} \cup A_{n_2} \cup \ldots \cup A_{n_k} = A_{n_0} = (1/n_0, 1] \neq (0,1] = X$, where $n_0 = \max\{n_1, n_2, \ldots, n_k\}$, and thus $\mathcal{A}$ has no finite subcovering for X. Hence, X is not compact.

Lemma: Let Y be a subspace of X. Then Y is compact if and only if every open covering of Y by sets open in X contains a finite subcollection covering Y.

Proof: ($\Longrightarrow$) Assume Y is a compact space.

Let $\mathcal{O} = \{O_\alpha\}$ be a collection of open subsets of X so that $Y \subseteq \bigcup_\alpha O_\alpha$. Then let $A_\alpha = Y \cap O_\alpha$, which are open subsets of Y. Then

$$Y \subseteq \left(\bigcup_\alpha O_\alpha\right) \cap Y = \bigcup_\alpha (O_\alpha \cap Y) = \bigcup_\alpha A_\alpha \subseteq Y$$

$\Rightarrow Y = \bigcup_\alpha A_\alpha$. Since Y is compact we have

$Y = A_{\alpha_1} \cup \dots \cup A_{\alpha_k}$, for some $\alpha_1, \dots, \alpha_k$, and $k \in \mathbb{N}$.

Then $Y = (Y \cap O_{\alpha_1}) \cup \dots \cup (Y \cap O_{\alpha_k}) = Y \cap (O_{\alpha_1} \cup \dots \cup O_{\alpha_k})$

$\Rightarrow Y \subseteq O_{\alpha_1} \cup \dots \cup O_{\alpha_k}$.

This finishes the proof of the "$\Longrightarrow$" part.

The other direction is left as an exercise. ∎

Theorem Every closed subspace of a compact space is compact.

Proof: Let Y be a subspace of a compact space X. Let $\mathcal{O} = \{O_\alpha\}$ be a collection of open subsets of X covering Y:

$$Y \subseteq \bigcup_\alpha O_\alpha.$$

must show: Y is contained in the union of finitely many $\mathcal{O}_\alpha$'s.

$$X = Y \cup (X \setminus Y) \subseteq \left(\bigcup_\alpha \mathcal{O}_\alpha\right) \cup (X \setminus Y) \subseteq X, \text{ where}$$

$X \setminus Y$ is also open. Hence, $X = \bigcup_\alpha \mathcal{O}_\alpha \cup (X \setminus Y)$.
Since X is compact we have

$$X = \mathcal{O}_{\alpha_1} \cup \dots \cup \mathcal{O}_{\alpha_k} \cup (X \setminus Y) \text{ for some } \alpha_1, \dots, \alpha_k.$$

Then $Y = Y \cap X = Y \cap \left[(\mathcal{O}_{\alpha_1} \cup \dots \cup \mathcal{O}_{\alpha_k}) \cup (X \setminus Y) \right]$

$$\Rightarrow Y = (Y \cap \mathcal{O}_{\alpha_1}) \cup \dots \cup (Y \cap \mathcal{O}_{\alpha_k}) \cup (\underbrace{Y \cap (X \setminus Y)}_{= \phi})$$

$\Rightarrow Y = (Y \cap \mathcal{O}_{\alpha_1}) \cup \dots \cup (Y \cap \mathcal{O}_{\alpha_k})$.

$\Rightarrow Y \subseteq \mathcal{O}_{\alpha_1} \cup \dots \cup \mathcal{O}_{\alpha_k}$ so that Y is compact.

Theorem: Every compact subspace of a Hausdorff space is closed.

Proof: Let Y be a compact subspace of a Hausdorff space X.

must show: $X \setminus Y$ is open.

Take any point $x_0 \in X \setminus Y$. For any $y \in Y$, $x_0 \neq y$. Since X is Hausdorff there are open subsets U_y and V_y so that

$$x_0 \in U_y, \quad y \in V_y \quad \text{and} \quad U_y \cap V_y = \phi.$$

Since $Y = \bigcup_{y \in Y} \{y\} \subseteq \bigcup_{y \in Y} V_y$, we see that the collection of open subsets of X, $\{V_y\}_{y \in Y}$,

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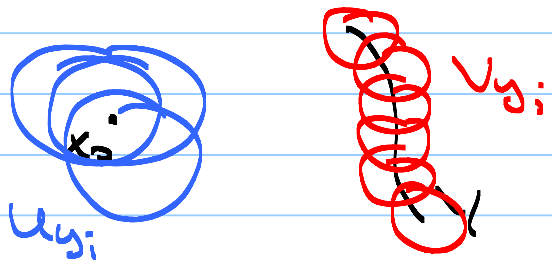
covers Y. Since Y is compact, $Y \subseteq V_{y_1} \cup V_{y_2} \cup \dots \cup V_{y_k}$ for some $y_1, \dots, y_k \in Y$. Let $V = V_{y_1} \cup \dots \cup V_{y_k}$ and $U = U_{y_1} \cap \dots \cap U_{y_k}$. Then U is an open subset containing x_0. Moreover,



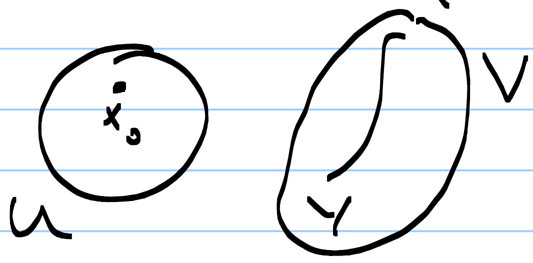
$$U \cap V = \left(\bigcap_i U_{y_i}\right) \cap \left(\bigcup_j V_{y_j}\right)$$

$$= \bigcup_j \left[\left(\bigcap_i U_{y_i}\right) \cap V_{y_j}\right]$$

$$\left(\bigcap_i U_{y_i}\right) \cap V_j \subseteq U_{y_j} \cap V_{y_j} = \emptyset \quad \Big| \quad = \emptyset.$$

Hence, $x_0 \in \bigcap U_{y_i} \subseteq (X \setminus \bigcup V_{y_i}) \subseteq X \setminus Y$, so that $X \setminus Y$ is open. This finishes the proof. ■

Lemma: If Y is a compact subspace of a Hausdorff space X and $x_0 \notin Y$, there are disjoint open subsets U and V of X so that $x_0 \in U$ and $Y \subseteq V$.



Theorem The image of a compact space under a continuous map is compact.

Proof: $f : X \to Y$, X compact, f continuous.

must show: $f(X)$ is a compact subspace.

Let $\mathcal{O} = \{O_\alpha\}$ be an open cover for $f(X)$:

$$f(X) \subseteq \bigcup_\alpha O_\alpha, \quad O_\alpha \subseteq Y \text{ open } \forall \alpha.$$

Then let $U_\alpha \doteq f^{-1}(O_\alpha)$ and note that

$$X = f^{-1}(f(X)) \subseteq f^{-1}\left(\bigcup_\alpha O_\alpha\right) = \bigcup_\alpha f^{-1}(O_\alpha) \subseteq X$$

$$X = \bigcup_\alpha f^{-1}(O_\alpha) = \bigcup_\alpha U_\alpha,$$ where each $U_\alpha = f^{-1}(O_\alpha)$ is open in X.

Since X is compact the open cover $\{U_\alpha\}_\alpha$ of X has a finite subcover, say

$$X = U_{\alpha_1} \cup \dots \cup U_{\alpha_k}, \text{ for some } \alpha_1, \dots, \alpha_k.$$

Then we have $X = f^{-1}(O_{\alpha_1}) \cup \dots \cup f^{-1}(O_{\alpha_k})$ so that

$$f(X) \subseteq f(f^{-1}(O_{\alpha_1}) \cup \dots \cup f^{-1}(O_{\alpha_k}))$$

$$\subseteq O_{\alpha_1} \cup \dots \cup O_{\alpha_k}.$$

Therefore, $f(X)$ is a compact subspace. ∎

Theorem: Let $f: X \to Y$ be a bijective continuous function. If X is compact and Y is Hausdorff then f is a homeomorphism.

Proof: Let $g = f^{-1}: Y \to X$. It is enough to show that g is continuous.

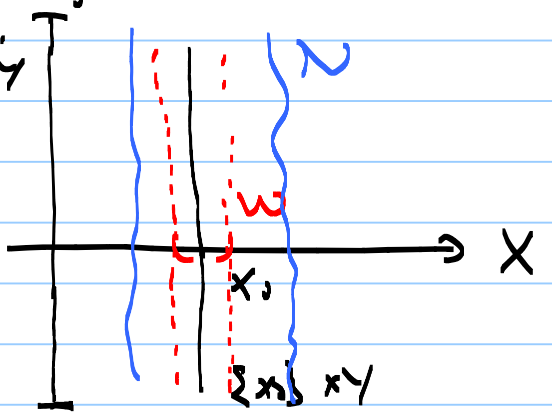
Take any closed set $A \subseteq X$.
must show: $g^{-1}(A)$ is closed in Y. However

$g^{-1}(A) = f(A)$, where A is a closed set in X. Since X is compact A is a compact subspace. Thus, $f(A)$ is a compact subspace of Y. Finally, since Y is Hausdorff $f(A)$ is a closed subset of Y. This finishes the proof. ■

Theorem: The product of finitely many compact spaces is compact.

Proof: **Step 1:** Suppose that X and Y are spaces where Y is compact. Suppose that N is an open subset of $X \times Y$ containing a slice $\{x_0\} \times Y$. Then there is some W neighborhood of x_0 in X so that $W \times Y \subseteq N$.

The product $W \times Y$ is called a tube around $x_0 \times Y$.



Consider the slice $\{x_0\} \times Y$. Note that $\{x_0\} \times Y$ is the image of the continuous (inclusion) map

$Y \longrightarrow X \times Y$, $y \longmapsto (x_0, y)$, $y \in Y$, and thus $\{x_0\} \times Y$ is compact.

For any point $y \in Y$ choose open subsets U_y and V_y with $x_0 \in U_y \subseteq X$, $y \in V_y \subseteq Y$ so that

$(x_0, y) \in U_y \times V_y \subseteq N$. Since Y is compact and $Y = \bigcup_{y \in Y} V_y$ then $\exists$ $y_1, \ldots, y_k \in Y$ so that

$Y = V_{y_1} \cup \ldots \cup V_{y_k}$. Set $W = U_{y_1} \cap \ldots \cap U_{y_k}$.

<u>Claim</u> $\{x_1\} \times Y \subseteq W \times Y \subseteq N$.

<u>proof</u>: $W \times Y = (U_{y_1} \cap \ldots \cap U_{y_k}) \times (V_{y_1} \cup \ldots \cup V_{y_k})$

$$= \bigcup_{i=1}^{k} (U_{y_1} \cap \ldots \cap U_k) \times V_{y_i}$$

$$\subseteq \bigcup_{i=1}^{k} (U_{y_i} \times V_{y_i}) \subseteq N.$$

The above statement is also known as
"Tube Lemma".

