METU Mathematics Department - Fall 2020
MATH 349 - Introduction to Mathematical Analysis
Exercise Problems - List A

(1) For any real number $\lambda \in \mathbb{R}$ choose a positive real number $a_\lambda > 0$. Show that the subset A below is unbounded:

$$A = \{a_{\lambda_1} + \cdots + a_{\lambda_k} \mid k \in \mathbb{N},\ \lambda_1, \cdots, \lambda_k \in \mathbb{R} \text{ are all distinct.}\}\ .$$

What if the indices λ were from integers but not from reals?

(2) Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing function. Show that the set of points, at which f is discontinuous, is at most countable. (Hint: Use the above problem!)

(3) Note that the interval $[0, 1]$ with addition modulo one is an abelian group.
a. Let H be a proper subgroup of $[0, 1]$, which is closed as a subset of $[0, 1]$. Show that H is finite.
b. Show that the subgroup generated by $\sqrt{2}$ in $[0, 1]$ is dense. Conclude that $\{m + n\sqrt{2} \mid m, n \in \mathbb{Z}\}$ is dense in $\mathbb{R}$.

(4) (W. Rudin, Principles of Math. Analy. page 22, Problem 6) Fix any real number $b > 1$. For any rational number $r = p/q$, $p, q \in \mathbb{Z}$, define b^r as $b^r \doteq (b^p)^{1/q}$, the qth positive root of b^p.
a. Prove that for rational numbers $r_1,\ r_2,\ b^{r_1+r_2} = b^{r_1}\ b^{r_2}$.

Now for any positive real number x let $B(x) = \{b^r \mid r \in \mathbb{Q},\ x \geq r\}$. Finally define b^x as $\sup B(x)$.

b. Show that for any rational number $r = p/q$ the two definitions of b^r agrees. In other words, show $\sup B(r) = (b^p)^{1/q}$.
c. Prove that for real numbers $x_1,\ x_2,\ b^{x_1+x_2} = b^{x_1}\ b^{x_2}$.

(5) Consider the sequence $f_n : [0, 1] \to \mathbb{R},\ f_n(x) = x^n$ in the complete metric space of bounded functions equipped with the supremum metric, $B([0, 1], \|\cdot\|_{\sup})$. Is it a Cauchy sequence? Show that the same sequence is convergent in $B([0, 1/2], \|\cdot\|_{\sup})$

(6) Let $(f_n) \in C(X) \cap B(X)$ be a convergent sequence, where $C(X) \cap B(X)$ is the metric space of real valued bounded and continuous functions on metric space (X, d). Let (x_n) be

convergent sequence in (X, d). Show that $f_n(x_n)$ is convergent in $(\mathbb{R}, \ |\cdot|)$.

(7) Show that a metric space (X, d) is compact if and only if any continuous real valued function on X has a maximum.

(8) A continuous map $f : (X, d_1) \to (Y, d_2)$ is called proper if the inverse image of any compact set under f is also compact. Show that a proper map is a closed map; i.e., the image of any closed set in X is closed in Y.

(9) Show that any nonconstant polynomial map on $\mathbb{R}$ or $\mathbb{C}$ is proper.

(10) Let $f, \ g \in \mathbb{C}[z]$ be two non constant monic polynomials. Consider the homotopy function

$$F : [0, 1] \times \mathbb{C} \to \mathbb{C} \ , \ (t, z) \mapsto (1 - t) \ f(z) + t \ g(z) \ .$$

Show that F is proper if and only if $\deg(f) = \deg(g)$.

In case of real polynomials the following holds: For monic real polynomials $f, \ g \in \mathbb{R}[x]$ the homotopy function

$$F : [0, 1] \times \mathbb{R} \to \mathbb{R} \ , \ (t, x) \mapsto (1 - t) \ f(x) + t \ g(x)$$

is proper if and only if $\deg(f) = \deg(g) \pmod 2$.

(11) Show that any bijection $f : [0, 1) \to (0, 1)$ has infinitely many discontinuity. Find such a bijection!

