Differential Geometry

Note Title 1.02.2020

"Elementary Differential Geometry"
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§1.1. Euclidean Space:

Definition: Euclidean 3-space $\mathbb{R}^3$ is the set of all ordered triples of real numbers. Such a triple $p=(p_1,p_2,p_3)$ is called a point of $\mathbb{R}^3$.

Summation and scalar multiplication are defined as follows:

$$p+q=(p_1,p_2,p_3)+(q_1,q_2,q_3)$$
$$=(p_1+q_1,p_2+q_2,p_3+q_3)$$ and

$$ap=a(p_1,p_2,p_3)=(ap_1,ap_2,ap_3).$$

Definition: The natural coordinate functions x,y,z on $\mathbb{R}^3$ are defined as follows:

$x:\mathbb{R}^3\to\mathbb{R}$, $x(p_1,p_2,p_3)=p_1$,

$y: \mathbb{R}^3 \to \mathbb{R}$, $y(p_1, p_2, p_3) = p_2$ and

$z: \mathbb{R}^3 \to \mathbb{R}$, $z(p_1, p_2, p_3) = p_3$

Recall that partial derivatives of a function $f: \mathbb{R}^3 \to \mathbb{R}$ are defined as follows:

$$\frac{\partial f}{\partial x}(p) = \lim_{h \to 0} \frac{f(p_1+h, p_2, p_3) - f(p_1, p_2, p_3)}{h}$$

$$\frac{\partial f}{\partial y}(p) = \lim_{h \to 0} \frac{f(p_1, p_2+h, p_3) - f(p_1, p_2, p_3)}{h}$$

$$\frac{\partial f}{\partial z}(p) = \lim_{h \to 0} \frac{f(p_1, p_2, p_3+h) - f(p_1, p_2, p_3)}{h},$$

provided that these limits exist. Note that in case they exist partial derivatives are also functions on $\mathbb{R}^3$. For example,

$$\frac{\partial f}{\partial x}: \mathbb{R}^3 \to \mathbb{R}.$$

Similarly, we may define higher partial derivatives, if they exists.

For example,

$$\frac{\partial^2 f}{\partial x \partial y} \doteq \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \quad \text{and}$$

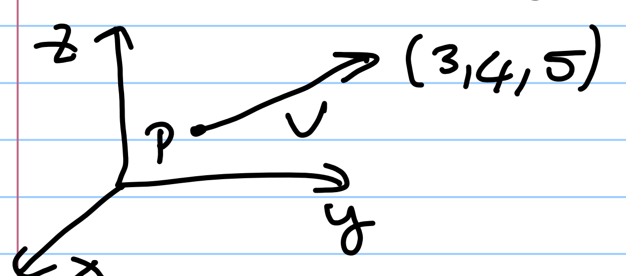
$$\frac{\partial^3 f}{\partial x^2 \partial z} \doteq \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right)\right).$$

Definition: A function $f: \mathbb{R}^3 \to \mathbb{R}$ will be called differentiable if all partial derivatives of all orders exist.

§1.2. Tangent Vectors:

Definition: A tangent vector v_p of $\mathbb{R}^3$ consists of two points of $\mathbb{R}^3$, its vector part v and its point of application p.

Ex $v_p = (2,3,2)_{(1,1,3)}$, $v = (2,3,2)$
$p = (1,1,3)$



v_p is called a tangent vector

to $\mathbb{R}^3$ at the point p. The set of all tangent vectors to $\mathbb{R}^3$ at a given point p is denoted as $T_p\mathbb{R}^3$.

$T_p\mathbb{R}^3$ is a vector space:

$V_p + U_p \doteq (V+U)_p, \quad aV_p \doteq (aV)_p.$

<u>Remark:</u> We cannot add V_p and U_q if $p \neq q$!

<u>Definition:</u> A vector field V on $\mathbb{R}^3$ is a function that assigns to each point p of $\mathbb{R}^3$ a tangent vector $V(p)$ to $\mathbb{R}^3$ at p.

<u>Example:</u> Let U_1, U_2 and U_3 be the vector fields defined as

$U_1(p) = (1,0,0)$, $U_2(p) = (0,1,0)$ and $U_3(p) = (0,0,1)$, for all $p \in \mathbb{R}^3$. The collection U_1, U_2, U_3 is called the natural frame field on $\mathbb{R}^3$.

Remark: If $V(p)=(v_1(p),v_2(p),v_3(p))$ is a vector field on $\mathbb{R}^3$ then we can write $V(p)$ as

$$V(p)=v_1(p)(1,0,0)+v_2(p)(0,1,0)+v_3(p)(0,0,1)$$
$$=v_1(p)U_1(p)+v_2(p)U_2(p)+v_3U_3(p).$$

§1.3. Directional Derivatives:

Definition: Let f be a real valued differentiable function on $\mathbb{R}^3$, and let v_p be a tangent vector to $\mathbb{R}^3$. Then the number

$$v_p[f]=\frac{d}{dt}(f(p+tv))_{t=0}$$

is called the derivative of f with respect to v_p.

Lemma: If f and v_p are as follows, then $v_p[f]=\sum_{i=1}^{3} v_i \frac{\partial f}{\partial x_i}(p)$.

Proof $v_p[f]=\frac{d}{dt}(f(p+tv))_{t=0}$

$$\Rightarrow v_p[f] = \frac{\partial f}{\partial x_1}(p)\,\frac{\partial(p_1+tv_1)}{\partial t}\Big|_{t=0}$$

$$+\frac{\partial f}{\partial x_2}(p)\,\frac{\partial(p_2+tv_2)}{\partial t}\Big|_{t=0}$$

$$+\frac{\partial f}{\partial x_3}(p)\,\frac{\partial(p_3+tv_3)}{\partial t}\Big|_{t=0}$$

$$=\frac{\partial f}{\partial x_1}(p)\,v_1+\frac{\partial f}{\partial x_2}(p)\,v_2+\frac{\partial f}{\partial x_3}(p)\,v_3.$$

<u>Theorem</u> Let f and g be differentiable functions on $\mathbb{R}^3$, v_p and w_p be tangent vectors and a, b be real numbers. Then

1) $(av_p+bw_p)[f] = a\,v_p[f]+b\,w_p[f]$
2) $v_p[af+bg] = a\,v_p[f]+b\,v_p[g]$
3) $v_p[fg] = v_p[f]\,g(p)+f(p)\,v_p[g]$.

<u>Corollary</u> If V and W are vector fields and f, g, a, b are as above then,

1) $(fV+gW)[h] = fV[h]+gW[h]$
2) $V[af+bg] = a\,V[f]+b\,V[g]$

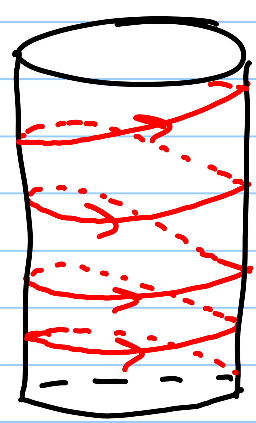
3) $V[fg] = V[f]\,g + f\,V[g]$,
for any differentiable function h on $\mathbb{R}^3$.

§1.4. Curves in $\mathbb{R}^3$:

Definition: A curve in $\mathbb{R}^3$ is a differentiable function $\alpha : I \to \mathbb{R}^3$ from an open interval I into $\mathbb{R}^3$.

Example 1) Straight line.
$\alpha(t) = p + tq$, $t \in \mathbb{R}$, is the line through a point $p \in \mathbb{R}^3$ in the direction of $q \in \mathbb{R}^3$.

2) Helix. $\alpha(t) = (a\cos t, a\sin t, bt)$ where $a > 0$ and $b \neq 0$.



3) $\alpha: \mathbb{R} \to \mathbb{R}^3$, $\alpha(t) = (e^t, e^{-t}, \sqrt{2}t)$

4) $\alpha: \mathbb{R} \to \mathbb{R}^3$, $\alpha(t) = (3t - t^3, 3t^2, 3t + t^3)$.

<u>Definition</u>: Let $\alpha: I \to \mathbb{R}^3$ be a curve in $\mathbb{R}^3$ with $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. For each number t in I, the velocity vector of α at t is the tangent vector

$$\alpha'(t) = \left(\frac{d\alpha_1}{dt}(t), \frac{d\alpha_2}{dt}(t), \frac{d\alpha_3}{dt}(t)\right)_{\alpha(t)}$$

at the point $\alpha(t)$.

<u>Example</u>: $\alpha(t) = (a\cos t, a\sin t, bt)$. Then $\alpha'(t) = (-a\sin t, a\cos t, b)$.

<u>Definition</u>: Let $\alpha: I \to \mathbb{R}^3$ be a curve. If $h: J \to I$ is a differentiable function on an open interval J, then the composite function $\beta = \alpha(h): J \to \mathbb{R}^3$ is a curve called a reparametrization of α by h.

Lemma: If $\beta = \alpha(h)$ is a reparametrization of α, where $h: J \to I$, $s \mapsto h(s)$, $s \in J$, then

$$\beta'(s) = \left(\frac{dh}{ds}(s)\right) \alpha'(h(s)).$$

Lemma: Let α be a curve in $\mathbb{R}^3$ and f a differentiable function on $\mathbb{R}^3$. Then

$$\alpha'(t)[f] = \frac{d(f(\alpha))}{dt}(t).$$

§1.5. 1-Forms:

Definition: A 1-form ϕ on $\mathbb{R}^3$ is a real valued function on the set of all tangent vectors to $\mathbb{R}^3$ such that ϕ is linear at each point, that is $\phi(av + bw) = a\phi(v) + b\phi(w)$, for all $a, b \in \mathbb{R}$ and tangent vectors v, w

at the same point of $\mathbb{R}^3$.
Note that for any point $p \in \mathbb{R}^3$ the 1-form defines a linear function $\phi_p : T_p\mathbb{R}^3 \to \mathbb{R}$, $v \mapsto \phi_p(v)$ so that ϕ_p is in the dual space of the vector space $T_p\mathbb{R}^3$.

Moreover, if V is a vector field on $\mathbb{R}^3$ then $\phi(V) : \mathbb{R}^3 \to \mathbb{R}$, $p \mapsto \phi_p(V_p)$, is a function.

Note that if ϕ and ψ are two 1-forms then $a\phi + b\psi$ is also a 1-form:
$$(a\phi + b\psi)(v_p) = a\,\phi(v_p) + b\,\psi(v_p).$$

__Definition:__ If f is a differentiable function then the differential
$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \text{ of } f$$
is a 1-form on $\mathbb{R}^3$ such that

$df(v_p) = v_p[f]$ for all tangent vectors v_p.

<u>Example</u> $x_i : \mathbb{R}^3 \to \mathbb{R}$, $i = 1, 2, 3$, are functions and thus dx_i, $i = 1, 2, 3$, are 1-form such that

$$dx_i(v_p) = v_p[x_i] = \sum_{j=1}^{3} v_j \frac{\partial x_i}{\partial x_j}(p) = v_i.$$

<u>Lemma:</u> Any 1-form on $\mathbb{R}^3$ has the form $\phi = \sum_{i=1}^{3} f_i \, dx_i$, where

$f_i = \phi(U_i)$. f_i's are called the Euclidean coordinat functions of ϕ.