<u>Step 2</u>: Now we prove the theorem.
First let's consider two compact spaces X, Y.

<u>must show</u> $X \times Y$ is compact.

Let $\mathcal{A} = \{A_\alpha\}$ be an open cover for $X \times Y$.
Let $x_0 \in X$ be fixed. For any $y \in Y$ there is
some A_{α_y} so that $(x_0, y) \in A_{\alpha_y}$.

$\{x_0\} \times Y = \bigcup_{y \in Y} \{(x_0, y)\} \subseteq \bigcup_{y \in Y} A_{\alpha_y}$ and hence $\{A_{\alpha_y}\}_{y \in Y}$
is an open cover for the compact subspace
$\{x_0\} \times Y$. Thus $\{x_0\} \times Y \subseteq A_{\alpha_{y_1}} \cup \ldots \cup A_{\alpha_{y_k}}$

for some $y_1, \ldots, y_k \in Y$. Now by the Tube
Lemma (Step 1) there is some W_{x_0}, an open
subset of X with $x_0 \in W$ and

$$W_{x_0} \times Y \subseteq A_{\alpha_{y_1}} \cup \cdots \cup A_{\alpha_{y_k}}.$$

Now note that the collection open open subsets $\{W_{x_0}\}_{x_0 \in X}$ is an open cover for the compact space X. Then $X = W_{x_1} \cup W_{x_2} \cup \cdots \cup W_{x_m}$ where each $W_{x_i} \times Y$ is covered by finitely many elements of the collection $\mathcal{A} = \{A_\alpha\}$.

So, $X \times Y = \left(\bigcup_{i=1}^{m} W_{x_i}\right) \times Y = \bigcup_{i=1}^{m} (W_{x_i} \times Y)$ is covered by finitely many A_α's.

This finishes the proof of the theorem. ∎

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Definition: A collection $\mathcal{C}$ of subsets of X is said to have the finite intersection property if for every finite subcollection $\{C_1, \dots, C_n\}$ of $\mathcal{C}$, the intersection $C_1 \cap \dots \cap C_n$ is nonempty.

Theorem: Let X be a topological space. Then X is compact if and only if for every collection $\mathcal{C}$ of closed sets in X having the finite intersection property, the intersection $\bigcap_{C \in \mathcal{C}} C$ of all the elements of $\mathcal{C}$ is nonempty.

Proof: ($\Longrightarrow$) Assume X is compact.

Let $\mathcal{C}$ be any collection of closed subsets of X having the finite intersection property.

<u>must show</u> $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$.

Assume on the contrary that $\bigcap_{C \in \mathcal{C}} C = \emptyset$.

Let $U_C = X \setminus C$, $C \in \mathcal{C}$, which is an open set

so that $\bigcup_{C \in \mathcal{C}} U_C = \bigcup_{C \in \mathcal{C}} (X \setminus C) = X \setminus \left(\bigcap_{C \in \mathcal{C}} C \right) = X \setminus \emptyset = X$.

Hence, the collection $\mathcal{U} = \{U_C\}_{C \in \mathcal{C}}$ is an open cover for the compact space X. Thus X can be covered by finitely many elements of $\mathcal{U}$, say $U_{C_1}, \dots, U_{C_n}$:

$$X = U_{C_1} \cup \dots \cup U_{C_n} = (X \setminus C_1) \cup \dots \cup (X \setminus C_n)$$

$$= X \setminus (C_1 \cap \dots \cap C_n).$$

Hence, $C_1 \cap \dots \cap C_n = \emptyset$, which is a contradiction because the collection $\mathcal{C}$ has the finite intersection property. Therefore, we must have

$$\bigcap_{C \in \mathcal{C}} C \neq \emptyset.$$

($\Leftarrow$) Left as an exercise. ∎

<u>Remark:</u> Assume that X is compact and we have a sequence of closed nonempty sets

$$C_1 \supseteq C_2 \supseteq C_3 \supseteq \cdots \supseteq C_n \supseteq C_{n+1} \supseteq \cdots$$

Note that $C_{n_1} \cap \cdots \cap C_{n_k} = C_{n_k}$ if $n_1 \leq n_2 \leq \cdots \leq n_k$. So that $C_{n_1} \cap \cdots \cap C_{n_k} = C_{n_k} \neq \emptyset$.
So the collection $\mathcal{C} = \{C_n\}_{n=1}^{\infty}$ has the finite intersection property. Since X is compact by the above theorem we see that

$$\bigcap_{n=1}^{\infty} C_n \neq \emptyset.$$

<u>Non-examples:</u> 1) $X = \mathbb{R}_{std}$, $C_n = [n, \infty)$, $n = 1, \ldots, \infty$.

C_n closed, nonempty and $C_{n+1} \subseteq C_n$. However,

$$\bigcap_{n=1}^{\infty} C_n = \emptyset.$$

2) $X = (0, 1]$, $C_n = (0, 1/n]$ closed nonempty subsets.

$C_{n+1} \subseteq C_n$, for all n. However, $\bigcap_{n=1}^{\infty} C_n = \emptyset$.

§27. Compact Subsets of the Real Line.

Theorem: A subset A of $\mathbb{R}_{std}$ is compact if and only if A is closed and bounded.

Proof: ($\Rightarrow$) Assume that A is compact. Since $\mathbb{R}_{std}$ is Hausdorff every compact subset is closed. Hence, A must be closed.

To show A is bounded consider the open cover $\mathcal{U} = \{U_n\}$, $U_n = (-n, n)$ of the real line

$\bigcup_{n=1}^{\infty} U_n = \mathbb{R} \supseteq A$. Since A is compact A is covered by finitely many elements of this open cover, say $A \subseteq U_{n_1} \cup U_{n_2} \cup \dots \cup U_{n_k}$, where $n_1 < n_2 < \dots < n_k$.

$\Rightarrow A \subseteq U_{n_k} = (-n_k, n_k)$, so that A is bounded.

($\Leftarrow$) Now assume that A is a closed and bounded subset of $\mathbb{R}_{std}$.

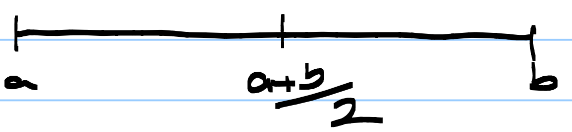
must show: A is compact.

Let $\mathcal{O} = \{O_\alpha\}_{\alpha \in \Lambda}$ be an open cover for $\mathbb{R}_{std}$.

must show: $A \subseteq O_{\alpha_1} \cup \dots \cup O_{\alpha_k}$ for some $\alpha_1, \dots, \alpha_k$.

Since A is bounded $A \subseteq [a, b]$ for some interval $[a, b]$.

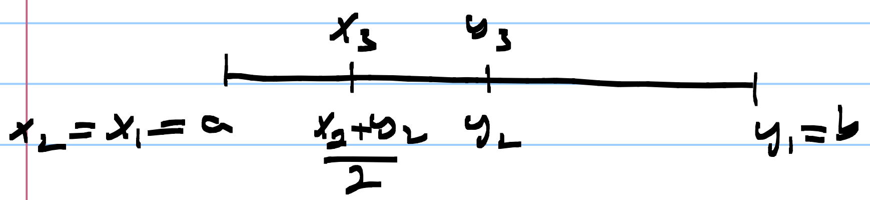
Assume on the contrary that A cannot be covered by finitely many O_α's.



Since $A = \left(A \cap \left[a, \frac{a+b}{2}\right]\right) \cup \left(A \cap \left[\frac{a+b}{2}, b\right]\right)$

we see that one of these sets cannot be covered by finitely many O_α's.

Let $x_1 = a$, $y_1 = b$. Let's say the one that cannot be covered by finitely many O_α's is $[x_2, y_2] \cap A$.



Repeating this process we obtain two sequences of real numbers (x_n) and (y_n) so that

1) $a = x_1 \leq x_2 \leq x_3 \leq \dots$
2) $b = y_1 \geq y_2 \geq y_3 \geq \dots$
3) $x_n < y_n$
4) $A \cap [x_n, y_n]$ cannot be covered by finitely many O_α's.
5) $y_n - x_n = \frac{b-a}{2^{n-1}}$.

By (4) $A \cap [x_n, y_n] \neq \emptyset$. So choose some $a_n \in A \cap [x_n, y_n]$.

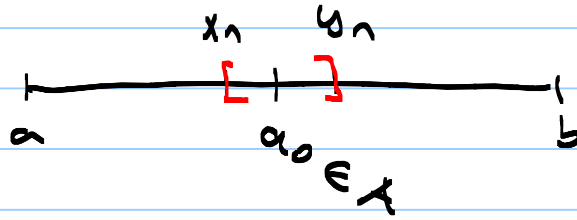
$a = x_1 < x_2 < x_3 \dots \quad \dots y_3 \leq y_2 \leq y_1 = b$

Since (x_n) is an increasing and bounded sequence it is convergent. Similarly, (y_n) is convergent.

However, $y_n - x_n = \frac{b-a}{2^{n-1}}$ so that

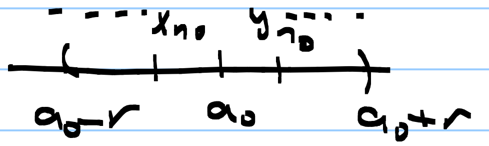
$$\lim_n y_n - \lim_n x_n = \lim_n (y_n - x_n) = \lim_n \frac{b-a}{2^{n-1}} = 0.$$

So, $\lim x_n = \lim y_n$. However, $x_n \le a_n \le y_n$ for all n. By the squeezing lemma $\lim a_n$ exists.
Since $a_n \in A$ and A is a closed subset of $\mathbb{R}$ then $a_0 = \lim a_n \in A$.



Since $a_0 \in A \subseteq \bigcup_\alpha O_\alpha$, then there is some α_0 so that $a_0 \in O_{\alpha_0}$. Since O_{α_0} is open there is some $r > 0$ so that

$$(a_0 - r, a_0 + r) \subseteq O_{\alpha_0}.$$



Since $\lim x_n = a_0 = \lim y_n$ there is some $n_0 \in \mathbb{N}$ so that $[x_{n_0}, y_{n_0}] \subseteq (a_0 - r, a_0 + r) \subseteq O_{\alpha_0}$. So

$A \cap [x_{n_0}, y_{n_0}] \subseteq O_{\alpha_0} \cap A \subset O_{\alpha_0}$, which is a contradiction, because by construction no $A \cap [x_n, y_n]$ can be covered by finitely many O_α's.

So A can be covered by finitely many O_α's, i.e., A is compact. ∎

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Corollary Every closed interval in $\mathbb{R}$ is compact.

Corollary A subset of $\mathbb{R}^n_{std}$ is compact if and only if it is closed and bounded.