(12) Let (f_n) be a sequence of Riemann integrable functions in $B([a, b])$ converging to some $f \in B([a, b])$. Show that f is also Riemann integrable and

$$\int_a^b f(x) \ dx = \lim_{n\to\infty} \int_a^b f_n(x) \ dx.$$

(13) Let $f_n : [0, 1] \to \mathbb{R}$ be a sequence of function defined by

$$f_n(x) = \begin{cases} n(1 - nx) \ , & \text{if} \quad 0 < x < \dfrac{1}{n}, \\ 0 \ , & \text{if} \ \ x = 0 \ \text{ or } \ \dfrac{1}{n} \le x \le 1. \end{cases}$$

a. Determine the function $f : [0,1] \to \mathbb{R}$, to which (f_n) converge pointwisely.

b. Compute the integral $\displaystyle\int_0^1 f_n(x)\ dx$.

c. Does (f_n) converge uniformly to f on $[0,1]$? Explain your answer.

(14) Show that the subset $A = \{(\cos n, \sin n) \mid n \in \mathbb{Z}\}$ is dense in the unit circle $S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. (Hint: Regarding the points of the plane as complex number note that the map

$$f(n) \doteq e^{in} = \cos n \ + \ i \sin n$$

is a group homomorphism from $(\mathbb{Z}, +)$ to $(\mathbb{C}^*, \cdot)$).

(15) Prove that there is a unique bounded continuous real valued function $f : [0,\infty) \to \mathbb{R}$ such that

$$f(s) = 1 + \int_0^s e^{-t^2}\ f(st)\ dt$$

for all $s \in [0, \infty)$.

(16) Let (f_n) be a bounded sequence in $C[0,1]$ and

$$g_n(x) = \int_0^x f_n(t)\ dt$$

prove that there is a subsequence of (g_n) which converges to a continuous function uniformly.

(17) Let X be a complete metric space. If

$$X = \cup_{n=1}^{\infty} X_n$$

prove that for some m, the closure $\overline{X}_m$ has non empty interior.

(18) If X is a complete metric space which has no isolated point prove that X is uncountable.

(19) Show that the set of irrational numbers is not a union of countably many closed subsets of $\mathbb{R}$.

(20) Show that any connected metric space containing at least two points is uncountable.

(21) Show that any open subset of reals is a countable disjoint union open intervals.

(22) For any nonempty subsets A and B of real numbers define

$$A + B = \{a + b \mid a \in A,\ b \in B\}.$$

Show that $A+B$ is compact if both A and B are both compact. Also show that $A+B$ is closed if A is compact and B is closed.

(23) If $X \setminus \partial A$ is connected, prove that either $IntA$ or $ExtA$ is empty. Use this to show that $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \neq 1\}$ is not connected.

(24) Construct a sequence of rational numbers (a_n) so that for any real number x there is a subsequence (a_{k_n}) with $\lim a_{k_n} = x$.

(25) Prove that a precompact metric space is separable, i.e., has a countable dense subset. Are all separable metric spaces precompact?

List B

(1) **a)** Show that any continuous function $f : \mathbb{R} \to \mathbb{Z}$ is constant.

b) Show that the function $f : \mathbb{Q}^* \to \mathbb{Z}$, $f(x) = [x/\sqrt{2}]$ (greatest integer part) is continuous and onto.

c) Show that any uniformly continuous function $f : \mathbb{Q} \to \mathbb{Z}$ is constant.

(2) **a**) Show that the function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$d(x, y) = |\sigma(x) - \sigma(y)|,$$

where $\sigma(x) = x$ if $x \leq 0$ and $\sigma(x) = \dfrac{x}{1+x}$ if $x > 0$, defines a metric.

b) Show that the subset $[a, \infty)$ is bounded for any a, while as $(-\infty, \infty)$ is not bounded.

c) Determine the balls $B(-3, 1)$, $B(1, 1/2)$, $B(1, 2)$ and $B(2, 2)$.

(3) Prove that for any two subsets of real numbers A and B, which are bounded from above the subset $A \cup B$ is bounded from above and $\sup(A \cup B) = \max\{\sup(A), \sup(B)\}$.

(4) Show that for any two subsets A and B of real numbers $(A \cup B)' = A' \cup B'$ and $(A \cap B)' \subseteq A' \cap B'$. Find A and B so that $(A \cap B)' \neq A' \cap B'$. (Here, A' denotes the set of accumulation points of A, the derived set of A.)

(5) **a.** If $a_n \leq b_n \leq c_n$ be sequences so that $\lim a_n = r = \lim c_n$, for some $r \in \mathbb{R}$. Show that $\lim b_n = r$.

b. If $\lim a_n = r$, $a_n \geq 0$, for all n, then show that $r \geq 0$ and $\lim \sqrt{a_n} = \sqrt{r}$.

c. If $\lim a_n = r$, then show that $\lim b_n = r$, where $b_n = \dfrac{a_1 + \cdots + a_n}{n}$.