<u>Proof:</u> Clearly any $\phi = \sum f_i \, dx_i$ is a 1-form.

Conversely, let ϕ be a 1-form. Let $f_i = \phi(U_i)$, $i = 1, 2, 3$. Then for any vector field

$V = a_1 U_1 + a_2 U_2 + a_3 U_3$, $a_i : \mathbb{R}^3 \to \mathbb{R}$, we have

$$\left(\phi - \sum_{i=1}^{3} f_i\, dx_i\right)(V) = \phi(V) - \sum_{i=1}^{3} f_i\, dx_i(V)$$

$$= \phi\left(\sum_{i=1}^{3} a_i U_i\right) - \sum_{i=1}^{3} f_i\, dx_i\left(\sum_{j=1}^{3} a_j U_j\right)$$

$$= \sum_{i=1}^{3} a_i \phi(U_i) - \sum_{i,j=1}^{3} f_i\, a_j\, dx_i(U_j)$$

$$= \sum_{i=1}^{3} a_i f_i - \sum_{i,j=1}^{3} f_i\, a_j\, \delta_{ij}$$

$$= 0.$$

Since this holds for any vector field V we deduce that

$$\phi = \sum_{i=1}^{3} f_i\, dx_i. \qquad \blacksquare$$

<u>Corollary</u> If f is a differentiable function on $\mathbb{R}^3$, then $df = \sum \frac{\partial f}{\partial x_i} dx_i$.

<u>Proof</u> By definition df satisfies $df(v_p) = v_p[f]$ for any vector field. Hence, $df(U_i) = U_i[f] = \frac{\partial f}{\partial x_i}$.

Hence, $df = \sum_{i=1}^{3} df(U_i) = \sum_{i=1}^{3} \frac{\partial f}{\partial x_i} dx_i$.

Lemma: For differentiable functions f and g, $d(fg) = g\,df + f\,dg$.

Proof $d(fg) = \sum_{i=1}^{3} \frac{\partial (fg)}{\partial x_i} dx_i$

$$\Rightarrow d(fg) = \sum_{i=1}^{3} \left(f \frac{\partial g}{\partial x_i} + g \frac{\partial f}{\partial x_i} \right)$$

$$= f \sum_{i=1}^{3} \frac{\partial g}{\partial x_i} + g \sum_{i=1}^{3} \frac{\partial f}{\partial x_i}$$

$$= f\,dg + g\,df.$$

Lemma Let $f: \mathbb{R}^3 \to \mathbb{R}$ and $h: \mathbb{R} \to \mathbb{R}$ be differentiable functions, then $d(h(f)) = h'(f)\,df$.

Proof; $d(h(f)) = \sum_{i=1}^{3} \frac{\partial (h(f))}{\partial x_i} dx_i$

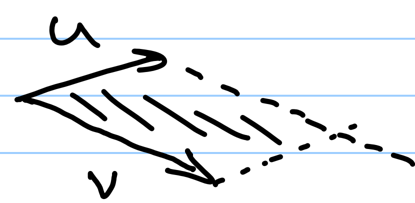
$$\Rightarrow d(h(f)) = \sum_{i=1}^{3} h'(f) \frac{\partial f}{\partial x_i} dx_i$$

$$= h'(f) \sum_{i=1}^{3} \frac{\partial f}{\partial x_i} dx_i$$

$= h'(f)\, df.$ ∎

§1.6. Differential Forms:

Consider the parallelogram formed by two vectors $v=(a,b)$ and $u=(c,d)$:



$$\text{Area} = u \times v = ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

Note that this can written as

$$\text{Area} = ad-bc = dx(v)\,dy(u) - dy(v)\,dx(u)$$
$$= (dx \otimes dy - dy \otimes dx)(u,v).$$

<u>Notation:</u> $dx \wedge dy \doteq dx \otimes dy - dy \otimes dx.$

<u>Proposition:</u> 1) $dx \wedge dy = -dy \wedge dx$

2) $dx \wedge dx = 0$

3) If $\phi = a_1 dx + b_1 dy + c_1 dz$

and $\psi = a_2 dx + b_2 dy + c_2 dz$, then

$$\phi \wedge \psi = (a_1 b_2 - a_2 b_1)\, dx \wedge dy$$
$$+ (a_1 c_2 - a_2 c_1)\, dx \wedge dz$$

$+(b_1c_2-b_2c_1)\,dy\wedge dz$

4) $\phi\wedge\psi=-\psi\wedge\phi$.

Ex $\phi=x\,dx-y\,dy$, $\psi=z\,dx+x\,dz$

$$\phi\wedge\psi=(x\,dx-y\,dy)\wedge(z\,dx+x\,dz)$$
$$=xz\,dx\wedge dx+x^2\,dx\wedge dz$$
$$-yz\,dy\wedge dx-xy\,dy\wedge dz$$
$$=x^2\,dx\wedge dz+yz\,dx\wedge dy-xy\,dy\wedge dz$$

Definition (Exterior Derivative)

If $\phi=\sum f_i\,dx_i$ is a 1-form on $\mathbb{R}^3$, the exterior derivative of ϕ is defined to be the 2-form

$$d\phi=\sum_{i=1}^{3} df_i\wedge dx_i.$$

Remark 0-form: function f

df is a 1-form.

If ϕ is a 1-form then $d\phi$ is a 2-form.

Example: Let $\phi = f_1 dx_1 + f_2 dx_2 + f_3 dx_3$.

Then
$$d\phi = \left(\frac{\partial f_1}{\partial x_2} - \frac{\partial f_2}{\partial x_1}\right) dx_1 \wedge dx_2$$
$$+ \left(\frac{\partial f_3}{\partial x_1} - \frac{\partial f_1}{\partial x_3}\right) dx_1 \wedge dx_3$$
$$+ \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\right) dx_2 \wedge dx_3.$$

Theorem: Let f, g be functions, ϕ, ψ 1-forms and a, b real numbers. Then

1) $d(a\phi + b\psi) = a\,d\phi + b\,d\psi$

2) $d(fg) = df\, g + f\, dg$

3) $d(f\phi) = df \wedge \phi + f\, d\phi$

4) $d(\phi \wedge \psi) = d\phi \wedge \psi - \phi \wedge d\psi$.

§1.7. Mappings

Definition: Given a function $F : \mathbb{R}^n \to \mathbb{R}^m$, let $f_1, \dots, f_m$ denote the real valued functions on $\mathbb{R}^n$ such that $F(p) = (f_1(p), \dots, f_m(p))$, for all $p \in \mathbb{R}^n$. f_i's are called the (Euclidean) coordinate functions of F, and we write $F = (f_1, \dots, f_m)$.

Definition: If $\alpha : I \to \mathbb{R}^n$ is a curve in $\mathbb{R}^n$ and $F : \mathbb{R}^n \to \mathbb{R}^m$ is a mapping then the composition $\beta = F(\alpha) : I \to \mathbb{R}^m$ is a curve in $\mathbb{R}^m$ called the image of α under F.

Definition: Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping. If v is a tangent vector to $\mathbb{R}^n$ at p, let $F_*(v)$ be the initial velocity of the vector of the curve $t \mapsto F(p+tv)$.

$F_*(v) = \frac{d}{dt}(F(p+tv))\Big|_{t=0}$. The map $F_*: T_p\mathbb{R}^n \to T_q\mathbb{R}^m$, $q=F(p)$, is called the tangent map of F at p.

<u>Proposition</u>: Let $F=(f_1, \ldots, f_m)$ be a mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$. If v is a tangent vector to $\mathbb{R}^n$ at p, then $F_*(v) = (v[f_1], \ldots, v[f_m])$ at $F(p)$.

<u>Proof</u>: $\beta(t) = F(p+tv)$
$= (f_1(p+tv), \ldots, f_m(p+tv))$.

By definition $F_*(v) = \beta'(0)$ and $\frac{d}{dt}(f_i(p+tv))\Big|_{t=0} = v[f_i]$. Thus,

$F_*(v) = (v[f_1], \ldots, v[f_m])_{\beta(0)}$. $\blacksquare$

<u>Corollary</u> If $F: \mathbb{R}^n \to \mathbb{R}^m$ is a mapping then at each $p \in \mathbb{R}^n$ the tangent map $F_{*p}: T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^m$ is a linear transformation.

Corollary Let $F: \mathbb{R}^n \to \mathbb{R}^m$ be a mapping. If $\beta = F(\alpha)$ is the image of a curve α in $\mathbb{R}^n$, then $\beta' = F_*(\alpha')$.

Proof Let $F = (f_1, f_2, \dots, f_m)$. Then
$\beta = F(\alpha) = (f_1(\alpha), f_2(\alpha), \dots, f_m(\alpha))$.
Thus $F_*(\alpha') = (\alpha'[f_1], \alpha'[f_2], \dots, \alpha'[f_m])$.
Finally, since $\alpha'[f_i] = \frac{df_i(\alpha)}{dt}$ we get

$$F_*(\alpha'(t)) = \left(\frac{df_1(\alpha)}{dt}(t), \frac{df_2(\alpha)}{dt}(t), \dots, \frac{df_m(\alpha)}{dt}(t)\right)$$
$$= \beta'(t). \blacksquare$$

Let $\{U_j\}_{j=1}^n$ and $\{\bar{U}_i\}_{i=1}^m$ be the natural fields of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Then

Corollary If $F = (f_1, f_2, \dots, f_m)$, then

$$F_*(U_j(p)) = \sum_{i=1}^m \frac{\partial f_i}{\partial x_j}(p)\, \bar{U}_i(F(p)).$$

Proof: For any vector v_p we have

$F_*(v_p) = (v_p[f_1], \ldots, v_p[f_m])$ and thus $F_*(U_j(p)) = (U_j(p)[f_1], \ldots, U_j(p)[f_m])$

$$\Rightarrow F_*(U_j(p)) = \left(\frac{\partial f_1}{\partial x_j}(p), \frac{\partial f_2}{\partial x_j}(p), \ldots, \frac{\partial f_m}{\partial x_j}(p)\right)_{F(p)}$$

$$= \sum_{i=1}^{m} \left(0, \ldots, \frac{\partial f_i}{\partial x_j}(p), \ldots 0\right)_{F(p)}$$

$$= \sum_{i=1}^{m} \frac{\partial f_i}{\partial x_j}(p)\, \overline{U_i}(F(p)).$$

<u>Remark</u>: Recall that the matrix $\left(\frac{\partial f_i}{\partial x_j}(p)\right)_{\substack{1\le i\le m \\ 1\le j\le n}}$ is called the Jacobian matrix of F at p.

<u>Definition</u>: A mapping $F: \mathbb{R}^n \to \mathbb{R}^m$ is called regular if the tangent map F_{*p} is one-to-one at all points $p \in \mathbb{R}^n$.

Recall that the followings are equivalent:

1) F_{*p} is one-to-one,

2) $F_*(v_p) = 0$ implies $v_p = 0$,

3) The Jacobian matrix of F at p has rank n, the dimension of the domain of $\mathbb{R}^n$ of F.