Proof: Let $A \subseteq \mathbb{R}^n_{std}$ be a compact subset. The proof we gave for $\mathbb{R}_{std}$ works here also so that A is closed and bounded.

For the other direction, let A be a closed and bounded subset. Then

$$A \subseteq [a_1, b_1] \times \cdots \times [a_n, b_n].$$

Since each $[a_i, b_i]$ is compact (the previous Corollary) this product $[a_1, b_1] \times \cdots \times [a_n, b_n]$ is compact. Finally, since A is a closed subset of the compact subspace $[a_1, b_1] \times \cdots \times [a_n, b_n]$ A is also compact. ∎

Theorem (Extreme Value Theorem)

Let $f : X \longrightarrow \mathbb{R}_{std}$ be a continuous map, where X is a compact space. Then there are points $c, d \in X$ so that

$$f(c) \leq f(x) \leq f(d), \text{ for all } x \in X.$$

Proof: Since X is compact and f is continuous $f(X)$ is a compact subset of $\mathbb{R}_{std}$. Therefore, $f(X)$ is a closed and bounded subset of $\mathbb{R}_{std}$. Let $m = \inf f(X)$ and $M = \sup f(X)$.

Since $f(X)$ is closed $m, M \in f(X)$. Hence, there are $c, d \in X$ such that

$$f(c) = m \text{ and } f(d) = M.$$

Now, for any $x \in X$, $m = \inf f(X) \leq f(x) \leq \sup f(X) = M$. ∎

§ 28. Limit Point Compactness:

<u>Theorem</u>: Let (X,d) be a metric space. Then the following conditions are equivalent.

1) X is compact

2) X is sequentially compact. In other words, every sequence (x_n) in X has a convergent subsequence (x_{k_n}).

3) (X,d) is complete and precompact.

<u>Definition</u>: 1) (X,d) is complete means that every Cauchy sequence in (X,d) is convergent.

2) (X,d) is precompact means that for every $\epsilon > 0$ there are finitely many balls $B(x_1,\epsilon), \dots, B(x_k,\epsilon)$ so that

$$X = B(x_1,\epsilon) \cup \dots \cup B(x_k,\epsilon).$$

<u>Remark</u>: The proof of the above theorem is basically the proof of the Theorem that

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states "Compact subsets of $\mathbb{R}_{std}$ are the ones that are closed and bounded."

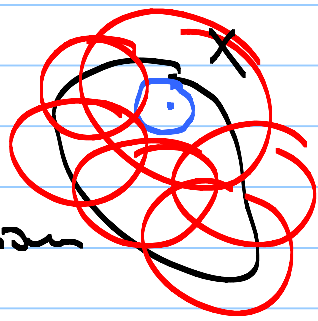
Lebesgue Number: Let $\mathcal{A}$ be an open cover of the metric space (X,d). If X is compact, there is some $\delta>0$ such that for any $x\in X$ the ball $B(x,\delta)\subseteq A$, for some element $A\in\mathcal{A}$.
The number $\delta>0$ is called a Lebesgue number of the covering $\mathcal{A}$ of the compact space (X,d).

Proof: Let $x\in X$. Since $\mathcal{A}$ is an open cover, $x\in A$ for some $A_x\in\mathcal{A}$. A_x is an open set and thus there is some $r_x>0$ so that $B(x,r_x)\subseteq A_x$.

Now, clearly we have $X=\bigcup_{x\in X}\{x\}\subseteq\bigcup_{x\in X}B(x,r_x/2)\subseteq X$

so that $X=\bigcup_{x\in X}B(x,r_x/2)$. Since X is compact finitely many such balls cover X, say

$$X=B(x_1,r_{x_1}/2)\cup\cdots\cup B(x_n,r_{x_n}/2).$$



Let $\delta=\min\{r_{x_1}/2,\ldots,r_{x_n}/2\}>0$.

Let $x\in X$. Then $x\in B(x_i,r_{x_i}/2)$, for some $i\in\{1,\ldots,n\}$. We know that

$$x\in B(x_i,r_{x_i}/2)\subseteq B(x_i,r_{x_i})\subseteq A_{x_i}.$$

Claim: $B(x,\delta)\subseteq B(x_i,r_{x_i})\subseteq A_{x_i}$.

Proof Let $y\in B(x,\delta)$.

Then $d(y, x_i) \le d(y,x) + d(x, x_i)$
$< \delta + r_{x_i}/2 \le r_{x_i}/2 + r_{x_i}/2 = r_{x_i}$.

$\Rightarrow y \in B(x_i, r_{x_i})$. Hence, $B(x,\delta) \subseteq B(x_i, r_{x_i}) \subseteq A_{x_i}$.

§29. Local Compactness.

Definition: Let X be a space and $x \in X$. We say that X is locally compact at x if there is a compact subspace C and an open subset U so that

$$x \in U \subseteq C \subseteq X.$$

In other words, x has a neighborhood whose closure is compact.

X is said to be a locally compact space if X is locally compact at every point $x \in X$.

Example: 1) Any compact space is locally compact.

2) $\mathbb{R}_{std}$ is locally compact, because for any $x \in \mathbb{R}$ we have $x \in (x-1, x+1) \subseteq [x-1, x+1]$, which is compact.

3) $\mathbb{R}^n_{std}$ is locally compact.

4) $\mathbb{R}^\omega_{std} = \mathbb{R}^{\mathbb{N}}_{std}$ is not locally compact, because any open set in $\mathbb{R}^\omega$ contains an open set of the form $(a_1, b_1) \times \cdots \times (a_n, b_n) \times \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \times \cdots$.

The following theorem is a characterization of locally compact spaces.

Theorem: Let X be a space. Then X is locally compact Hausdorff if and only if there is a space Y satisfying the following conditions:

1) X is a subspace of Y.
2) The $Y \setminus X$ consists of a single point.
3) Y is a compact Hausdorff space.

If Y and Y' are two spaces satisfying these conditions, then there is a homeomorphism of Y onto Y' that equals the identity map on X.

Proof: **Step 1:** **Uniqueness:** Let Y and Y' be two spaces satisfying the conditions 1, 2 and 3.
Let $h: Y \to Y'$ defined by

$$h(x) = \begin{cases} x & \text{if } x \in X \subseteq Y \\ y' & \text{if } x = y \end{cases}, \text{ where } \begin{array}{l} Y = X \cup \{y\} \text{ and} \\ Y' = X \cup \{y'\}. \end{array}$$

Let $U \subseteq Y'$ be an open set.
Case 1 $y' \notin U$, then $U \subseteq X$. Then $h^{-1}(U) = U$ is open in X and thus open in Y, because X is open in Y. X is open in Y because $X = Y \setminus \{y\}$, where $\{y\}$ is closed since Y is Hausdorff.

Case 2 $y' \in U$. Let $C = Y' \setminus U$, which is a closed set in Y'. Hence, C is compact in Y' and hence in X.

$h^{-1}(C) = C$, which is compact in X.
then $h^{-1}(C)$ is compact in Y. Since Y is Hausdorff the compact subset $h^{-1}(C) = C$ is closed in Y.
Hence, $h^{-1}(U) = Y \setminus C$ is open in Y.
Thus, h is continuous.

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The inverse map $h^{-1}: Y' \to Y$ is basically the same map. The same argument shows that h^{-1} is also continuous. Thus, h is a homeomorphism.

Step: Now let X be a locally compact space. Will construct a space Y satisfying the three conditions.

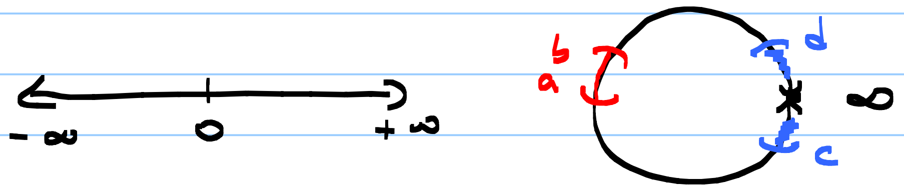
Let $Y = X \cup \{\infty\}$, $(\infty \notin X)$.

We declare that the open subsets of Y are the open subsets of X and subsets of Y containing ∞, whose complements are compact subsets of X:

$U \subseteq Y$ open $\iff$ 1) $U \subseteq X$ open
2) $\infty \in U$, $X \setminus U = C \subseteq X$ compact subset of X.

Remark. $X = \mathbb{R}_{std}$, $Y = \mathbb{R}_{std} \cup \{\infty\}$.

$\mathcal{B} = \{(a,b) \mid a,b \in \mathbb{R}\} \cup \{\{\infty\} \cup (-\infty, c) \cup (d, \infty) \mid c < d \in \mathbb{R}\}$



We need to show that this really defines a topology on Y, left as an exercise.

Then we claim that Y is compact: Let $\mathcal{A}$ be an cover for $Y = X \cup \{\infty\}$. Let $A_0 \in \mathcal{A}$ be so that $\infty \in A_0$. Then $Y \setminus A_0 = X \setminus A_0$ is compact in X.

Since $Y = \bigcup_{A \in \mathcal{A}} A$ we have $X \setminus A_0 \subseteq \bigcup_{A \in \mathcal{A}} A$ and

since $X \setminus A_0$ is compact $X \setminus A_0 \subseteq A_1 \cup \dots \cup A_n$ for some $A_1, \dots, A_n \in \mathcal{A}$. Then $X \subseteq A_0 \cup \dots \cup A_n$.

Finally, since $\infty \in A_0$, $Y = X \cup \{\infty\} \subseteq A_0 \cup A_1 \cup \dots \cup A_n$.
Hence, Y is compact.

<u>Y is Hausdorff:</u> Let $x_1, x_2 \in Y$ with $x_1 \neq x_2$.

If $x_1, x_2 \in X$ then since X is Hausdorff there are open subsets $U, V \subseteq X$ so that $x_1 \in U$, $x_2 \in V$ and $U \cap V = \emptyset$. Since, U, V are also open in Y, we are done in this case.
Otherwise, without loss of generality we may assume that $x_1 \in X$ and $x_2 = \infty$. Since X is locally compact there are some U, C subsets of X so that $x_1 \in U \subset C \subseteq X$, where U is open in X and C is compact in X.
Then $V = (X \setminus C) \cup \{\infty\} \subseteq Y$ is open in Y so that $x_2 = \infty \in V$ and $U \cap V = \emptyset$.
Hence Y is also Hausdorff.

<u>Step 3:</u> Suppose that there is a space Y satisfying the conditions 1, 2, 3. Then we must show $X = Y \setminus \{\infty\}$ is a locally compact Hausdorff subspace.
Since Y is Hausdorff by assumption, the subspace X is Hausdorff.
Given $x \in X$ choose <u>disjoint</u> open sets U and V of Y with $x \in U$ and $\infty \in V$. Such U and V exist since Y is Hausdorff and $x \neq \infty$.
Then the subset $C = Y \setminus V$ is a compact subset of X and $U \subseteq Y \setminus V$ since $U \cap V = \emptyset$.
Hence, $x \in U \subseteq Y \setminus V = C \subseteq X$, where U is open and C is compact in X. Thus, X is a locally compact Hausdorff space. ∎

Definition. If Y is a compact Hausdorff space and X is a proper subspace of Y whose closure equals Y, then Y is said to a compactification of X.
If $Y \setminus X$ equals a single point, then Y is called the one-point compactification of X.