(6) Show that any function $f : \mathbb{Z} \to \mathbb{R}$ is continuous and every continuous functions $f : \mathbb{R} \to \mathbb{Z}$ is constant, where both spaces have the absolute value metric.

(7) Either prove or give counter examples for the following statements: Let (X, d) be a metric space. Then, for any subsets A, B of X we have,

a. $Int(A) \cup Int(B) \subseteq Int(A \cup B)$;
b. $Int(A) \cap Int(B) = Int(A \cap B)$;
c. $\overline{A} \cup \overline{B} = \overline{A \cup B}$;
d. $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$;

e. $\overline{Int(A)} = \overline{A}$;

f. $Int(\overline{A}) = Int(A)$.

(8) Prove that for any two subsets of real numbers A and B, which are bounded from below the subset

$$A + B = \{a + b \mid a \in A,\ b \in B\}$$

is bounded from below and $\inf(A + B) = \inf(A) + \inf(B)$.

(9) Let $f : X \to Y$ be a function from the set X to the set Y. For any subset $B \subseteq Y$ define the inverse image of B under f as $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$. Prove the following statements:

For any family of substes $\{A_\alpha\}_{\alpha \in \Lambda}$ in X,

a) $f(\cup_{\alpha \in \Lambda} A_\alpha) = \cup_{\alpha \in \Lambda} f(A_\alpha)$;

b) $f(\cap_{\alpha \in \Lambda} A_\alpha) \subseteq \cap_{\alpha \in \Lambda} f(A_\alpha)$. Also find a counter example for

$$f(\cap_{\alpha \in \Lambda} A_\alpha) = \cap_{\alpha \in \Lambda} f(A_\alpha)\ .$$

On the other hand, for any family of subsets $\{B_\alpha\}_{\alpha \in \Lambda}$ of Y,

c) $f^{-1}(\cup_{\alpha \in \Lambda} B_\alpha) = \cup_{\alpha \in \Lambda} f^{-1}(B_\alpha)$;

d) $f^{-1}(\cap_{\alpha \in \Lambda} B_\alpha) = \cap_{\alpha \in \Lambda} f^{-1}(B_\alpha)$.

(10) Prove that a function $f : X \to Y$ is continuous if and only if $f(\overline{A}) \subseteq \overline{f(A)}$.
Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 0$, for all $x < 0$ and $f(x) = 1$ for all $x \geq 0$. Find a subset A so that $f(\overline{A}) \not\subseteq \overline{f(A)}$.

(11) Prove that for any two nonempty subsets of positive real numbers A and B with $\inf(A) \neq 0$, $\inf(B) \neq 0$ prove that $\inf(A \cdot B) = \inf(A) \cdot \inf(B)$, where

$$A \cdot B = \{ab \mid a \in A,\ b \in B\}.$$

(12) Let P be the set of irrational numbers in the interval $[0,1]$. Compute P', the set of accumulation points of P.

(13) Let S be a subset of real numbers. An element $x \in S$ is called an *isolated point* of S if there is a positive real number $\epsilon > 0$ so that $(x-\epsilon, x+\epsilon) \cap S$ is finite. Prove that if x is an isolated point of S then x is an accumulation point of $\mathbb{R} \setminus S$.

(14) Let $f : [0,1] \to [0,1]$ be a function so that $f(A)$ is closed, whenever $A \subseteq [0,1]$ is closed. Set $A_1 = [0,1]$ and $A_{n+1} = f(A_n)$, for $n \geq 1$. Show that $\bigcap_{n=1}^{\infty} A_n$ is a nonempty closed set.

(15) Let (a_n) and (b_n) be bounded sequences of real numbers. Show that $\liminf(a_n + b_n) \geq \liminf(a_n) + \liminf(b_n)$.

(16) Let (a_n) be a sequence of real numbers so that $\lim na_n = L$. Show that $\lim a_n = 0$.

(17) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 0$, for all $x < 0$ and $f(x) = 1$ for all $x \geq 0$. Clearly, $f \in B(\mathbb{R})$. Show that the ball $B(f, 1/3)$ contains no continuous function. Find a continuous function in the ball $B(f, 2/3)$.