<u>Theorem:</u> (Inverse Function Theorem) Let $F: \mathbb{R}^n \to \mathbb{R}^n$ be a mapping. If F_{*p} is one-to-one at a point $p \in \mathbb{R}^n$, then there is an open subset $\mathcal{U}$ containing p so that F is a diffeomorphism from $\mathcal{U}$ onto an open subset $V \subseteq \mathbb{R}^n$.

CHAPTER 2: Frame Fields

§2.1. Dot Product:

For points $p=(p_1,p_2,p_3)$ and $q=(q_1,q_2,q_3)$ in $\mathbb{R}^3$ the dot product $p\cdot q$ of p and q is defined by the formula

$$p\cdot q = p_1q_1+p_2q_2+p_3q_3.$$

Note that the dot product is

- symmetric: $p\cdot q = q\cdot p$
- bilinear: $(ap+bq)\cdot r = a(p\cdot r)+b(q\cdot r)$
- positive definite: $p\cdot p \geq 0$ and $p\cdot p = 0$ if and only if $p=0$.

Defn Norm of a vector is defined as $\|p\| = (p\cdot p)^{1/2}$.

Defn: If p and q are points in $\mathbb{R}^3$, the Euclidean distance from p to q is $d(p,q)=\|p-q\|$.

Remark: We also know that

for vectors $v_p, u_p \in T_p\mathbb{R}^3$
$v_p \cdot u_p = \|v_p\| \, \|u_p\| \cos\theta$, where θ is the angle between $v_p, u_p \in T_p\mathbb{R}^3$.
So $v_p \cdot u_p = 0$ if and only $\theta = \pi/2$.
In this case, we say that v_p and u_p are orthogonal.

<u>Definition</u>: A set e_1, e_2, e_3 of three mutually orthogonal unit vectors tangent to $\mathbb{R}^3$ at p is called a frame at the point p.

So in this case, $e_i \cdot e_j = \delta_{ij}$ for all $i, j = 1, 2, 3$.

<u>Proposition</u>. If e_1, e_2, e_3 is a frame at $p \in \mathbb{R}$ and $v_p \in T_p\mathbb{R}^3$, then

$$v = (v \cdot e_1)e_1 + (v \cdot e_2)e_2 + (v \cdot e_3)e_3$$

Definition: If $e_1 = (a_{11}, a_{12}, a_{13})$, $e_2 = (a_{21}, a_{22}, a_{23})$ and $e_3 = (a_{31}, a_{32}, a_{33})$ is a frame at a point $p \in \mathbb{R}^3$, then the matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is called the attitude matrix of the frame.

Definition: For tangent vector $v_p = (v_1, v_2, v_3)$ and $u_p = (u_1, u_2, u_3)$ their cross product $v_p \times u_p$ is defined to be the vector

$$\underline{v}_p \times \underline{u}_p = \begin{vmatrix} U_1(p) & U_2(p) & U_3(p) \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{vmatrix}.$$

Note that $v \times u = -u \times v$.

Lemma: $\|u \times v\|^2 = \|u\|^2 \|v\|^2 - (u \cdot v)^2$

Proof: Direct computation gives the result. $=$

Now using $(u \cdot v)^2 = \|u\|^2 \|v\|^2 \cos^2\theta$

gives, $\|u\times v\|^2 = \|u\|^2\|v\|^2\sin^2\theta$.

§2.2. Curves: Let $\alpha: I \to \mathbb{R}^3$ be a curve, with $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$ for some functions $\alpha_i : I \to \mathbb{R}$.

Then the velocity vector of $\alpha(t)$ at t is defined to be the vector $\alpha'(t) = (\alpha_1'(t), \alpha_2'(t), \alpha_3'(t))$ and its speed at t is defined to be the norm of the velocity vector $\|\alpha'(t)\|$.

The arc length of $\alpha(t)$ from $t=a$ to $t=b$ is defined to be the integral $\int_a^b \|\alpha'(t)\|\, dt$.

A curve $\alpha(t)$ is called unit speed if $\|\alpha'(t)\| = 1$ for all t.

Note that for a unit speed

curve $\alpha(t)$ the arclength

$$\int_a^b \|\alpha'(t)\| dt = \int_a^b 1 \cdot dt = b-a.$$

In this case, we say that $\alpha(t)$ is arc-length parametrization.

<u>Theorem:</u> If α is a regular curve in $\mathbb{R}^3$, then there exists a reparametrization β of α so that β has unit speed.

<u>Proof:</u> If $\alpha: \mathbb{R} \to \mathbb{R}^3$, let $s(t) = \int_a^t \|\alpha'(u)\| du$

so $\frac{ds}{dt} = \|\alpha'(t)\| > 0$ since α is regular. Now by the Inverse Function Theorem $s = s(t)$ has inverse $t = t(s)$.

Now $\beta(s) = \alpha(t(s))$ is the required curve: $\beta'(s) = \alpha'(t(s)) \frac{dt}{ds}$ and thus

$$\|\beta'(s)\| = \|\alpha'(t(s))\| \, |dt/ds|$$

$$= \frac{ds}{dt} \left| \frac{dt}{ds} \right| = \frac{ds}{dt} \frac{dt}{ds} = 1$$

because both ds/dt and dt/ds are positive. $\blacksquare$

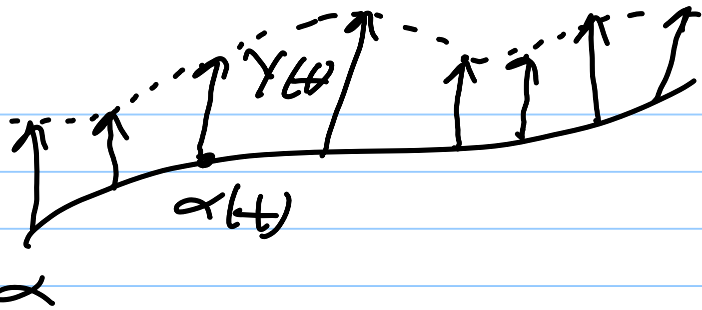
<u>Example</u> $\alpha(t) = (a\cos t, a\sin t, bt)$.
Then $\alpha'(t) = (-a\sin t, a\cos t, b)$ and thus $s(t) = \int_{t_0=0}^{t} \|\alpha'(t)\|\,dt = t\sqrt{a^2+b^2}$

($t_0 = 0$ say)

Hence, $t(s) = \dfrac{s}{c}$, $c = \sqrt{a^2+b^2}$.

So, $\beta(s) = \alpha(t(s)) = \left(a\cos\dfrac{s}{c}, a\sin\dfrac{s}{c}, \dfrac{bs}{c}\right)$

<u>Definition:</u> A vector field Y on the curve $\alpha: I \to \mathbb{R}^3$ is a function that assigns to each $t \in \mathbb{R}$ a tangent vector $Y(t)$ to $\mathbb{R}^3$ at $\alpha(t)$.



$$Y(t) = (y_1(t), y_2(t), y_3(t))_{\alpha(t)}$$
$$= \sum_{i=1}^{3} y_i(t)\, U_i(\alpha(t)).$$

The derivative of a vector field is defined as

$$Y'(t) = (y_1'(t), y_2'(t), y_3'(t))$$

<u>Lemma:</u> 1) A curve α is constant if and only if its velocity vector is zero.

2) A nonconstant curve α is a straight line if and only if its acceleration is zero $\alpha'' = 0$

3) A vector field Y on a curve is parallel if and only if its derivative is zero, $Y' = 0$.

§2.3. The Frenet Formulas

Let $\beta: I \to \mathbb{R}^3$ be a unit speed curve. Let $T = \beta'$ be the unit tangent vector field on β. Then $\|T(s)\|^2 = 1$ and thus

$$0 = \frac{d}{d}(1) = \frac{d}{d}\|T(s)\|^2$$
$$= \frac{d}{ds}\left(T(s)\cdot T(s)\right)$$
$$= T'(s)\cdot T(s) + T(s)\cdot T'(s)$$
$$= 2T'(s)\cdot T(s)$$

and thus $T'(s) \perp T(s)$, for all $\in I$. Thus $T'(s)$ is orthogonal β. The real valued function $\kappa(s) = \|T'(s)\|$ is called the curvature of β. Note that it measures the rate at which the tangent vector bends at s, because the length of $T(s)$ is constant.

The vector field N on β given by $N(s) = T'(s)/\kappa(s)$ is a unit vector field, called the principal normal vector field of β.

Finally, $B = T \times N$ on β is called the binormal vector field of β.

<u>Lemma</u>: Let β be a unit-speed curve in $\mathbb{R}^3$ with $\kappa > 0$. Then the vector fields T, N and B on β are unit vector fields that are mutually orthogonal at each point. We call T, N, B the Frenet frame field on β.

Summary

$T' = B$, $N = T'/K$, $B = T \times N$ so that $T \cdot T = N \cdot N = B \cdot B = 1$ and all other dot products are zero.

Since T, N, B is a frame any vector field on $\mathcal{B}$ is a linear combination of T, N and B. Hence, T', N' and B' can be written as a linear combination of T, N, B.

Indeed, we know already that

$$\boxed{T'(s) = KN = 0 \cdot T + KN + 0 \cdot B.}$$

What about $B'(s)$?

Again $\|B(s)\| = 1$ and thus

$$0 = \frac{d}{ds}\langle B(s), B(s)\rangle = 2B(s) \cdot B'(s).$$

So $B'(s) \perp B(s)$. Also since $B = T \times N$ we have $B \cdot T = 0$ and thus $\frac{d}{ds}(B\cdot T)=0 \Rightarrow B'\cdot T + B\cdot T'=0$

$\Rightarrow B'\cdot T = -B\cdot T' = -B\cdot \kappa N = 0$

since $B \perp N$.

Now since $B' \perp B$ and $B' \perp T$

we see that $B' \parallel N$. Hence,

there is a smooth function $\tau(s)$

so that $B' = -\tau N$.

<u>Theorem</u> (Frenet formulas)

If $\beta: I \to \mathbb{R}^3$ is a unit-speed

curve with curvature $\kappa > 0$ and

torsion τ, then

$T' = \kappa N$, $N' = -\kappa T + \tau B$ and

$B' = -\tau N$.

<u>Proof</u>: We just need to prove

the statement $N' = -\kappa T + \tau B$.

Note that we can N' as

$N' = (N' \cdot T)\,T + (N' \cdot N)\,N + (N' \cdot B)\,B.$

<u>$N' \cdot T = ?$</u> Since $N \cdot T = 0 \ \forall s$, we get $N' \cdot T + N \cdot T' = 0$ and thus $N' \cdot T = -N \cdot T' = -N \cdot \kappa N = -\kappa$.

Also, $N \cdot B = 0 \ \forall s$ and thus $N' \cdot B + N \cdot B' = 0 \Rightarrow$
$N' \cdot B = -N \cdot B' = -N \cdot (-\tau N) = \tau$.

Finally, $N \cdot N = 1$ implies $N \cdot N' = 0$.

Thus, $N' = -\kappa T + \tau B$.

<u>Remark:</u> One is tempted to write

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

<u>Example:</u> $\beta(s) = \left(a\cos\frac{s}{c}, a\sin\frac{s}{c}, \frac{bs}{c}\right)$, where $c = (a^2+b^2)^{1/2}$, $a > 0$. Then

$T(s) = \beta'(s) = \left(-\frac{a}{c}\sin\frac{s}{c}, \frac{a}{c}\cos\frac{s}{c}, \frac{b}{c}\right)$.