Ex 1) $X = \mathbb{R}$, $Y = [0,1]$
$= (0,1) \subseteq [0,1] = Y$, $\overline{X} = Y$

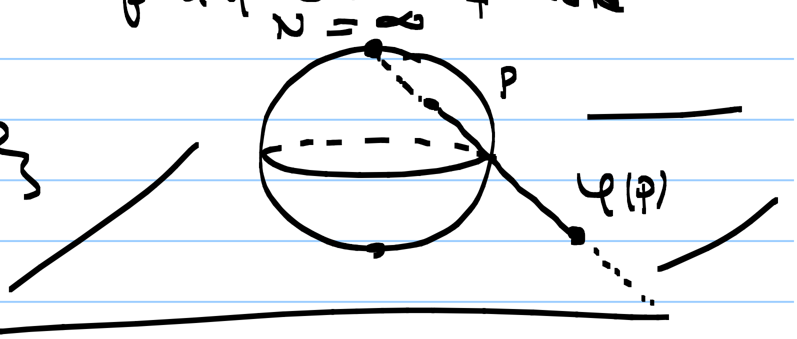
Hence, $[0,1]$ is a compactification for $\mathbb{R}$.

2) The one point compactification of $\mathbb{R}$ is the circle S^1.

3) $X = \mathbb{R}^2_{std} = \mathring{D}^2$ open disc. $\overline{X} = \overline{\mathring{D}^2} = D^2$ is a compactification of $\mathbb{R}^2_{std}$.

4) The one-point compactification of $\mathbb{R}^2_{std}$ is the sphere S^2.

$S^2 = \mathbb{R}^2 \cup \{+\infty\}$

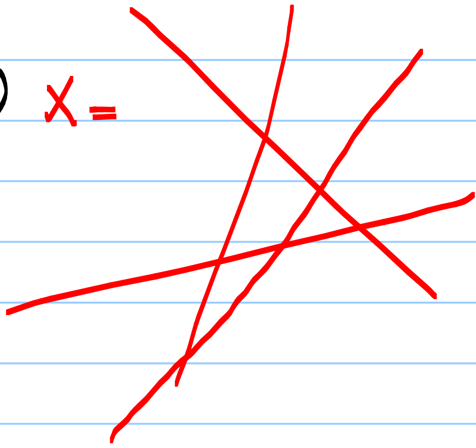


$\infty = N = (0,0,1)$

$\varphi : S^2 \setminus \{N\} \longrightarrow \mathbb{R}^2$, $p \longmapsto \varphi(p)$, is a homeomorphism.

$\varphi((x,y,z)) = \left(\frac{2x}{1-z}, \frac{2y}{1-z}\right)$?

Similarly, the one point compactification of $\mathbb{R}^n_{std}$ is S^n.

5) $X =$  $\subseteq \mathbb{R}^2_{std}$

$\overline{X} = \overline{\mathbb{R}^2_{std}} = S^2$

∞

Theorem: Let X be a Hausdorff space. Then X is locally compact if and only if given $x \in X$, and given a neighborhood U of x, there is a neighborhood V of x such that $\overline{V}$ is compact and $\overline{V} \subseteq U$.

Proof Assume that X is locally compact. Let $x \in X$ and U a neighborhood of x.

must find: a neighborhood V of x s.t. $\overline{V}$ is compact and $\overline{V} \subseteq U$.

Let $Y = X \cup \{\infty\}$ be the one point compactification of X. Since $U \subseteq X$ is open it is also open in Y. $C = Y \setminus U$ is closed in Y and hence compact in Y.

Now we have

$$x \in X, \quad C \subseteq Y \text{ compact}, \quad x \notin C.$$

Then there are open sets V and W in Y so that $x \in V$, $C \subseteq W$ and $V \cap W = \emptyset$.

In particular, $V \subseteq Y \setminus W \subseteq Y \setminus C = U$.

Note that since W is open $Y \setminus W$ is closed in Y.

So, $V \subseteq Y \setminus W \Rightarrow \overline{V} \subseteq Y \setminus W \subseteq Y \setminus C = U$

so that $\overline{V} \cap U = \emptyset$.

Hence, $x \in V \subseteq \overline{V} \subseteq U$.

The other direction is left as an exercise. ■

Corollary: Let X be a locally compact space, let A be a subspace of X. If A is closed in X or open in X, then A is locally compact.

Proof: First assume that A is closed in X. Given $x \in A$ and a neighborhood U of x in A. Then there is an open subset V in X so that $U = V \cap A$. Since X is locally compact there is a compact set C s.t. $x \in C \subseteq V$. Then $x \in C \cap A \subseteq V \cap A = U$, where $C \cap A$ is compact in A. Hence, A is locally compact.
The case A is open is left as an exercise. ∎

Corollary A space X is homeomorphic to an open subspace of a compact Hausdorff space if and only if X is locally compact Hausdorff.

Proof follows from the above theorem and corollary

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CHAPTER 4: Countability and Separation Axioms

Definition: A space X is said to have a countable basis at a point $x \in X$ if there is a countable collection $\mathcal{B}$ of neighborhood of x such that every neighborhood of x contains at least one of the elements of $\mathcal{B}$. A space that has a countable basis at each of its points is said to satisfy the first countability axiom, or to be first countable.

Example: A metric space (X,d) is first countable. For any $x \in X$, consider the collection

$$\mathcal{B} = \{B(x, 1/n) \mid n = 1, 2, \ldots\}.$$

So if U is a neighborhood of $x \in X$ then there is a ball $B(x,r)$ with $B(x,r) \subseteq U$. Choose $n \in \mathbb{N}$ so that $0 < \frac{1}{n} < r$, then $x \in B(x, 1/n) \subseteq B(x,r) \subseteq U$. Hence, (X,d) is first countable.

Theorem: Let X be a topological space.

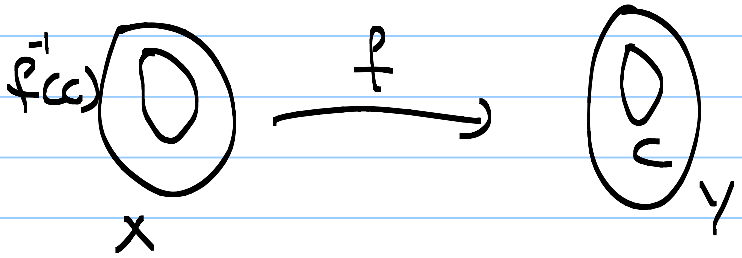
a) Let A be a subset of X. If there is a sequence of points of A converging to x, then $x \in \bar{A}$; the converse holds if X is first countable.

b) Let $f: X \to Y$ be a function. If f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n) \to f(x)$. The converse holds if X is first countable.

Proof of b) Assume that for every sequence (x_n) with $\lim x_n = x$ we have $\lim f(x_n) = f(x)$. We assume X is first countable.

must show: f is continuous.

Suppose not. Then there is a closed subset C of Y so that $f^{-1}(C)$ is <u>not</u> closed in X.



Then there is some $x_0 \in \overline{f^{-1}(C)} \setminus f^{-1}(C)$. By Part (a) there is a sequence $(x_n) \subseteq f^{-1}(C)$ so that $\lim x_n = x_0$. By assumption, $\lim f(x_n) = f(x_0)$ so that $f(x_0) \in \overline{\{f(x_n) \mid n=1,2,\ldots\}} \subseteq \overline{f(f^{-1}(C))} \subseteq \overline{C} = C$ so that $f(x_0) \in C$ and thus $x_0 \in f^{-1}(C)$, a contradiction. Hence, f is continuous.

a) ($\Rightarrow$) is easy and done before.

($\Leftarrow$) Assume that X is first countable and $A \subseteq X$ any subset. Let $\underline{\underline{a \in \overline{A}}}$.

<u>must construct</u> a sequence (x_n) in A with $\lim x_n = a$.

Since X is first countable there is a countable basis $\mathcal{B} = \{U_n \mid n=1,\ldots\}$ of X at a. Choose a sequence (x_n) as follows:

$x_1 \in U_1 \cap A,\ x_2 \in U_1 \cap U_2 \cap A, \ldots, x_n \in U_1 \cap U_2 \cdots \cap U_n \cap A.$

<u>Claim</u> $\lim x_n = a$.

<u>Proof</u> Let $a \in V$ be a neighborhood. Then there is some $n_0 \in \mathbb{N}$ so that $a \in U_{n_0} \subset V$. If $n \geq n_0$ then $x_n \in U_1 \cap U_2 \cap \cdots \cap U_{n_0} \cap \cdots \cap U_n \subseteq U_{n_0} \subset V$. Thus, $\lim x_n = a$. ∎

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Definition. If a space X has a countable basis for its topology, then X is said to satisfy the second countability axiom, or to be second-countable.

Example: 1) $\mathbb{R}^n_{std}$ the countable collection

$\mathcal{B} = \{ B(x,r) \mid x \in \mathbb{Q}^n, r \in \mathbb{Q}^+ \}$. Since $\mathbb{Q}$ is countable $\mathcal{B}$ is countable.

Let $x \in \mathbb{R}^n_{std}$ and $x \in U$ an open subset. Then there is an $\rho > 0$ so that

$$B(x,\rho) \subseteq U.$$

$y \in \mathbb{Q}^n$

$r \in \mathbb{Q}$

$x \in B(y,r) \subseteq B(x,\rho) \subseteq U.$

2) $\mathbb{R}^\omega_{std}$ is second countable.

$$\mathcal{B} = \{ (a_1,b_1) \times \cdots \times (a_n,b_n) \times \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \times \cdots \mid a_i, b_i \in \mathbb{Q}, n \in \mathbb{N} \}$$

Clearly, $\mathcal{B}$ is countable and it is a basis for $\mathbb{R}^\omega_{std}$.

3) In the uniform topology $\mathbb{R}^\omega$ is first countable since $\mathbb{R}^\omega$ is a metric space but it is not second countable. (Will be done later!)

Theorem. A subspace of a first countable space is first countable, a countable product of first countable spaces is first countable. A subspace of a second countable space is second countable, and a countable product of second countable spaces is second countable

Proof. Assume that X is first countable and $A \subseteq X$ a subspace. If $a \in A \subseteq X$ then let $\mathcal{B}_a = \{B_n \mid n \in \mathbb{N}\}$ be a basis X at the point a. Then the collection $\mathcal{B}_a^A = \{B_n \cap A \mid n \in \mathbb{N}\}$ be a basis for A at the point $a \in A$: let $U \subseteq A$ be an open subset containing a. Then $U = V \cap A$ for some open $V \subseteq X$. Since $\mathcal{B}_a$ is a basis for X at a, there is some $B_n \in \mathcal{B}_a$ so that $a \in B_n \subseteq V$. Then $a \in B_n \cap A \subseteq V \cap A = U$.