(18) Consider the sequence defined by $a_1 = 1$ and $a_{n+1} = \sqrt{6 + a_n}$, for all $n \geq 1$. Show that $\lim a_n = 3$.

(19) Show that any continuous bijection $f : \mathbb{R} \to \mathbb{R}$ is a homeomorphism.

(20) Determine the interior, the exterior, the boundary and the closure of the following subsets:

a. $\mathbb{Z} \subset \mathbb{R}$,
b. $\mathbb{Z} \subset \mathbb{Q}$,
c. $\mathbb{Q} \subset \mathbb{R}$,
d. $\mathbb{Z} \times \mathbb{Q} \subset \mathbb{R} \times \mathbb{R}$,

e. $\mathbb{Z} \times \mathbb{Q} \subset \mathbb{R} \times \mathbb{Q}$,
f. $\{(x, y) \mid x, y \in \mathbb{R},\ x < y^2\}$ in $\mathbb{R}^2$,
g. $Y \subseteq X$, where X is any set equipped with the discrete metric and Y is any subset.
h. $\{f \in C(\mathbb{R}) \cap B(\mathbb{R}) \mid f(0) = 1\} \subset B(\mathbb{R})$, where $B(\mathbb{R})$ and $C(\mathbb{R})$ denotes the set of bounded and continuous functions on the real line, respectively and $B(\mathbb{R})$ is endowed with the supremum metric.

(21) Let $f : (\mathbb{R}, |.|) \to (\mathbb{R}, |.|)$ be defined by $f(x) = 5x$. Show that f is a homeomorphism but not an isometry.

(22) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x - 1$. Show that f is uniformly continuous on $\mathbb{R}$.

(23) Show that $\mathbb{Z}$ is not compact.

(24) Determine whether the following sets of $\mathbb{R}^2$ are complete and compact. Explain.
(i) $\{(x, y) : 1 \leq x^2 + y^2 \leq 4\}$,
(ii) $\{(x, y) : 1 < y < 3\}$,
(iii) $\{(x, y) : 2x + y = 1\}$,
(iv) $\{(x, y) : 0 < x < 1,\ y = 0\}$,
(v) $\{(x, y) : 0 \leq x \leq 1,\ y = e^x\}$.

(25) Let (X, d) be a compact connected metric space and $f : X \to \mathbb{R}$ be a continuous function. Prove that $f(X)$ is a finite closed interval.

(26) Let (X, d) be a metric space and $t \in X$. Let $f : X \to \mathbb{R}$ be defined by $f(x) = d(x, t)$. Show that f is uniformly continuous on X.

(27) Let (X, d) be a metric space. Show that if (x_n), (y_n) are Cauchy sequence of X, then $(d(x_n, y_n))$ converges in $\mathbb{R}$.

(28) A metric space which has a countable dense subset is called separable. Prove that the cartesian product of two separable metric spaces is also separable.

(29) Let (X, d) be a bounded metric space and d' be the metric on X defined by $d'(x, y) = \frac{d(x,y)}{1+d(x,y)}$, for all $x, y \in X$. Show that these two metrics are equivalent.

(30) **a)** Prove that the complement of $IntE$ is the closure of the complement of E.

b) Do E and $\bar{E}$ always have the same interiors?

c) Do E and $IntE$ always have the same closures?

(31) In each case below give an example of a bounded set A in $\mathbb{R}$.
(i) $\sup(A) \in A$,
(ii) $\sup(A) \notin A$,
(iii) $\sup(A)$ is an accumulation point of A,
(iv) $\sup(A)$ is not an accumulation point A.

(32) Let (a_n) be a bounded sequence. Show that $\limsup(-a_n) = -\liminf(a_n)$.

(33) Evaluate, wherever possible, the limit of the following sequence:
(i) $(\frac{e^n}{\pi^n})$,
(ii) $(\sin(\pi/n))$,
(iii) $(\frac{n!}{n^n})$,
(iv) $\frac{(-1)^n}{n}$.

(34) Let A be a precompact subset of a complete metric space X and $f : X \to Y$ a continuous map. Show that its restriction to A, $f : A \to Y$, is uniformly continuous.

(35) Let E_n be sequence of nonempty closed subsets of a compact metric space so that $E_{n+1} \subseteq E_n$, for each n. Prove that

$$\cap_{n=1}^{\infty} E_n \neq \emptyset.$$

This list contains several questions from the course textbook entitled *An Introduction to Real Analysis* by Tosun Terzioğlu.