Hence, $T'(s) = \left(-\frac{a}{c^2}\cos\frac{s}{c}, -\frac{a}{c^2}\sin\frac{s}{c}, 0\right)$

and thus $\kappa(s) = a/c^2$.

Also, $N(s) = \dfrac{T'(s)}{\kappa(s)} = \left(-\cos\frac{s}{c}, -\sin\frac{s}{c}, 0\right)$

and thus $B(s) = T(s) \times N(s)$

$$= \left(\frac{b}{c}\sin\frac{s}{c}, -\frac{b}{c}\cos\frac{s}{c}, \frac{a}{c}\right).$$

Thus $B'(s) = \left(\frac{b}{c^2}\cos\frac{s}{c}, \frac{b}{c^2}\sin\frac{s}{c}, 0\right)$.

Now since, $B' = -\tau N$ we get

$\tau(s) = b/c^2$.

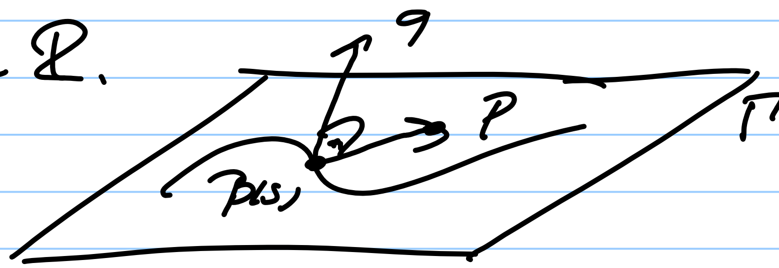
Definition: A curve in $\mathbb{R}^3$ that lies in a single plane of $\mathbb{R}^3$ is called a plane curve.

Proposition: Let β be a unit-speed curve in $\mathbb{R}^3$ with $\kappa > 0$. Then β is a plane curve if and only if $\tau = 0$.

Proof: Suppose β is a plane curve, say lies in the plane P. ~~If~~ p is a point in Γ and q is a normal vector to Γ then the product

$$(\beta(s) - p) \cdot q = 0 \text{ for all}$$

$s \in \mathbb{R}$.



Taking derivative we obtain

$\beta'(s) \cdot q = 0$ and $\beta''(s) \cdot q = 0$ $\forall s$.

Thus q is orthogonal to $T = \beta'$ and $N = \beta''/\kappa$. Hence, $q \parallel B$.

Now $\|B\| = 1$ implies that

$B = \pm \dfrac{q}{\|q\|}$. Thus $B' = 0$ and hence, $\tau = 0$.

Conversely, suppose $\tau = 0$. Thus $B' = 0$ and hence B is a constant (unit) vector, say $B(s) = q$, $\forall s$.

Now consider the real valued

function $f(s) = (\beta(s) - \beta(0)) \cdot B$, for all s.

$\frac{df}{ds} = \beta' \cdot B = T \cdot B = 0$. However, note that $f(0) = 0$ and thus $f(s) = 0$ $\forall s$. So $(\beta(s) - \beta(0)) \cdot B = 0$ for all $s \in \mathbb{R}$. Thus, $\beta(s)$ lies in the plane Γ containing the point $\beta(0)$ and perpendicular to the point B. This finishes the proof. ∎

<u>Lemma</u>: If β is a unit-speed curve with constant curvature $\kappa > 0$ and torsion zero, then β is a part of a circle of radius $1/\kappa$.

<u>Proof</u>: $\tau = 0$ implies that β is a plane curve.

Consider the curve $\gamma = \beta + (1/\kappa) N$

Then by the assumption and the Frenet formulas, we obtain

$$\gamma' = \beta' + \frac{1}{\kappa} N' = T + \frac{1}{\kappa}(-\kappa T) = 0.$$

So the function $\gamma(s)$ is constant, say $\beta(s) + \frac{1}{\kappa} N(s) = c$, for some c. Now the distance from $\beta(s)$ to c is $d(c, \beta(s)) = \| c - \beta(s) \| = \| \frac{1}{\kappa} N(s) \| = \frac{1}{\kappa}$ for all $s \in \mathbb{R}$. Hence, $\beta(s)$ is on the circle with center c and radius $1/\kappa$. ∎

<u>Proposition</u>: If a curve $\beta(s)$ lies on a sphere of radius $a > 0$ then we have $\kappa \geq 1/a$.

<u>Proof</u>: Suppose $\beta(s)$ lies on a sphere of radius a. Then shifting the origin we may assume that

$\beta(s)\cdot\beta(s) = a^2$, for all s. Thus $2\beta'(s)\cdot\beta(s) = 0$ and hence $\beta(s)\cdot T = 0$. Differentiating one more time we get $\beta'(s)\cdot T + \beta(s)\cdot T' = 0$ and hence, $T\cdot T + \kappa\,\beta(s)\cdot N = 0$, which implies $\kappa\beta\cdot N = -1$.

Now by Schwarz inequality, $|\beta\cdot N| \le \|\beta\|\,\|N\| = a$. Since $\kappa \ge 0$ we see that $\kappa = |\kappa| = \dfrac{1}{|\beta\cdot N|} \ge \dfrac{1}{a}$.

§2.4. Arbitrary-Speed Curves

Let $\alpha = \alpha(t)$ be any smooth curve, not necessarily unit speed.
Let $\bar{\alpha} = \bar{\alpha}(s)$ be the unit speed reparametrization of α. So, $\alpha(t) = \bar{\alpha}(s(t))$. If $\bar{\kappa}, \bar{\tau}, \bar{T}, \bar{N}$ and $\bar{B}$ are defined for $\bar{\alpha}$ then we define for α the $(s = s(t))$

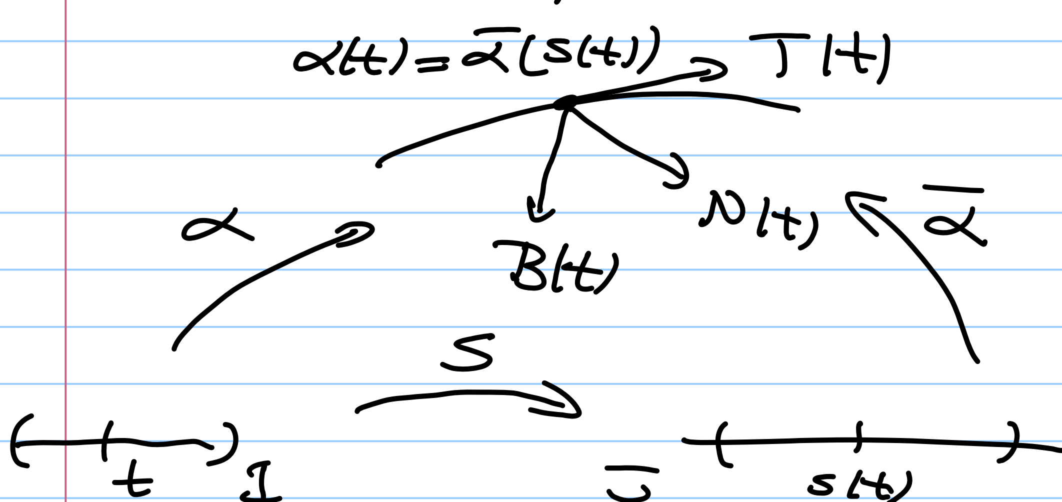
curvature function: $\kappa(t) = \bar{\kappa}(s(t))$

torsion function: $\tau = \bar{\tau}(s)$

unit tangent vector field: $T = \bar{T}(s)$

Principal Normal vector field; $N = \bar{N}(s)$

Binormal vector field: $B = \bar{B}(s)$.



Lemma: Let α be a regular curve in $\mathbb{R}^3$ with $\kappa > 0$ and speed function $v = v(t)$. Then, we have

$$T' = \kappa v N$$
$$N' = -\kappa v T \qquad + \tau v B$$
$$B' = \qquad -\tau v N.$$

Proof: Assume the above setup. Since $T(t) = \bar{T}(s(t))$. Then $T'(t) = \bar{T}'(s)\frac{ds}{dt} = \bar{T}'v$. By the usual Frenet formula we have $\bar{T}'(s) = \bar{\kappa}(s)\bar{N}(s) = \kappa N$. So $T' = \bar{T}'v = \kappa v N$. The rest can be done in a similar fashion. ∎

Lemma: If α is a regular curve with speed function v, then the velocity and acceleration of α are given by $\alpha' = vT$, $\alpha'' = v'T + \kappa v^2 N$.

Proof: $\alpha(t)=\bar{\alpha}(s(t))$.

$$\alpha'(t)=\bar{\alpha}'(s)\frac{ds}{dt}=v\bar{T}(s)=vT.$$

So, $$\alpha''(t)=\frac{dv}{dt}T+vT'$$
$$=\frac{dv}{dt}T+\kappa v^2 N,$$ by the above lemma. ∎

Theorem: Let α be a regular curve. Then

$T=\alpha'/\|\alpha'\|$

$N=B\times T$, $\kappa=\|\alpha'\times\alpha''\|/\|\alpha'\|^3$

$B=\alpha'\times\alpha''/\|\alpha'\times\alpha''\|$

$$\tau=(\alpha'\times\alpha'')\cdot\alpha'''/\|\alpha'\times\alpha''\|^2.$$

Proof Clearly, $T=\alpha'/\|\alpha'\|$. Now by previous lemma

$$\alpha'\times\alpha''=(vT)\times\left(\frac{dv}{dt}T+\kappa v^2 N\right)$$
$$=v\frac{dv}{dt}\underbrace{\cancel{T\times T}}_{0}+\kappa v^3\overbrace{T\times N}^{B}$$
$$=\kappa v^3 B.$$

Also, $\|\alpha'\times\alpha''\|=\|\kappa v^3 B\|=\kappa v^3$

So, $$B=\frac{\alpha'\times\alpha''}{\kappa v^3}=\frac{\alpha'\times\alpha''}{\|\alpha'\times\alpha''\|}$$ and

$\kappa = \|\alpha' \times \alpha''\| / \nu^3 = \|\alpha' \times \alpha''\| / \|\alpha'\|^3$

For the statement about the torsion we need to compute $(\alpha' \times \alpha'') \cdot \alpha'''$. Since, $\alpha' \times \alpha'' = \kappa \nu^3 B$ it is enough to compute the B component of α'''. Now

$$\alpha''' = \left(\frac{d\nu}{dt} T + \kappa \nu^2 N\right)' = \kappa \nu^2 N' + \cdots$$
$$= \kappa \nu^3 \tau B + \cdots$$

Hence, $(\alpha' \times \alpha'') \cdot \alpha''' = \kappa^2 \nu^6 \tau$ so that

$$\tau = \frac{(\alpha' \times \alpha'') \cdot \alpha'''}{(\kappa \nu^3)^2} = \frac{(\alpha' \times \alpha'') \cdot \alpha'''}{\|\alpha' \times \alpha''\|^2}$$

<u>Example</u>: $\alpha(t) = (3t - t^3, 3t^2, 3t + t^3)$.