Let X_n, $n \in \mathbb{N}$, be a sequence of first countable spaces. Let $X = \prod_{n=1}^{\infty} X_n$. Take any $x = (x_n) \in X$.

Then $x_n \in X_n$, for all n. Let $\mathcal{B}_n = \{B_n^m \mid m \in \mathbb{N}\}$ be a countable basis for X_n at x_n.

Let $\mathcal{B} = \{B_{i_1}^1 \times \cdots \times B_{i_n}^n \times X_{n+1} \times X_{n+2} \times \cdots \mid B_{i_j}^j \in \mathcal{B}_j\}$

$x = (x_n) \in U \subseteq X$ open. $(x_n) \in U = U_1 \times \cdots \times U_n \times X_{n+1} \times \cdots$

$x_k \in U_k$ $k = 1, \to n$.

$\Rightarrow x_k \in B_{i_k}^k \subseteq U_k$.

Hence, $\mathcal{B}$ is a countable basis for X at (x_n).

Definition: A subset A of a space X is said to be dense if $\bar{A} = X$.

Theorem. Suppose X has a countable basis. Then

a) Every open covering of X contains a countable collection covering X.

b) There exists a countable subset of X that is dense in X.

Proof: Let $\{B_n\}$ be a countable basis for X.

a) Let $\mathcal{A}$ be an open covering of X. For any $n \in \mathbb{N}$ choose an element A_n of $\mathcal{A}$ so that $B_n \subseteq A_n$, if it is possible. Then the collection $\mathcal{A}'$ of such A_n's is clearly countable. Given $x \in X$, since $\mathcal{A}$ is an open for X there is some $A \in \mathcal{A}$ s.t. $x \in A$. Since A is open there is some basis element B_n with $x \in B_n \subseteq A$. Then $x \in B_n \subseteq A_n$. Then x lies in some element of the subcollection $\mathcal{A}'$. Hence, $\mathcal{A}'$ is an open subcover for X.

b) For any $n \in \mathbb{N}$ choose $x_n \in B_n$. Let
$D = \{x_n \mid n \in \mathbb{N}\}$. Then D is dense in X:

Let $x \in X$ and $U \subseteq X$ any open subset containing x. Then there is a basis element B_n so that $x \in B_n \subseteq U$. So $x_n \in B_n \cap D \subseteq U \cap D$ and hence $U \cap D \neq \emptyset$. So, $x \in \overline{D}$ and thus

$X = \overline{D}$. ∎

Definition: A space for which every open covering contains a countable subcovering is called a Lindelöf space.

A space having a countable dense subset is called separable.

Remark: The above theorem implies that a second countable space is Lindelöf and separable.

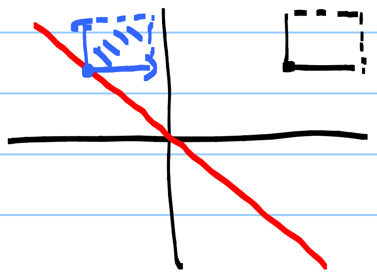
Example: The product of two Lindelöf spaces may **not** be Lindelöf.

Consider $\mathbb{R}_\ell$ (real line with lower limit topology)

Claim $\mathbb{R}_\ell$ is Lindelöf (Example 3 at page 192).

Claim $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is **not** Lindelöf.

$L = \{(x,-x) \mid x \in \mathbb{R}_\ell\}$. Note that L is closed in $\mathbb{R}_\ell \times \mathbb{R}_\ell$.



$\mathbb{R}_\ell^2$ has a basis of the form $[a,b) \times [c,d)$

Sorgenfrey plane

$\mathbb{R}_\ell^2$ is covered by the open subsets $\mathbb{R}_\ell^2 \setminus L$ and products of the form $[a,b) \times [-a,d)$.

This cover has no countable subcover because each $([a,b) \times [-a,d)) \cap L = \{(a,-a)\}$ and L is uncountable.

Hence, $\mathbb{R}_\ell^2$ is **not** Lindelöf. ∎

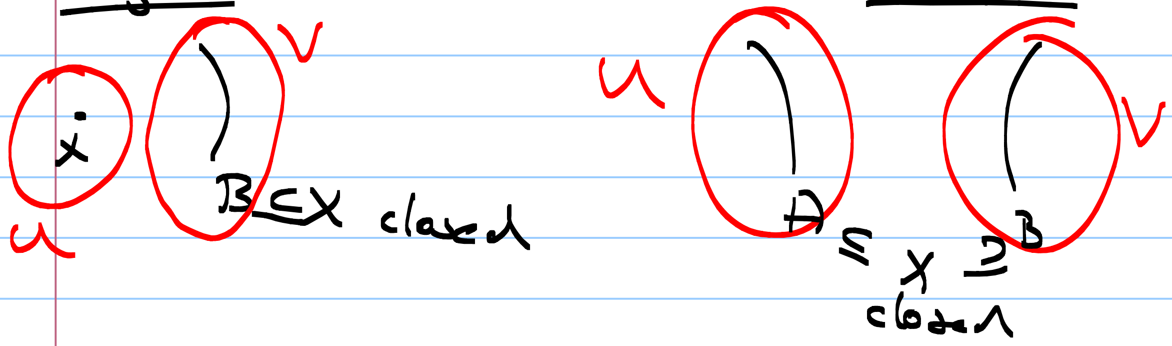
§31. The Separation Axioms:

Definition: Suppose that one point sets are closed in X (i.e. X is T_1). Then X is said to be regular if for each pair consisting of a point x and a closed set B disjoint from x, there exist disjoint open sets containing x and B, respectively.

The space X is called normal if for each pair A, B of disjoint closed sets of X, there exist disjoint open sets containing A and B, respectively.

X T_1, $\{x\} \subseteq X$ closed, for all $x \in X$.

Regular | Normal



Remark: Clearly Normal $\Rightarrow$ Regular $\Rightarrow$ Hausdorff $\Rightarrow T_1$

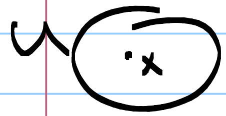
The following is a useful characterization of being regular / normal.

Lemma: Let X be a topological space. Let one-point sets in X be closed.

a) X is regular if and only if given a point x of X and a neighborhood U of x, there is a neighborhood V of x s.t. $\overline{V} \subseteq U$.

b) X is normal if and only if given a closed set A in X and an open set U containing A there is an open set V containing A and $\overline{V} \subseteq U$.

Proof: a) First assume that X is regular, $x \in X$ and U a neighborhood of x. Let $A = X \setminus U$, which is a closed set disjoint from x.



Since X is regular there are disjoint open sets V and W so that $x \in V$ and $A \subseteq W$.

$x \in V$ and $V \cap W = \emptyset \Rightarrow V \subseteq X \setminus W \subseteq X \setminus A = U$.

Moreover, since $V \subseteq X \setminus W$ and $X \setminus W$ is closed we see that $\overline{V} \subseteq X \setminus W$ and thus
$\overline{V} \subseteq X \setminus W \subseteq X \setminus A = U$.

$$x \in V \subseteq \overline{V} \subseteq U.$$

For the other direction, let $x \in X$ and $A \subseteq X$ be a closed set with $x \notin A$. Let $U = X \setminus A$ then $x \in U$ and U is open. So by the assumption there is an open set V s.t.

$$x \in V \subseteq \overline{V} \subseteq U.$$

Let $W = X \setminus \overline{V}$, then $V \cap W = V \cap (X \setminus \overline{V}) = \emptyset$.

Moreover, $x \in V$ and $A = X \setminus U \subseteq X \setminus \overline{V} = W$.
Hence, X is regular.

(b) is similar and left as an exercise.

Theorem: a) A subspace of a Hausdorff space is Hausdorff, a product of Hausdorff spaces is Hausdorff.
b) A subspace of a regular space is regular, a product of regular spaces is regular.

Proof (a) is left as an exercise.

b) Let Y be a subspace of a regular space X. Then one point subsets of Y are also closed. Let $x \in Y$ and $B \subseteq Y$ a closed subset disjoint from x: $x \notin B$. $\overline{B} \cap Y = B$ since B is closed. Hence, $x \notin \overline{B}$. Since $\overline{B}$ is the closure of B in X, it is a closed subset of X, disjoint from x. Since X is regular there are disjoint open subsets U and V of X with

$$x \in U \text{ and } \overline{B} \subseteq V.$$

Then $x \in U \cap Y$ and $B = \overline{B} \cap Y \subseteq V \cap Y$ satisfying

$(U \cap Y) \cap (V \cap Y) = U \cap V \cap Y = \phi$. Thus, Y is regular.

Product: Let $\{X_\alpha\}$ be a family of regular spaces,

$X = \prod X_\alpha$. By (a) X is Hausdorff since each X_α is Hausdorff. Thus X is T_1 so that one point sets are closed.

Take any $x = (x_\alpha) \in X$ and let U be a neighborhood of x. Choose a basis element

$\prod U_\alpha \subseteq U$ containing $x = (x_\alpha)$:

$x_\alpha \in U_\alpha$, where $U_\alpha = X_\alpha$ for all but finitely many α, say $\alpha_1, \ldots, \alpha_n$.

For each α_i, $i = 1, \ldots, n$, choose open subset $V_{\alpha_i} \subseteq X_{\alpha_i}$, so that

$$x_{\alpha_i} \in V_{\alpha_i} \subseteq \overline{V}_{\alpha_i} \subseteq U_{\alpha_i} \subseteq X_{\alpha_i}.$$

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If $\alpha \notin \{\alpha_1, \ldots, \alpha_n\}$, let $V_\alpha = U_\alpha = X_\alpha$.

Now, $x = (x_\alpha) \in \underbrace{\prod_\alpha V_\alpha}_{x \in V} \subseteq \underbrace{\overline{\prod_\alpha V_\alpha}}_{\overline{V}} = \prod \overline{V_\alpha} \subseteq \prod U_\alpha = U$

$$x \in V \subseteq \overline{V} \subseteq U.$$

Hence, $X = \prod X_\alpha$ is regular.

§32. Normal Spaces:

Theorem: Every regular space with countable basis is normal.

Proof: Let X be a regular space with countable basis $\mathcal{B}$. Let A and B be disjoint closed subsets of X.

Aim: Find disjoint open subsets $\tilde{U}$ and $\tilde{V}$ of X such $A \subseteq \tilde{U}$ and $B \subseteq \tilde{V}$.

Let $x \in A$, then $x \notin B$. Since B is closed and $x \in X \setminus B$, which is open there is some V_x open so that $x \in V_x \subseteq \overline{V_x} \subseteq X \setminus B$, because X is regular. Then choose a basis element $C \in \mathcal{B}$ s.t. $x \in C_x \subseteq V_x$.

Then $A = \bigcup_{x \in A} \{x\} \subseteq \bigcup_{x \in A} C_x \subseteq \bigcup_{x \in A} V_x \subseteq \bigcup_{x \in A} \overline{V_x} \subseteq X \setminus B$.

Moreover $\overline{C_x} \subset \overline{V_x} \subset X \setminus B$. Hence, A is covered by countably many basis elements whose closure not intersecting B. Say that countable set is $\{V_n\}$.

$$A \subseteq \bigcup_{n=1}^{\infty} V_n, \quad \overline{V_n} \subseteq X \setminus B \text{ so that } \overline{V_n} \cap B = \emptyset.$$

Similarly, we can find a countable collection of basis elements say $\{U_n\}$ so that $B \subseteq \bigcup_{n=1}^{\infty} U_n$ and $\overline{U_n} \cap A = \emptyset$.

Let $U = \bigcup U_n$ and $V = \bigcup V_n$. Then $A \subseteq U$ and $B \subseteq V$. We would be finished if we had $U \cap V = \emptyset$. However, in general they may intersect. So we'll modify them as follows:

$$U_n' = U_n \setminus \bigcup_{i=1}^{n} \overline{V}_i \quad \text{and} \quad V_n' = V_n \setminus \bigcup_{i=1}^{n} \overline{U}_i,$$ which are both open and $A \subseteq \bigcup_{n=1}^{\infty} V_n'$ and $B \subseteq \bigcup_{n=1}^{\infty} U_n'$

since $\overline{V}_n \cap B = \emptyset = \overline{U}_n \cap A$, for all n.

The following claim will finish the proof of the theorem.

<u>Claim</u>: $U' \cap V' = \emptyset$, where $U' = \bigcup_{n=1}^{\infty} U_n'$ and $V' = \bigcup_{n=1}^{\infty} V_n'$.

<u>Proof of the claim</u>: Suppose not and let $x \in U' \cap V'$.

Then $x \in U_j' \cap V_k'$ for some j and k.

Suppose that $j \leq k$. Since $x \in U_j'$ we have $x \in U_j$. Also since $j \leq k$ and $V_k' = V_k \setminus \bigcup_{i=1}^{k} \overline{U}_i$ and hence $x \notin V_k'$, a contradiction.

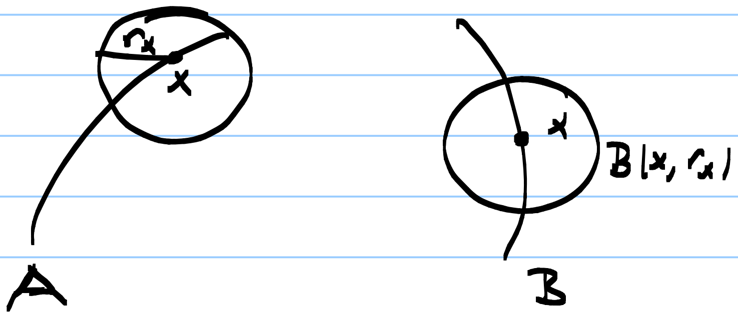
Similarly, if $k \leq j$ then we obtain a similar contradiction. This finishes the proof of the claim. ∎

<u>Theorem</u>: Every metrizable space is normal.

<u>Proof</u>: Let (X, d) be a metric space. Let A and B be disjoint closed subsets of X. For any $x \in A$ choose $r_x > 0$ so that $B(x, r_x) \cap B = \emptyset$; such $r_x > 0$ exists since $x \in X \setminus B$ and $X \setminus B$ is open. Similarly, for any $x \in B$, there is some $r_x > 0$ st. $B(x, r_x) \cap A = \emptyset$.

Now $A \subseteq \bigcup_{x \in A} B(x, r_x/2) \doteq U$, $B \subseteq \bigcup_{x \in B} B(x, r_x/2) \doteq V$

so that $B(x, r_x) \cap B = \emptyset$ if $x \in A$ and
$B(x, r_x) \cap A = \emptyset$ if $x \in B$.



<u>Claim</u>: $U \cap V = \emptyset$.

<u>Proof</u>: Suppose not. So there $\exists$ some $x \in U \cap V$.

Then $x \in B(y, r_y/2) \cap B(z, r_z/2)$, for some $y \in A$ and $z \in B$.

Then $d(y,z) \leq d(y,x) + d(x,z)$
$< r_y/2 + r_z/2$.

<u>Case 1</u> $r_y \leq r_z$. Then $d(y,z) < \frac{r_z}{2} + \frac{r_z}{2} = r_z$, so

that $y \in B(z, r_z) \implies A \cap B(z, r_z) \neq \emptyset$, a contradiction.

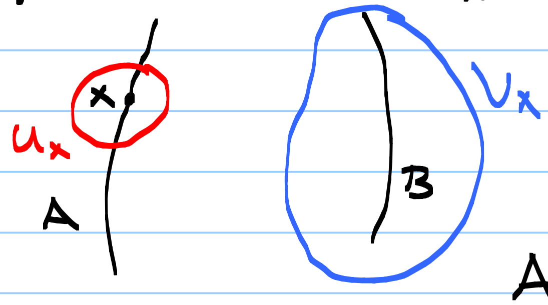
<u>Case 2</u> $r_z \leq r_y$. Then $d(y,z) < \frac{r_y}{2} + \frac{r_y}{2} = r_y$ so that

$z \in B(y, r_y) \implies B \cap B(y, r_y) \neq \emptyset$, again a contradiction.

Thus, $U \cap V = \emptyset$ and hence X is normal. ■

<u>Theorem</u>: Every compact Hausdorff space is normal.

Proof: Let X be a compact Hausdorff space. Let A and B be disjoint closed subsets of X. For any $x \in A$ we have $x \notin B$. By the Midterm 2 question, since X is regular, there are disjoint open subsets $x \in U_x$ and $B \subseteq V_x$.



Since X is compact and A is closed, A is also compact and thus $A \subseteq U_{x_1} \cup \dots \cup U_{x_n}$, for some $x_1, \dots, x_n \in A$.

Let $U = U_{x_1} \cup \dots \cup U_{x_n}$ and $V = V_{x_1} \cap \dots \cap V_{x_n}$, which are both open so that

$A \subseteq U$, $B \subseteq V$.

Claim: $U \cap V = \emptyset$.

proof
$$U \cap V = (U_{x_1} \cup \dots \cup U_{x_n}) \cap (V_{x_1} \cap \dots \cap V_{x_n}) \subseteq \underbrace{(U_{x_1} \cap V_{x_1})}_{= \emptyset} \cup \underbrace{(U_{x_2} \cap V_{x_2})}_{= \emptyset} \cup \dots \cup \underbrace{(U_{x_n} \cap V_{x_n})}_{= \emptyset}$$

$\Rightarrow U \cap V = \emptyset$.

Hence, X is normal. ■

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§33. The Urysohn Lemma

Theorem (Urysohn Lemma)
Let X be a normal space; let A and B be disjoint closed subsets of X. Then there is a continuous map $f: X \longrightarrow [0,1] \subseteq \mathbb{R}_{std}$ so that

$f(x) = 0$, for all $x \in A$ and $f(x) = 1$, for all $x \in B$.

Remark: If $f: X \to [0,1]$ is a continuous map and A and B subsets with $f(A) = \{0\}$, $f(B) = \{1\}$. Then, let $U = f^{-1}(-\infty, 1/2)$ and $V = f^{-1}(1/2, \infty)$ which are disjoint open sets in X so that $A \subseteq U$ and $B \subseteq V$.

Proof of the theorem: Proof contains some steps.

Step 1: Let $P = \mathbb{Q} \cap [0,1]$ be the set of rational numbers in $[0,1]$. Since P is countable we can write $P = \{r_1, r_2, \ldots, r_n, \ldots\}$ so that $r_1 = 1$ and $r_2 = 0$.

Aim: For any $p \in P$ we'll construct an open set so that whenever $p < q$ $(p, q \in P)$ we'll have $\overline{U_p} \subset U_q$.

First define $U_1 \doteq X \setminus B$. Since $A \subseteq X \setminus B = U_1$ and X is regular there is some open set U_0 so that

$$A \subseteq U_0 \subseteq \overline{U_0} \subseteq U_1.$$

For general n, let P_n denote the set consisting of the first n rational numbers in the sequence. $P_n = \{r_1, r_2, \ldots, r_n\}$. Suppose that U_p is defined for all rational numbers p in P_n satisfying

the condition
$$p<q \implies \overline{U_p} \subset U_q.$$

Let $r = r_{n+1}$ the next rational number in $\mathbb{P}$.
Now we'll define U_r:

Note that $\mathbb{P}_{n+1} = \mathbb{P}_n \cup \{r\}$, which is a finite subset of $[0,1]$. Write these finitely many numbers in increasing order
$r_1 = 0 < \cdots < p < r < q < \cdots < 1 = r_2$.

Suppose that p is the immediate predecessor of r in $\mathbb{P}_{n+1}$ and q is the immediate successor of r in $\mathbb{P}_{n+1}$. Then sets U_p and U_q are already defined and $\overline{U_p} \subset U_q$ (since $p,q \in \mathbb{P}_n$).
Since X is normal we can find an open set U_r of X s.t.
$$\overline{U_p} \subset U_r \quad \text{and} \quad \overline{U_r} \subset U_q.$$

Claim: Every pair of elements of $\mathbb{P}_{n+1}$ satisfy the following condition:

$(*)$ $\quad p<q \implies \overline{U_p} \subseteq U_q.$

Proof: If both $p,q \in \mathbb{P}_n$ then $\overline{U_p} \subseteq U_q$ holds by induction.

$p < r < q$

If one of them is r and the other one is a point say $s \in \mathbb{P}_n$ then either $s \leq p$ or $s \geq q$.

If $s \leq p$ then, since $s,p \in \mathbb{P}_n$ we have
$$\overline{U_s} \subseteq U_p \subset U_r.$$

If $s \geq q$ then, since $s, q \in I_n$ we have

$$\overline{U_r} \subset U_q \subset \overline{U_q} \subseteq U_s.$$

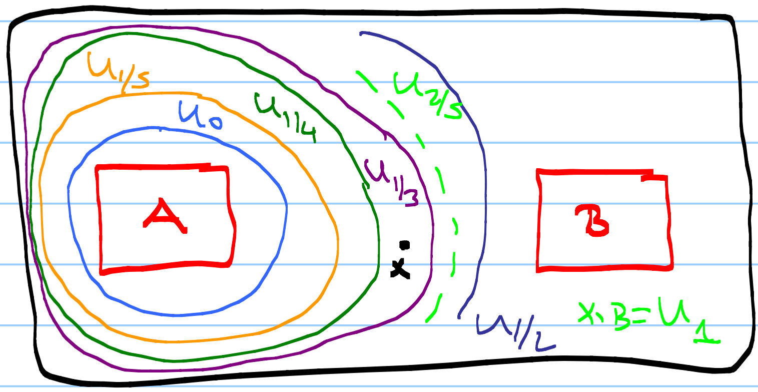
<u>Conclusion:</u> For every pair of points in I_n the relation (*) holds, for all n.