$\alpha'(t) = 3(1 - t^2, 2t, 1 + t^2)$

$\alpha''(t) = 6(-t, 1, t)$, $\alpha'''(t) = 6(-1, 0, 1)$,

so $\alpha'(t) \cdot \alpha'(t) = 18(1 + 2t^2 + t^4)$ and

$\nu(t) = (\alpha'(t) \cdot \alpha'(t))^{1/2} = \sqrt{18}\,(1 + t^2)$.

Also using the definition of cross product

$$\alpha'(t)\times\alpha''(t) = \begin{vmatrix} u_1 & u_2 & u_3 \\ 1-t^2 & 2t & 1+t^2 \\ -t & 1 & t \end{vmatrix}$$

$$= 18(-1+t^2, -2t, 1+t^2)$$

Hence, $\|\alpha'(t)\times\alpha''(t)\| = 18\sqrt{2}(1+t^2)$.

Now, $(\alpha'\times\alpha'')\cdot\alpha''' = 6\cdot 18\cdot 2$

So, $T = \dfrac{1}{\sqrt{2}(1+t^2)}(1-t^2, 2t, 1+t^2)$

$N = \dfrac{1}{1+t^2}(-2t, 1-t^2, 0)$

$B = \dfrac{1}{\sqrt{2}(1+t^2)}(-1+t^2, -2t, 1+t^2)$

$\kappa = \tau = \dfrac{1}{3(1+t^2)^2}$.

An Application of Frenet Formulas

The spherical image of a unit speed curve β is defined by $\sigma(s) \doteq T(s) = \beta'(s)$. Thus $\|\sigma(s)\| = 1$ so that $\sigma(s)$ lies on the unit sphere Σ.

Example: For example for the

unit sphere helix

$B(s) = \left(\frac{a}{c}\cos\frac{s}{c}, \frac{a}{c}\sin\frac{s}{c}, \frac{bs}{c}\right)$,

$c = \sqrt{a^2+b^2}$ the curve $\sigma(s)$ is given

by $\sigma(s) = \left(-\frac{a}{c}\sin\frac{s}{c}, \frac{a}{c}\cos\frac{s}{c}, \frac{b}{c}\right)$.

It follows that the image of σ lies

in the intersection of Σ with the

plane $z = b/c$.

The curve $\sigma(s)$ is not unit speed

and indeed $\sigma' = T' = \kappa N$ so that

its speed is the curvature κ of B.

(Use the Frenet formulas for

general curves)

$$\sigma'' = (\kappa N)' = \frac{d\kappa}{ds} N + \kappa N'$$

$$= -\kappa^2 T + \frac{d\kappa}{ds} N + \kappa\tau B, \text{ we get}$$

$$\sigma' \times \sigma'' = -\kappa^3 N \times T + \kappa^2 \tau N \times B$$
$$= \kappa^2(\kappa B + \tau T).$$

Finally, the curvature of σ is

$$\kappa_\sigma = \frac{\|\sigma' \times \sigma''\|}{\|\sigma'\|^3} = \frac{\kappa^2 \|\kappa B + \tau T\|}{\kappa^3}$$

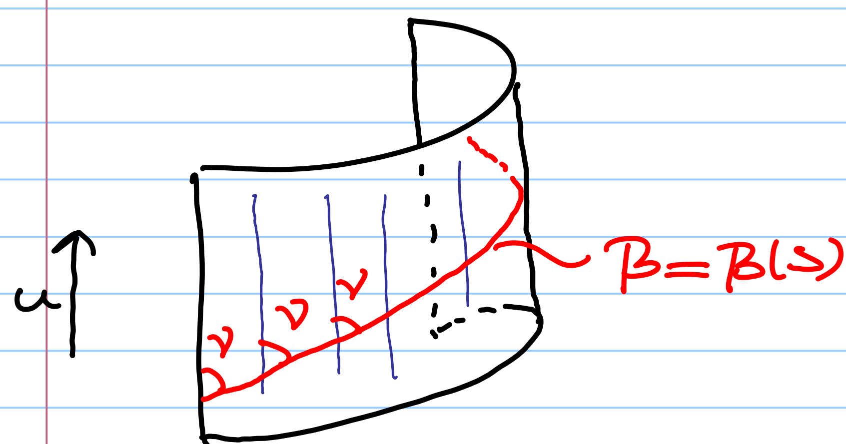
$$= \frac{\sqrt{\kappa^2+\tau^2}}{\kappa} = \left(1+\left(\frac{\tau}{\kappa}\right)^2\right)^{1/2} \geq 1.$$

In particular, κ_σ depends on the ratio τ/κ.

<u>Definition</u>: A regular curve α in $\mathbb{R}^3$ is a cylindrical helix provided the unit tangent T of α has constant angle ϑ with some fixed unit vector u; that is $T(t) \cdot u = \cos\vartheta$, for all t. Since the angle is independent of the length of $T(t)$ we may assume that the curve is unit speed, say $\beta = \beta(s)$. So we assume $T \cdot u = \cos\vartheta$.

Consider the real valued function $h(s) = (\beta(s) - \beta(0)) \cdot u$. Then

$\frac{dh}{ds} = \beta' \cdot u = T \cdot u = \cos\nu$, so that β is rising at a constant rate relative to the arc length and $h(s) = s\cos\nu$. Hence, we get $h(t) = s(t)\cos\nu$.



<u>Theorem:</u> A regular curve α with $\kappa > 0$ is a cylindrical helix if only if the ratio τ/κ is constant.

<u>Proof:</u> We may assume that α has unit speed. We have seen above that if α is a cylindrical helix with $T \cdot u = \cos\nu$, then

$0=(T\cdot u)'=T'\cdot u=\kappa N\cdot u$. Since $\kappa>0$, we see that $N\cdot u=0$. Thus for each s, u lies in the plane determined by $T(s)$ and $B(s)$. So we may write u as $u=\cos\vartheta T+\sin\vartheta B$. Now take the derivative of this expression and apply Frenet formulas to obtain $0=(\kappa\cos\vartheta-\tau\sin\vartheta)N$. So, $\tau\sin\vartheta=\kappa\cos\vartheta\Rightarrow\cot\vartheta=\tau/\kappa$.

Conversely, assume that τ/κ is constant. Choose ϑ so that $\cot\vartheta=\tau/\kappa$. Let $U=\cos\vartheta T+\sin\vartheta B$. Then $U'=(\kappa\cos\vartheta-\tau\sin\vartheta)N=0$. Let $u=\frac{U}{\|U\|}$, then $T\cdot u=\cos\vartheta$, so that α is a cylindrical helix. $\blacksquare$

Summary: For a regular curve in $\mathbb{R}^3$ we have

$\kappa = 0 \iff$ straight line

$\tau = 0 \iff$ plane curve

$\kappa > 0$ constant and $\tau = 0 \iff$ circle

$\kappa > 0, \tau > 0$ both constant $\iff$ circular helix

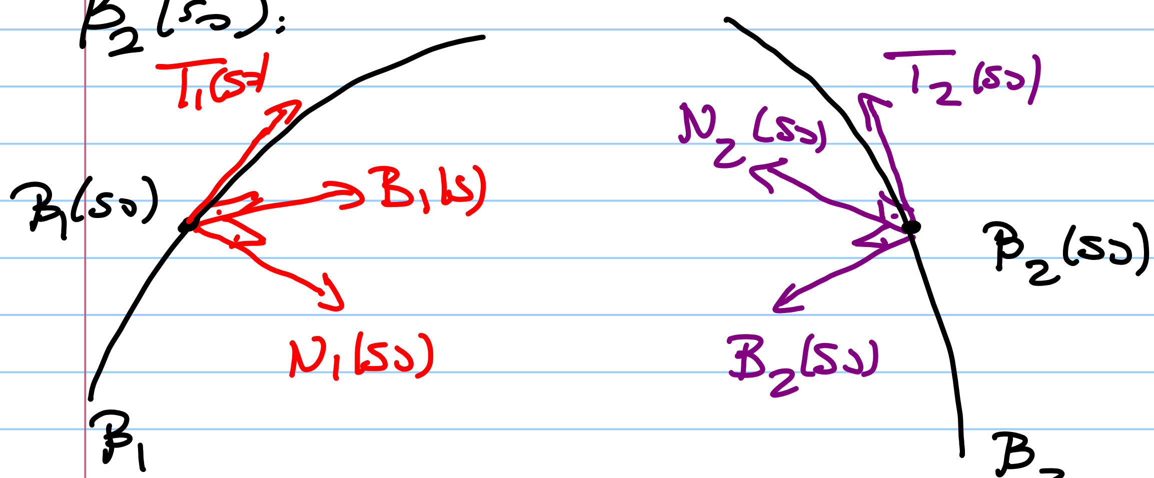
τ / κ constant and $\neq 0 \iff$ cylindrical helix.

<u>Fundamental Theorem of Curves</u>

Let $\beta_1 : I \longrightarrow \mathbb{R}^3$ and $\beta_2 : I \longrightarrow \mathbb{R}^3$ be two unit speed regular curves having the same curvature $\kappa(s)$ and torsion $\tau(s)$ functions. Then there is a rigid motion φ of $\mathbb{R}^3$ so that $\beta_2(s) = \varphi(\beta_1(s))$, for all $s \in I$.

<u>Proof</u>: Pick any $s_0 \in I$ and consider

the Frenet frames of β_1 and β_2 at s_0, say $\{T_1(s_0), N_1(s_0), B_1(s_0)\}$ at $\beta_1(s_0)$ and $\{T_2(s_0), N_2(s_0), B_2(s_0)\}$ at $\beta_2(s_0)$:



First choose a translation $\varphi_1 : \mathbb{R}^3 \to \mathbb{R}^3$ with $\varphi_1(\beta_1(s_0)) = \beta_2(s_0)$ and an orthogonal linear map φ_2 (centered at $\beta_2(s_0)$) so that $\varphi_2(\varphi_1(T_1(s_0))) = T_2(s_0)$, $\varphi_2(\varphi_1(N_1(s_0))) = N_2(s_0)$ and $\varphi_2(\varphi_1(B_1(s_0))) = B_2(s_0)$. Let $\varphi = \varphi_2 \circ \varphi_1$, and call $\varphi_2(\beta_1(s))$, $\gamma(s)$. Since both frames are both right-handed φ_2 can be choosen to be a rotation.

One can see easily that the curvature and the torsion of a regular curve do not alter under rigid motions of $\mathbb{R}^3$. Thus the curvature and torsion of $\gamma(s)$ are still $\kappa(s)$ and $\tau(s)$. Hence, $\gamma(s)$ and $\beta_1(s)$ are both solutions of the same Initial Value Problem

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix}' = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

with

$$\begin{bmatrix} T(s_0) \\ N(s_0) \\ B(s_0) \end{bmatrix} = \begin{bmatrix} T_1(s_0) \\ N_1(s_0) \\ B_1(s_0) \end{bmatrix}.$$

However, from the theory of systems of O.D.E.'s we know that such a system has a unique solution (provided that $\kappa = \kappa(s)$ and $\tau = \tau(s)$ are continuous, which is the case). Hence, $\varphi(T_1(s)) = T_2(s)$

for all $s \in I$ (indeed, $\varphi(N_1(s)) = N_2(s)$ and $\varphi(B_1(s)) = B_2(s)$, for all $s \in I$).

Now, $\varphi(T_1(s)) = T_2(s)$ for all s, implies that $\beta_2'(s) = \gamma'(s)$ for all $s \in I$.

So, $(\beta_2 - \gamma)'(s) = 0$ $\forall s \in I$. Hence, $\beta_2 - \gamma$ is a constant vector.

In particular,

$$0 = \beta_2(s_0) - \gamma(s_0) = \beta_2(s) - \gamma(s), \forall s \in I.$$

$$\Rightarrow \beta_2(s) = \gamma(s) = \varphi(\beta_1(s)), \forall s \in I.$$

This finishes the proof. ∎

<u>Remark:</u> 1) Two curves α, β are called congruent if $\beta = F(\alpha)$ for some isometry. In this case, $\kappa_\alpha = \kappa_\beta$ and $\tau_\beta = \operatorname{sgn}(F)\tau_\alpha$.

2) For any two functions $\kappa > 0$ and τ on an interval there is a unit speed curve α having κ and τ as its curvature and torsion.

§2.5. Covariant Derivative.

Definition. Let W be a vector field on $\mathbb{R}^3$, and let v be a tangent vector to $\mathbb{R}^3$ at a point p. Then the covariant derivative of W with respect to v is the tangent vector $\nabla_v W = W(p+tv)'(0)$ at the point p.

Example. Let $W(x,y,z) = x^2 U_1 + yz U_3$, $v = (-1,0,2)$ and $p = (2,1,0)$. Then $p+tv = (2-t, 1, 2t)$. Hence,

$$W(p+tv) = W(2-t, 1, 2t)$$
$$= (2-t)^2 U_1 + 2t\, U_3.$$

Thus,
$$\nabla_v W = \frac{d}{dt}\left[(2-t)^2 U_1 + 2t\, U_3\right]\Big|_{t=0}$$
$$= \left[-2(2-t)\, U_1 + 2U_3\right]\Big|_{t=0}$$
$$= -4U_1 + 2U_3$$

Lemma: If $W = \sum_{i=1}^{3} w_i U_i$ is a vector field on $\mathbb{R}^3$, and v is tangent vector at p, then
$$\nabla_v W = \sum_{i=1}^{3} v[w_i]\, U_i(p).$$

Proof $W(p+tv) = \sum w_i(p+tv)\, U_i(p+tv)$

So, $\nabla_v W = (W(p+tv))'(0)$
$$= \sum_i \frac{d}{dt}(w_i(p+tv))\Big|_{t=0} U_i(p+tv)$$
$$= \sum_i v[w_i]\, U_i(p).$$

Theorem: Let v and w be tangent vectors to $\mathbb{R}^3$ at p and let Y and Z be vector fields on $\mathbb{R}^3$. Then for real numbers a, b and function f, we have

1) $\nabla_{av+bw} Y = a\nabla_v Y + b\nabla_w Y$

2) $\nabla_v(aY+bZ) = a\nabla_v Y + b\nabla_v Z$

3) $\nabla_v(fY) = v[f]\, Y(p) + f(p)\nabla_v Y$.

4) $v[Y \cdot Z] = \nabla_v Y \cdot Z(p) + Y(p) \cdot \nabla_v Z$.

If V is also a vector field then $\nabla_V W$ is defined to be the vector field given by $(\nabla_V W)(p) = \nabla_{V(p)} W$.

<u>Example</u> Let $V = (y-x)U_1 + xyU_3$ and $W = x^2 U_1 + yzU_2$. Then

$$\nabla_V W = V(p)[x^2]U_1 + V(p)[yz]U_2$$
$$= [(y-x)2x + xy \cdot 0]U_1$$
$$+ [(y-x)\cdot 0 + xy \cdot y]U_2$$
$$= 2x(y-x)U_1 + xy^2 U_2.$$

An immediate consequence of the above theorem is as below:

<u>Corollary</u> Let V, W, Y and Z be vector fields on $\mathbb{R}^3$. Then

1) $\nabla_{fV+gW} Y = f\nabla_V Y + g\nabla_W Y$, for all functions f and g.

2) $\nabla_V (aY+bZ) = a\nabla_V Y + b\nabla_V Z$

3) $\nabla_V (fY) = V[f]Y + f\nabla_V Y$, for all functions f.

4) $V[Y\cdot Z] = \nabla_V Y \cdot Z + Y\cdot \nabla_V Z$.

§26. Frame Fields

Recall that three vector fields E_1, E_2 and E_2 on $\mathbb{R}^3$ are said to constitute a frame field if $E_i \cdot E_j = \delta_{ij}$ for all $i, j \in \{1,2,3\}$.

Example: 1) Let r, ϑ, z be the cylindrical coordinates on $\mathbb{R}^3$. Then

$E_1 = \cos\vartheta\, U_1 + \sin\vartheta\, U_2$,

$E_2 = -\sin\vartheta\, U_1 + \cos\vartheta\, U_1$ and

$E_3 = U_3$ constitude so called the cylindrical frame field on $\mathbb{R}^3$.

2) Similarly, if ρ, ϑ and φ are the spherical coordinates on $\mathbb{R}^3$ then

$F_1 = \cos\varphi(\cos\vartheta\, U_1 + \sin\vartheta\, U_2) + \sin\varphi\, U_3$

$F_2 = -\sin\vartheta\, U_1 + \cos\vartheta\, U_2$, and

$F_3 = -\sin\varphi(\cos\vartheta\, U_1 + \sin\vartheta\, U_2) + \cos\varphi\, U_3$

form so called the spherical frame field on $\mathbb{R}^3$.

How to obtain:

$x = \rho\cos\varphi\cos\nu$, $y = \rho\cos\varphi\sin\nu$

$z = \rho\sin\varphi$

$$\frac{\partial}{\partial\rho}(x,y,z) = (\cos\varphi\cos\nu, \cos\varphi\sin\nu, \sin\varphi)$$
$$= \cos\varphi(\cos\nu\, u_1 + \sin\nu\, u_2) + \sin\varphi\, u_3$$

$\left\|\frac{\partial}{\partial\rho}(x,y,z)\right\| = 1$ and thus

$$F_1 = \cos\varphi(\cos\nu\, u_1 + \sin\nu\, u_2) + \sin\varphi\, u_3$$

$$\frac{\partial}{\partial\varphi}(x,y,z) = (-\rho\sin\varphi\cos\nu, -\rho\sin\varphi\sin\nu, \rho\cos\varphi)$$

so $F_3 = \dfrac{\frac{\partial}{\partial\varphi}(x,y,z)}{\left\|\frac{\partial}{\partial\varphi}(x,y,z)\right\|}$

$$= -\sin\varphi(\cos\nu\, u_1 + \sin\nu\, u_2) + \cos\varphi\, u_3.$$

Finally, $\frac{\partial}{\partial\nu}(x,y,z) = (-\rho\cos\varphi\sin\nu, \rho\cos\varphi\cos\nu, 0)$ and thus

$$F_2 = \frac{\partial/\partial\nu(x,y,z)}{\|\partial/\partial\nu(x,y,z)\|} = -\sin\nu\, u_1 + \cos\nu\, u_2.$$

Lemma: Let E_1, E_2 and E_3 be a frame field on $\mathbb{R}^3$. Then

1) If V is a vector field on $\mathbb{R}^3$, then $V = \sum f_i E_i$, where $f_i = V \cdot E_i$ are called the coordinate functions of V with respect to E_1, E_2 and E_3.

2) If $V = \sum f_i E_i$ and $W = \sum g_i E_i$, then $V \cdot W = \sum_i f_i g_i$ and in particular, $\|V\| = \left(\sum_i f_i^2\right)^{1/2}$.

CHAPTER 3. Euclidean Geometry

§3.1. Isometries of $\mathbb{R}^3$:

Definition: An isometry of $\mathbb{R}^3$ is a mapping $F:\mathbb{R}^3 \to \mathbb{R}^3$ such that $d(F(p),F(q)) = d(p,q)$, for all $p,q \in \mathbb{R}^3$.

Examples 1) Translations: Let $a \in \mathbb{R}^3$ and define $T:\mathbb{R}^3 \to \mathbb{R}^3$ as $T(p) = a+p$ for all $p \in \mathbb{R}^3$.

2) Rotation about a coordinate axis.

For example ϑ radian counter-clockwise rotation about the z-axis is given by $C(p) = C(p_1,p_2,p_3)$

$$= (p_1\cos\vartheta - p_2\sin\vartheta,\ p_1\sin\vartheta + p_2\cos\vartheta,\ p_3).$$

Lemma: If F and G are isometries of $\mathbb{R}^3$, then so is $F \circ G$.

Lemma: 1) If S and T are translations then $S \circ T = T \circ S$ is also a translation.

2) If T is a translation by the vector a then T^{-1} is the translation by the vector a^{-1}.

3) Given two points $p, q \in \mathbb{R}^3$ there is a unique translation T s.t. $T(p) = q$.

<u>Lemma</u> If $C: \mathbb{R}^3 \to \mathbb{R}^3$ is a linear orthogonal transformation then C is an isometry of $\mathbb{R}^3$.

<u>Lemma</u>: If $F: \mathbb{R}^3 \to \mathbb{R}^3$ is an isometry with $F(0) = 0$ then F is an orthogonal transformation.

<u>Proof</u>: Since $F(0) = 0$ and

$$
\begin{aligned}
\|F(p)\| &= d(0, F(p)) \\
&= d(F(0), F(p)) \\
&= d(0, p) \\
&= \|p\|,
\end{aligned}
$$

so that F preserves the norm. Hence, we see that

$$\|F(p)-F(q)\| = \|F(p-q)\| = \|p-q\|$$

for all $p, q \in \mathbb{R}^3$. So

$$(F(p)-F(q))\cdot(F(p)-F(q)) = (p-q)\cdot(p-q)$$

$$\Rightarrow \|F(p)\|^2 + \|F(q)\|^2 - 2F(p)\cdot F(q)$$
$$= \|p\|^2 + \|q\|^2 - 2\,p\cdot q,$$

which implies that

$F(p)\cdot F(q) = p\cdot q$. In other words,

F preserves the inner product.

Thus, to finish the proof we just need to show that F is linear.

Let $u_1 = (1,0,0)$, $u_2 = (0,1,0)$ and $u_3 = (0,0,1)$.

Since $u_i \cdot u_j = \delta_{ij}$ we see that

$F(u_i)\cdot F(u_j) = u_i \cdot u_j = \delta_{ij}$. So

$\{F(u_1), F(u_2), F(u_3)\}$ is also an orthonormal frame.

Let $p \in \mathbb{R}^3$. Then $p = (p\cdot u_1)u_1 + (p\cdot u_2)u_2$
$+ (p\cdot u_3)u_3$.

Also, $F(p) = (F(p)\cdot F(u_1))F(u_1)$

$+ (F(p)\cdot F(u_2))F(u_2)$

$+ (F(p)\cdot F(u_3))F(u_3)$.