In particular, we've constructed U_p, for any $p \in P$ satisfying (*).

For example, we may choose P as follows:

$$P = \{1, 0, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{4}, \tfrac{3}{4}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{4}{5}, \tfrac{1}{6}, \tfrac{5}{6}, \tfrac{1}{7}, \dots\}$$

$$\overline{U_0} \subset U_{1/2} \subset \overline{U_{1/2}} \subset U_1$$



<u>Step 2</u> For any $p \in P = [0,1] \cap \mathbb{Q}$ we've defined U_p. Now we extend this to all rationals in $\mathbb{R}$ as follows:

$$U_p = \emptyset \quad \text{if} \quad p < 0 \quad \text{and} \quad U_p = X \quad \text{if} \quad p > 1.$$

Note that still we have the relation

$$p < q \Rightarrow \overline{U_p} \subset U_q.$$

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Step 3 Given a point $x \in X$ define $\mathbb{Q}(x)$ to be the set of all rational numbers p s.t. $x \in U_p$:

$$\mathbb{Q}(x) = \{ p \in \mathbb{Q} \mid x \in U_p \}.$$

Note that $\mathbb{Q}(x)$ does not contain negative rational numbers since if $p < 0$ then $U_p = \emptyset$ so that $x \notin U_p$. Also note that $\mathbb{Q}(x)$ contains all $p \in \mathbb{Q}$ with $p > 1$ since in this case $U_p = X$.

Define $f(x)$ as $f(x) = \inf \mathbb{Q}(x) = \inf\{ p \mid x \in U_p \}$. Note that since $\mathbb{Q}(x)$ does not negative reals $\inf \mathbb{Q}(x)$ exists.

Step 4: f is the desired function. In other words, f is continuous, $f(x) = 0$ for all $x \in A$, $f(x) = 1$, for all $x \in B$.

Proof: If $x \in A$ then $x \in U_p$ for all $p \geq 0$. This is because if $x \in A \subseteq U_0$ and thus if $p \geq 0$ then $\overline{U_0} \subseteq U_p$. Then $\mathbb{Q}(x)$ equals the set of all nonnegative rational and thus $f(x) = \inf \mathbb{Q}(x) = 0$.
Similarly, if $x \in B$ then $x \notin U_1 = X \setminus B$, and $x \in U_p = X$ for $p > 1$. Thus $\mathbb{Q}(x)$ consists of all rationals larger than 1. So $f(x) = \inf \mathbb{Q}(x) = 1$.

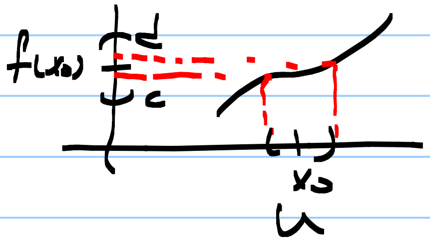
<u>Next we show that f is continuous:</u>

<u>Claim:</u> 1) $x \in \overline{U_r} \Rightarrow f(x) \leq r$, 2) $x \notin U_r \Rightarrow f(x) \geq r$.

<u>Proof of the claim:</u> 1) If $x \in \overline{U_r}$, then $x \in U_s$ for all $s > r$, since $\overline{U_r} \subset U_s$. Thus $\mathbb{Q}(x)$ contains all rationals greater than r, so that $f(x) = \inf \mathbb{Q}(x) \leq r$.

2) If $x \notin U_r$ then $x \notin U_s$ for any $s<r$, because $\overline{U_s} \subset U_r$. Thus $\mathbb{Q}(x)$ contains no rationals less than r. Thus $f(x) = \inf \mathbb{Q}(x) \geq r$. ∎

To prove that $f: X \to [0,1]$ is continuous let $x_0 \in X$ and (c,d) an open interval containing $f(x_0)$. Must find: some open subset U in X with $x_0 \in U$ and $f(U) \subseteq (c,d)$.



Choose rational p and q with $c < p < f(x_0) < q < d$.

Let $U = U_q \setminus \overline{U_p}$. Claim: $x_0 \in U$ and $f(U) \subseteq (c,d)$.

Since $f(x_0) < q$ by condition 2 of the above claim $x_0 \in U_q$. Since $f(x) > p$ by condition 1 of the above claim $x \notin \overline{U_p}$. So $x \in U_q \setminus \overline{U_p} = U$.

For the second statement let $x \in U$. Then $x \in U_q$ for some q and thus $f(x) = \inf \mathbb{Q}(x) \leq q$ since $q \in \mathbb{Q}(x)$. On the other hand, since $x \notin \overline{U_p}$ we have $x \notin U_p$ and thus by condition 2, $f(x) \geq p$. So $f(x) \in [p,q] \subseteq (c,d)$. Hence, $f(U) \subseteq (c,d)$, so that f is continuous. ∎

Definition: If A and B are two subsets of a topological space X, and if there is a continuous function $f: X \to [0,1]$ s.t. $f(A) = 0$ and $f(B) = 1$, then we say that A and B are separated by a continuous function.

So the Urysohn Lemma says that if X is a normal space then every of disjoint closed subsets can be separated by a continuous function.

<u>Definition</u> A space X is completely regular if one-point sets are closed in X and for each point x_0 and each closed subset A <u>not</u> containing x_0, there is a continuous function $f: X \to [0,1]$ s.t. $f(x_0)=1$ and $f(A)=0$.

By Urysohn Lemma any normal space is completely regular. Also a completely regular space is regular: $x_0 \in X$, $A \subseteq X$ closed, $x_0 \notin A$, $f: X \to [0,1]$ cont. $f(x_0)=0$, $f(A)=1$. Let $U = f^{-1}([0,1/2))$, $V = f^{-1}((1/2, 1])$, which are both open in X, $x_0 \in U$, $A \subseteq V$ and $U \cap V = \emptyset$.

§34. The Urysohn Metrization Theorem:

<u>Theorem</u>: Every regular space X with a countable basis is metrizable.

<u>Proof</u>: <u>Idea</u>: Embed X into a metrizable space so that X is a subspace of a metric space and hence X is metrizable.

Note that $\mathbb{R}^{\omega} = \mathbb{R}^{\mathbb{N}}$ with the product topology is metrizable.

We'll construct an embedding $F: X \to \mathbb{R}^{\omega}$.

$F(x) = (f_1(x), f_2(x), \ldots, f_n(x), \ldots)$

We'll prove this in two steps.

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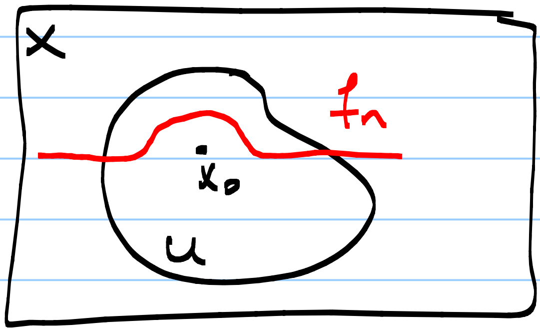
Step 1) Claim: There exists a countable collection of continuous functions $f_n : X \to [0,1]$ having the property that given any $x_0 \in X$ and any neighborhood U of x_0 there exists an index n such that f_n is positive at x_0 and vanishes outside U.



$f_n : X \to [0,1]$
$f_n(x_0) > 0$
$f_n(x) = 0$ if $x \notin U$.

$x_0 \in U$, $x_0 \notin A = X \setminus U$ which is closed. By the Urysohn lemma there is a function $f_{x_0,U} : X \to [0,1]$ (continuous) so that $f_{x_0,U}(x_0) = 1$ and $f_{x_0,U}(A) = 0$.

However, the collection $\{f_{x_0,U} \mid x_0 \in X,\ x_0 \in U \subseteq X\}$ may not be countable.

So we proceed as follows: let $\{B_n\}$ be a countable basis for X. For any pair n, m of indices for which $\overline{B_n} \subseteq B_m$, by the Urysohn lemma (applied to $\overline{B_n}$ and $X \setminus B_m$) there is a continuous function $g_{n,m} : X \to [0,1]$ so that $g_{n,m}(\overline{B_n}) = 1$, $g_{n,m}(X \setminus B_m) = 0$. (Recall that a regular space with countable basis is normal so that the Urysohn Lemma can be applied.)
Clearly, the collection $\{g_{n,m}\}$ is countable.

Subclaim: The collection $\{g_{n,m}\}$ satisfies our requirement.
If $x_0 \in X$ and $x_0 \in U$ open then there is a basis element B_m with $x_0 \in B_m \subseteq U$. Since X is regular there is another open set and thus another basis element B_n so that $x_0 \in B_n \subset \overline{B_n} \subset B_m$.

So for this pair n,m $g_{n,m}: X \to [0,1]$ is defined. Since $x_0 \in B_n \subseteq \overline{B_n}$, $g_{n,m}(x_0)=1$ and $g_{n,m}(X \setminus U) = 0$ since $x_0 \in B_m \subseteq U$ and $g_{n,m}(X \setminus B_m) = 0$.

This finishes the proof of sublemma and thus the proof of the claim.

Step 2: Renaming the countable collection we have a countable set of continuous functions $\{f_n\}$ of Step 1. Consider the function

$$F : X \longrightarrow \mathbb{R}^\omega, \quad x \longmapsto (f_1(x), f_2(x), \ldots, f_n(x), \ldots)$$

must show F is an embedding.

1) F is continuous because each f_i is continuous and $\mathbb{R}^\omega$ has the product topology.

2) F is injective: Let $x,y \in X$ with $x \neq y$. Then there is an open subset U so that $x \in U$ and $y \notin U$. So by the construction of f_n's of Step 1 there is some f_n so that $f_n(x_0) > 0$ and $f_n(X \setminus U) = 0$, so that $f_n(y) = 0$ since $y \in X \setminus U$. Hence, $F(x) \neq F(y)$, so that F is injective.

3) F is a homeomorphism onto its image $Z = F(X)$.
Note that $F : X \to Z = F(X)$ is a continuous bijection. Let $G = F^{-1} : Z \to X$.
must show: G is continuous.
Let $U \subseteq X$ be any open set. We need to show that the inverse image $G^{-1}(U) = F(U)$ is open in Z. Let $z_0 \in F(U)$. We'll find an open subset W of Z so that $z_0 \in W \subseteq F(U)$.
Choose an index N so that $f_N(x_0) > 0$

and $f_N(X \setminus U) = 0$, where $x_0 \in U$ with $F(x_0) = z_0$.