However, $F(p)\cdot F(u_i) = p\cdot u_i$

and thus

$$F(p) = (p\cdot u_1)F(u_1) + (p\cdot u_2)F(u_2) + (p\cdot u_3)F(u_3).$$

Finally, $F(p+q) = ((p+q)\cdot u_1)F(u_1)$

$+ ((p+q)\cdot u_2)F(u_2) + ((p+q)\cdot u_3)F(u_3)$

$= (\underline{p\cdot u_1} + \underline{q\cdot u_1})F(u_1) + (\underline{p\cdot u_2} + \underline{q\cdot u_2})$

$F(u_2) + (\underline{p\cdot u_3} + \underline{q\cdot u_3})F(u_3)$

$= F(p) + F(q)$.

Moreover, for any real number λ

$F(\lambda p) = (\lambda p\cdot u_1)F(u_1) + (\lambda p\cdot u_2)F(u_2)$

$+ (\lambda p\cdot u_3)F(u_3)$

$= \lambda F(p)$, so that F is linear. This finishes the proof. ∎

<u>Theorem</u> If F is an isometry of $\mathbb{R}^3$, then there is a unique translation T and a unique orthogonal transformation so that $F = T \circ C$.

<u>Proof</u> Let $F(0) = a$ and let T be the translation by the vector a. Then $T^{-1} \circ F$ is an isometry such that $(T^{-1} \circ F)(0) = T^{-1}(a) = 0$. So $T^{-1} \circ F$ must be an orthogonal transformation, say C. Hence $T^{-1} \circ F = C$ and thus

$$F = T \circ C.$$

For uniqueness part, let $T \circ C = \bar{T} \circ \bar{C}$ for some other translation $\bar{T}$ and orthogonal transformation $\bar{C}$. Then

$\overline{T}^{-1} \circ T = \overline{C} \circ C^{-1}$. We know that $\overline{T}^{-1} \circ T$ is a translation by the vector $\overline{T}^{-1} \circ T(0) = \overline{C} \circ C^{-1}(0)$

$$= \overline{C}(0)$$
$$= 0,$$
because both C and $\overline{C}$ are linear.

So $\overline{T}^{-1} \circ T$ is the translation by the zero vector. In other words $\overline{T}^{-1} \circ T = Id$, the identity.

Finally, $\overline{C} \circ C^{-1} = \overline{T}^{-1} \circ T = Id$ and thus $\overline{C} = C$. This finishes the proof. $\blacksquare$

§3.2 The Tangent Map of an Isometry

<u>Theorem</u>: Let F be an isometry of $\mathbb{R}^3$ with orthogonal part C, then
$F_*(v_p) = C(v)_{F(p)}$.

<u>Proof</u> Let $F = TC$, then

$$F_*(v_p) = T_*(C_*(v_p))$$
$$= T_*(C(v)_p)$$
$$= C(v)_p.$$

Second Proof:

$$F_*(v_p) = \frac{d}{dt} F(p+tv)\Big|_{t=0}$$
$$= \frac{d}{dt}(TC(p+tv))\Big|_{t=0}$$
$$= \frac{d}{dt}(C(p+tv)+a)\Big|_{t=0}$$
$$= \frac{d}{dt}(C(p)+tC(v)+a)\Big|_{t=0}$$
$$= C(v)_{F(p)}$$

<u>Corollary</u> Isometries preserve dot products of tangent vectors. That is, if v_p and w_p are tangent vectors

to $\mathbb{R}^3$ at p, and F is an isometry then $F_*(v_p)\cdot F_*(w_p)=v_p\cdot w_p$.

Proof $F_*(v_p)\cdot F_*(w_p)= C(v)_{F(p)}\cdot C(w)_{F(p)}$

$$= C(v)\cdot C(w)$$

$$= v\cdot w$$

$$= v_p\cdot w_p. \qquad \blacksquare$$

Theorem: If e_1, e_2, e_3 is a frame at a point $p\in\mathbb{R}^3$ and f_1, f_2, f_3 is another frame at a point q, then there is a unique isometry F of $\mathbb{R}^3$ so that $F(p)=q$ and $F_*(e_i)=f_i$, $i=1,2,3$.

§3.3. Orientation.

A frame e_1, e_2, e_3 at a point is called positively oriented if

$e_1 \times e_2 = e_3$ (Note that for any frame $e_1 \times e_2 = \pm e_3$).

Remark Frenet Frames (T, N, B) are always positively oriented, because $B = T \times N$.

Definition: For an isometry F of $\mathbb{R}^3$ $\operatorname{sgn}(F)$ is defined to be the sign $\operatorname{sgn} F = \det C$, where $F = TC$.

Lemma: If e_1, e_2, e_3 is a frame at some point of $\mathbb{R}^3$ and F is an isometry then

$$F_*(e_1) \cdot (F_*(e_2) \times F_*(e_3)) = (\operatorname{sgn} F)\, e_1 \cdot (e_2 \times e_3).$$

Definition: An isometry is said to be orientation-preserving if $\operatorname{sgn} F = \det C = +1$ and orientation reversing if $\operatorname{sgn} F = \det C = -1$, where C is the orthogonal part of F.

Examples: 1) All translations are orientation preserving, because if T is a translation then $T_* = Id$ and $\det(T_*) = \det(Id) = 1$.

2) Rotations are orientation preserving:
$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

3) Reflections are orientation reversing.

Lemma: Let e_1, e_2, e_3 be a frame at point p of $\mathbb{R}^3$. If $v = \sum v_i e_i$ and $w = \sum w_i e_i$, then

$$v \times w = \epsilon \begin{vmatrix} e_1 & e_2 & e_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}, \text{ where}$$

$\epsilon = e_1 \cdot (e_2 \times e_3)$.

<u>Proof</u>: Note that the following are equivalent:

i) $\epsilon = 1$, ii) $e_1 \times e_2 = e_3$,

iii) $e_2 \times e_3 = e_1$, iv) $e_3 \times e_1 = e_2$.

Also, $e_1 \cdot (e_2 \times e_3) = (v_2 w_3 - v_3 w_2)$.

This finishes the proof. ∎

<u>Theorem</u>: If F is an isometry then for any two tangent vectors v and w at a point of $\mathbb{R}^3$ we have

$$F_*(v \times w) = (\operatorname{sgn} F)\, F_*(v) \times F_*(w).$$

<u>Proof</u>: Let $v = \sum v_i U_i(p)$, $w = \sum w_i U_i(p)$ and $e_i = F_*(U_i(p))$.

Then $F_*(v) = \sum v_i e_i$ and

$$F_*(w) = \sum w_i e_i.$$

So by a straight computation we see that

$$F_*(v) \times F_*(w) = \epsilon \, F_*(v \times w), \text{ where}$$

$$\epsilon = e_1 \cdot (e_2 \times e_3) = F_*(U_1(p)) \cdot (F_*(U_2(p)) \times F_*(U_3(p))).$$

However, U_1, U_2 and U_2 are positively oriented and thus by the above lemma $\epsilon = \operatorname{sgn} F$. $\blacksquare$

§3.4. Euclidean Geometry

Theorem: Let β be a unit speed curve with $\kappa > 0$, and let $\bar{\beta} = F(\beta)$ be the image of β under an isometry F of $\mathbb{R}^3$. Then

$\bar{\kappa} = \kappa$, $\bar{T} = F_*(T)$

$\bar{\tau} = (\operatorname{sgn} F)\tau$, $\bar{N} = F_*(N)$

$\bar{B} = (\operatorname{sgn} F) F_*(B)$, where $\operatorname{sgn} F = \pm 1$ is the sgn of F.

Proof: $\|\bar{\beta}'\| = \|F_*(\beta')\| = \|\beta'\| = 1$ because F is an isometry.

Clearly, $\bar{T} = \bar{\beta}' = F_*(\beta') = F_*(T)$.

Also $\bar{\kappa} = \|\bar{\beta}''\| = \|F_*(\beta'')\| = \|\beta''\| = \kappa$,

and $\bar{N} = \dfrac{\bar{\beta}''}{\bar{\kappa}} = \dfrac{F_*(\beta'')}{\kappa} = F_*\left(\dfrac{\beta''}{\kappa}\right) = F_*(N)$.

Moreover,

$\bar{B} = \bar{T} \times \bar{N} = F_*(T) \times F_*(N) = (\operatorname{sgn} F)$

$F_*(T \times N)$

$= \mathrm{sgn}(f)\, F_*(B).$

Finally, using $\tau = -B' \cdot N = B \cdot N'$,

$$\overline{\tau} = \overline{B} \cdot \overline{N} = (\mathrm{sgn} f)\, F_*(B) \cdot F_*(N')$$

$$= (\mathrm{sgn} f)\, B \cdot N'$$

$$= (\mathrm{sgn} f)\, \tau .$$

CHAPTER 4: Calculus on a Surface

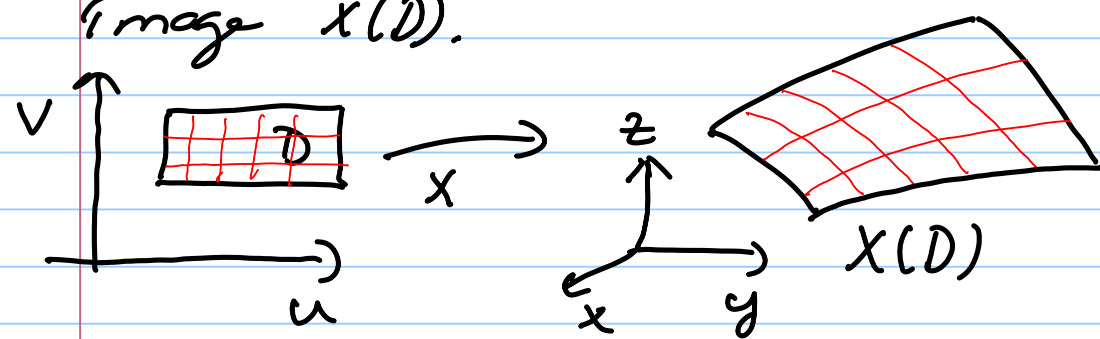
§4.1. Surfaces in $\mathbb{R}^3$:

A surface in $\mathbb{R}^3$ is a subset of $\mathbb{R}^3$ that looks like $\mathbb{R}^2$ locally. More specifically we have

Definition: A coordinate patch $x: D \to \mathbb{R}^3$ is a one to one regular mapping of an open set D of $\mathbb{R}^2$ into $\mathbb{R}^3$.

As usual regularity means that the derivative map $x_*: T_pD \to T_{x(p)}\mathbb{R}^3$ is one to one at each point $p \in D$.

We'll also ask that the coordinate patches to be proper maps. What we mean by a proper map in this case is that $x^{-1}: x(D) \to D$ is also a continuous map. So this is different than the being a proper map we have studied in Math 349. In particular, $x : D \to \mathbb{R}^3$ is a homeomorphism onto its image $x(D)$.



<u>Definition</u>: A surface in $\mathbb{R}^3$ is a subset M of $\mathbb{R}^3$ such that for each point p of M there exists a proper patch in M whose image contains a neighborhood of p in M.