Consider the projection $\pi_N : \mathbb{R}^\omega \to \mathbb{R}$ onto the N^{th} component ($\pi_N(x_1, \dots, x_n, \dots) = x_N$).

Let $V = \pi_N^{-1}((0,\infty)) = \mathbb{R} \times \mathbb{R} \times \dots \times \underset{N^{th}\text{ plane}}{(0,\infty)} \times \mathbb{R} \times \cdots$ which is open in $\mathbb{R}^\omega$.

Set $W = V \cap Z$, which is open in the subspace Z.

<u>Claim:</u> $z_0 \in W \subseteq F(U)$.

<u>Proof:</u> $\pi_N(z_0) = \pi_N(F(x_0)) = f_N(x_0) > 0$ and

thus $z_0 \in \pi_N^{-1}((0,\infty)) \cap Z = V \cap Z = W$.

So $z_0 \in W$.

Let $z \in W$, then $z = F(x)$ for some $x \in X$, and $\pi_N(z) \in (0,\infty)$.

Note that $\pi_N(z) = \pi_N(F(x)) = f_N(x)$, and f_N vanishes outside U and thus $x \in U$.

Thus, $z = F(x) \in F(U)$.

This finishes the proof. ∎

<u>Corollary</u> Any normal space with countable basis is metrizable.

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CHAPTER 5: The Tychonoff Theorem:

Lemma: Let X be a set; let $\mathcal{A}$ be a collection of subsets of X having the finite intersection property. Then there is collection $\mathcal{D}$ of subsets of X such that $\mathcal{D}$ contains $\mathcal{A}$, and $\mathcal{D}$ has the finite intersection property, and no collection of subsets of X that properly contains $\mathcal{D}$ has this property.

Proof: Proof uses so called Zorn's lemma: Given a set A that is strictly partially ordered, in which every simply ordered subset has an upper bound, A itself has a maximal element.

Some notation:

c is an element of X
C is a subset of X
$\mathcal{C}$ is a collection of subsets of X
$\mathbb{C}$ is a superset whose elements are collections of subsets of X.

We know that $\mathcal{A}$ is a collection of subsets of X having the finite intersection property. Let $\mathbb{A}$ denote the superset consisting of all collections $\mathcal{B}$ of subsets of X such that $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{B}$ has the finite intersection property.

Aim: $\mathbb{A}$ has a maximal element $\mathcal{D}$.

Let $\mathbb{B}$ is a subsuperset of $\mathbb{A}$ that is simply ordered by proper inclusion.

must show: $\mathbb{B}$ has an upper bound in $\mathbb{A}$.

Indeed, we'll show that the collection

$\mathcal{C} = \bigcup_{B \in \mathcal{B}} B$, is an element of $\mathbb{A}$; then it is the required upper bound on $\mathcal{B}$.
So we must show two things:
1) $\mathcal{A} \subset \mathcal{C}$
2) $\mathcal{C}$ has the finite intersection property.

1) Since each element of $\mathcal{B}$ contains $\mathcal{A}$, $\mathcal{C}$ contains $\mathcal{A}$.

2) Let $C_1, .., C_n$ be elements of $\mathcal{C}$. Since $\mathcal{C}$ is the union of the elements of $\mathcal{B}$, there is, for each i, an element $\mathcal{B}_i$ of $\mathcal{B}$ such that $C_i \in \mathcal{B}_i$. The superset $\{\mathcal{B}_1, \mathcal{B}_2, .., \mathcal{B}_n\}$ is contained in $\mathcal{B}$, so it is simply ordered by the relation of proper inclusion, say $\mathcal{B}_k$. Hence, $\mathcal{B}_i \subseteq \mathcal{B}_k$ for all $i = 1, 2, .., n$. The $C_1, .., C_n$ are all in the collection $\mathcal{B}_k$. Since $\mathcal{B}_k$ has the finite intersection property the intersection of the sets $C_1, .., C_n$ is nonempty as desired. ∎

Lemma: Let X be a set; let $\mathcal{D}$ be a collection of subsets of X that is maximal with respect to the finite intersection property. Then
a) Any finite intersection of elements of $\mathcal{D}$ is an element of $\mathcal{D}$.
b) If A is a subset of X that intersects every element of $\mathcal{D}$, then A is an element of $\mathcal{D}$.

Proof a) Let $B = D_1 \cap .. \cap D_m$ for some elements $D_1, .., D_m$ of $\mathcal{D}$.
Define $\mathcal{E} = \mathcal{D} \cup \{B\}$.
We'll show that $\mathcal{E}$ has the finite intersection

property, so that by the maximality of $\mathcal{D}$ we'll have $\mathcal{E}=\mathcal{D}$ so that $B\in\mathcal{D}$.

Take finitely many elements from $\mathcal{E}$, say $E_1, \dots, E_k$. If $E_i \neq B$ for all $i=1,\dots,k$, then $E_i \in \mathcal{D}$, for all $i=1,\dots,k$, and thus $E_1\cap\dots\cap E_k=\emptyset$ because $\mathcal{D}$ has the finite intersection property. If $B=E_i$ for some $i=1,\dots,k$, then

$$E_1\cap\dots\cap E_k = D_1'\cap\dots\cap D_{k-1}'\cap B \quad \text{for some}$$

$D_1', \dots, D_{k-1}'$ in $\mathcal{D}$.

So, $E_1\cap\dots\cap E_k = D_1'\cap\dots\cap D_{k-1}'\cap D_1\cap\dots\cap D_m$, which is again a finite intersection of elements of $\mathcal{D}$. Since $\mathcal{D}$ has the finite intersection property $E_1\cap\dots\cap E_k\neq\emptyset$.

b) Given A, define $\mathcal{E}=\mathcal{D}\cup\{A\}$. Clearly, $\mathcal{E}$ contains $\mathcal{D}$. We'll show that $\mathcal{E}$ has the finite intersection property and then by maximality of $\mathcal{D}$, we'll obtain $\mathcal{E}\subseteq\mathcal{D}$ so that $A\in\mathcal{D}$.

Take finitely many elements, say, $E_1,\dots,E_m$ from $\mathcal{E}$. If $E_i\neq A$ for all $i=1,\dots,m$, then $E_i\in\mathcal{D}$ for all $i=1,\dots,m$, and thus $E_1\cap\dots\cap E_m=\emptyset$ since $\mathcal{D}$ has the finite intersection property. Otherwise, without loss of generality assume that $E_1=A$. Then

$$E_1\cap\dots\cap E_m = A\cap D_1\cap\dots\cap D_k,$$ where $D_i\in\mathcal{D}$ and $D_i=E_j$ for some $j=2,\dots,m$.

By part (a), $D_1\cap\dots\cap D_k\in\mathcal{D}$. Now by the assumption, $A\cap(D_1\cap\dots\cap D_k)\neq\emptyset$, which finishes the proof.

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Theorem (Tychonoff Theorem)

An arbitrary product of compact spaces is compact in the product topology.

Proof: Let $X = \prod_{\alpha \in J} X_\alpha$, where each X_α is a compact space.

Let $\mathcal{A}$ be a collection of subsets of X having the finite intersection property. We prove that the intersection

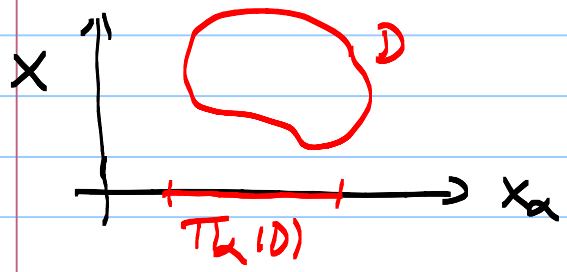
$$\bigcap_{A \in \mathcal{A}} \overline{A} \neq \emptyset,$$ which will imply that X is a compact space.

By the first lemma there is a maximal collection $\mathcal{D}$ containing $\mathcal{A}$ having the finite intersection property. Since, $\mathcal{A} \subseteq \mathcal{D}$ it is enough to prove that

$$\bigcap_{D \in \mathcal{D}} \overline{D} \neq \emptyset \text{ because } \bigcap_{D \in \mathcal{D}} \overline{D} \subseteq \bigcap_{A \in \mathcal{A}} \overline{A}.$$

Let $\alpha \in J$ be any index. Then the collection $\{\pi_\alpha(D) \mid D \in \mathcal{D}\}$, where $\pi_\alpha : \prod_{\gamma \in J} X_\gamma \to X_\alpha$ is the projection onto X_α, has the finite intersection property because

$$\emptyset \neq \pi_\alpha(D_1 \cap \dots \cap D_n) \subseteq \pi_\alpha(D_1) \cap \dots \cap \pi_\alpha(D_n).$$



Since X_α is compact the intersection

$$\bigcap_{D \in \mathcal{D}} \overline{\pi_\alpha(D)} \neq \emptyset$$ and

thus we may choose some $x_\alpha \in \bigcap_{D \in \mathcal{D}} \overline{\pi_\alpha(D)}$.

Then $x = (x_\alpha)_{\alpha \in J} \in X$.

Claim: $x \in \overline{D}$ for all $D \in \mathcal{D}$ so that $\bigcap_{D \in \mathcal{D}} \overline{D} \neq \emptyset$.

Proof of the claim: Consider any subbasis element of the form $\pi_\beta^{-1}(U_\beta)$ (for the product topology) containing x.
Since $x \in \pi_\beta^{-1}(U_\beta) \Rightarrow x_\beta = \pi_\beta(x) \in U_\beta$ so that U_β is a neighborhood of x_β in X_β.

By the definition of x_β, $x_\beta \in \overline{\pi_\beta(D)}$ (for any element D of $\mathcal{D}$) and thus $U_\beta \cap \pi_\beta(D) \neq \emptyset$.
Choose some $y_\beta \in U_\beta \cap \pi_\beta(D)$, for each $\beta \in J$.
Let $y = (y_\beta) \in X$.
Then $y \in \pi_\beta^{-1}(U_\beta) \cap D$.

So we've seen that for any $D \in \mathcal{D}$ and any subbasis element $\pi_\beta^{-1}(U_\beta)$ the intersection $\pi_\beta^{-1}(U_\beta) \cap D \neq \emptyset$. Now by part (b) of the second lemma $\pi_\beta^{-1}(U_\beta) \in \mathcal{D}$, for any $\beta \in J$.

Note that any basis element containing $x \in X$ is a finite intersection of subbasis elements of the form $\pi_\beta^{-1}(U_\beta)$, each containing x. Hence, by part (a) of the second lemma any basis element of $X = \prod_\alpha X_\alpha$ containing x belongs to the collection $\mathcal{D}$.
In particular, for any $D \in \mathcal{D}$ and any basis element U containing x, we have
$D \cap U \neq \emptyset$, since $D, U \in \mathcal{D}$ and $\mathcal{D}$ has the finite intersection property. So, $x \in \overline{D}$, for any $D \in \mathcal{D}$. This finishes the proof of the claim and the proof of the Tychonoff Theorem.