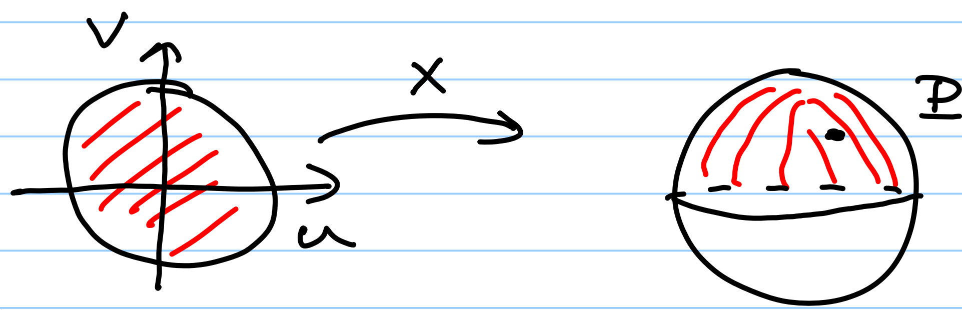
<u>Example</u>: The unit sphere Σ in $\mathbb{R}^3$,

$$\Sigma = \{ p = (p_1, p_2, p_3) \in \mathbb{R}^3 \mid p_1^2 + p_2^2 + p_3^2 = 1 \}.$$

Let $p = (p_1, p_2, p_3) \in \Sigma$ be a point.

Since, $p_1^2 + p_2^2 + p_3^2 = 1$ at least one

$p_i \neq 0$. First assume that $p_3 > 0$.
Then consider the function
$x : D \to \mathbb{R}^3$, given by
$x(u,v) = (u, v, \sqrt{1-u^2-v^2})$, where
$D = \{(u,v) \in \mathbb{R}^2 / u^2+v^2 < 1\}$ is the
unit open disc in $\mathbb{R}^2$. Note
that $p \in x(D)$ since $p_3 > 0$.



Note that the image $x(D)$ is
the Northern hemisphere.
Let's check that x is a
proper (surface) patch.

1) x is one to one. To see
this note that x^{-1} exists. Let
$\Sigma^+ = \{(x,y,z) \in \Sigma \mid z > 0\}$ the

Northern hemisphere and then

$x^{-1}: \Sigma^{+} \longrightarrow D$, $x^{-1}(x,y,z) = (x,y)$

is just the inverse of x:

$(x \circ x^{-1})(x,y,z) = x(x,y) = (x, y, \sqrt{1-x^2-y^2})$

$= (x,y,z)$

and $(x^{-1} \circ x)(u,v) = x^{-1}(u, v, \sqrt{1-u^2-v^2})$

$= (u,v)$.

2) x is proper. Clearly, x^{-1} is continuous since $x(x,y,z) = (x,y)$.

3) x is regular. To see this we compute its Jacobian matrix

$$J = \begin{pmatrix} \frac{\partial u}{\partial u} & \frac{\partial u}{\partial v} \\ \frac{\partial v}{\partial v} & \frac{\partial v}{\partial v} \\ \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \end{pmatrix},$$

where $f(u,v) = \sqrt{1-u^2-v^2}$

Clearly, the Jacobian matrix

has rank 2 and thus the derivative map $X_*: T_pD \to T_{X(p)}\mathbb{R}^3$ is onto.

Hence, X is a proper surface patch at points $p=(p_1,p_2,p_3)$ with $p_3>0$. Clearly, for other points, say for $p=(-1,0,0)$ we may use the surface patch given by $x: D \to \mathbb{R}^3$, $x(u,v)=(-\sqrt{1-u^2-v^2}, u, v)$.

Hence, it follows that the unit sphere Σ is a surface in $\mathbb{R}^3$.

Surface patches of the form $(u,v) \longmapsto (u,v,f(u,v))$ for some function f are called Monge patches.

Example: Let $f:\mathbb{R}^2 \to \mathbb{R}$ be a differentiable function and let $\Sigma = \{(x,y,z) \in \mathbb{R}^3 \mid z = f(x,y)\}$. In this case, $x:\mathbb{R}^2 \to \mathbb{R}^3$, $x(u,v) = (u,v,f(u,v))$ is a surface patch covering all of Σ. Showing that x is a surface path is similar to the above example. In particular, Σ is a surface in $\mathbb{R}^3$.

The following theorem provides many examples of surfaces.

Theorem: Let $g:\mathbb{R}^3 \to \mathbb{R}$ be a differentiable function and $c \in \mathbb{R}$ be a real number. Set

$$M = g^{-1}(c) = \{(x,y,z) \in \mathbb{R}^3 \mid g(x,y,z) = c\}.$$

Assume further that the

differential dg is not zero at any point p of M. Then M is a surface.

Proof. This is a simple application of the Implicit Function Theorem. Let $p \in M$ be a point. By the assumption

$$0 \neq dg(p) = \frac{\partial g}{\partial x}(p)\,dx + \frac{\partial g}{\partial y}(p)\,dy + \frac{\partial g}{\partial z}(p)\,dz$$

so that one of the partial derivatives of g is not zero. Without loss of generality let $\frac{\partial g}{\partial z}(p) \neq 0$. Let $p = (p_1, p_2, p_3)$. Then there is a differentiable function $h : D \to \mathbb{R}$, where D is an open set containing (p_1, p_2), satisfying

1) for each $(u, v) \in D$,

$g(u,v,h(u,v))=c$ so that
$(u,v,h(u,v)) \in M$ and

2) Points of the form $(u,v,h(u,v))$, with $(u,v)\in D$ "fills a neighborhood of p in M. In particular, the map $(u,v)\longmapsto (u,v,h(u,v))$ is a Monge patch for M.

Hence, M is a surface in $\mathbb{R}^3$. ∎

Back to the unit sphere example:

Let $g:\mathbb{R}^3\to\mathbb{R}$, $g(x,y,z)=x^2+y^2+z^2$, and $c=1$. Then

$M=g^{-1}(c)=\Sigma=\{(x,y,z)\in\mathbb{R}^3 / x^2+y^2+z^2=1\}$,

is a surface since

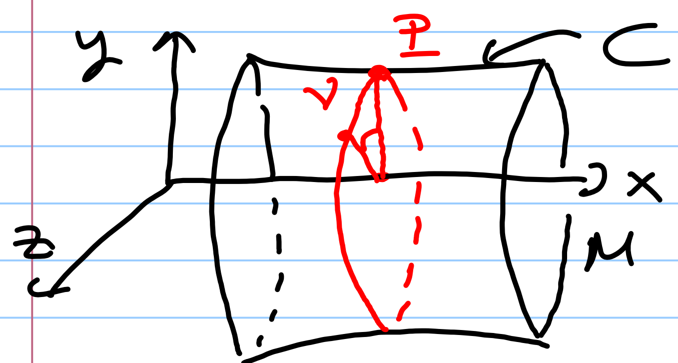
$dg(p)=2p_1\,dx+2p_2\,dy+2p_3\,dz\neq 0$

for all $p\in\Sigma$, since $p_1^2+p_2^2+p_3^2=1$.

Example: Surfaces of revolution.

Let C be a curve in the xy-plane. If $(a_1, a_2, 0) \in C$, then rotation this point about the x-axis we obtain the curve

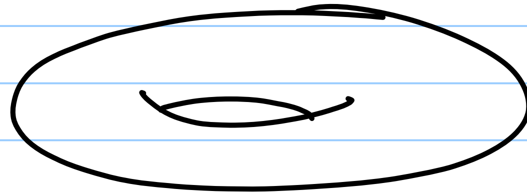
$(a_1, a_2 \cos v, a_2 \sin v)$, $v \in [0, 2\pi]$.



Let M be the surface obtained by revolving C about the x-axis. Let C be given by $y = f(x)$, $f: I \to \mathbb{R}^2$, $I = (a, b)$. Then $x: I \times [0, 2\pi] \to \mathbb{R}^3$ be given by

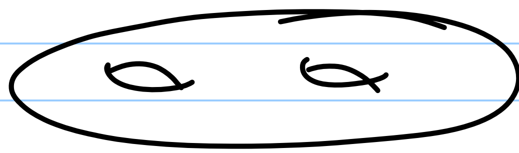
$$x(t, v) = (t, f(t)\cos v, f(t)\sin v)$$

is a proper surface patch.

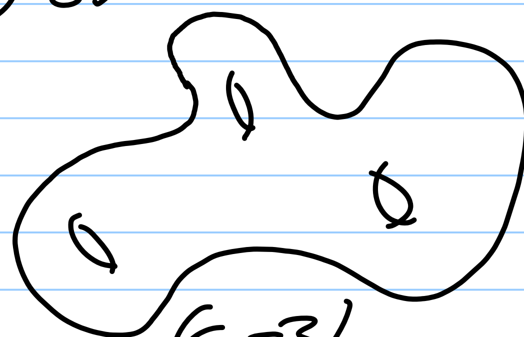


Torus
$(g=1)$

Other examples of surfaces:



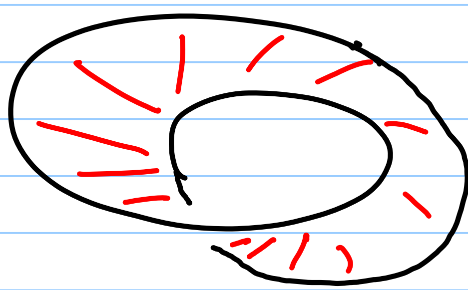
or



$(g=2)$

g: genus of the surface

$(g=3)$



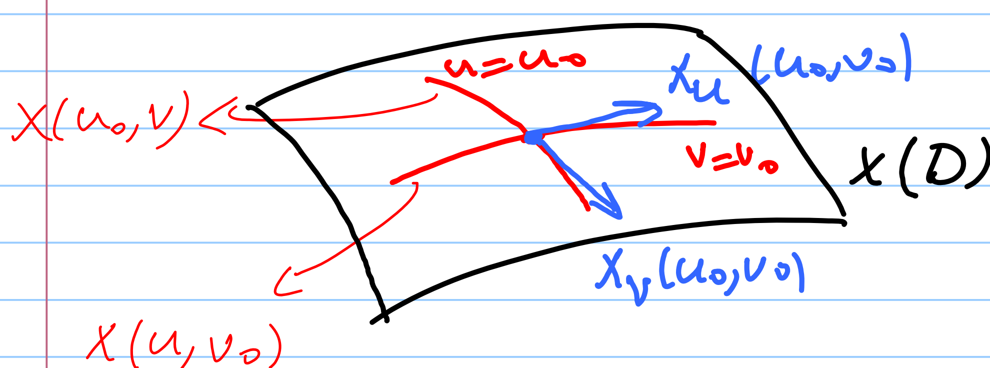
Möbius Band

§ Patch Computations:

Let $x: D \to \mathbb{R}^3$ be a coordinate patch and $(u_0, v_0) \in D$. Then the functions $u \mapsto x(u, v_0)$ and $v \mapsto x(u_0, v)$ define two curves passing through the point $x(u_0, v_0)$.

<u>Definition</u>: The velocity vectors of the above parametric curves are denoted by $\frac{d}{du}\left(x(u, v_0)\right)\Big|_{u=u_0} = x_u(u_0, v_0)$

and $\frac{d}{dv}\left(x(u_0, v)\right)\Big|_{v=v_0} = x_v(u_0, v_0)$.



In coordinates if

$x(u,v) = (x_1(u,v), x_2(u,v), x_3(u,v))$ then

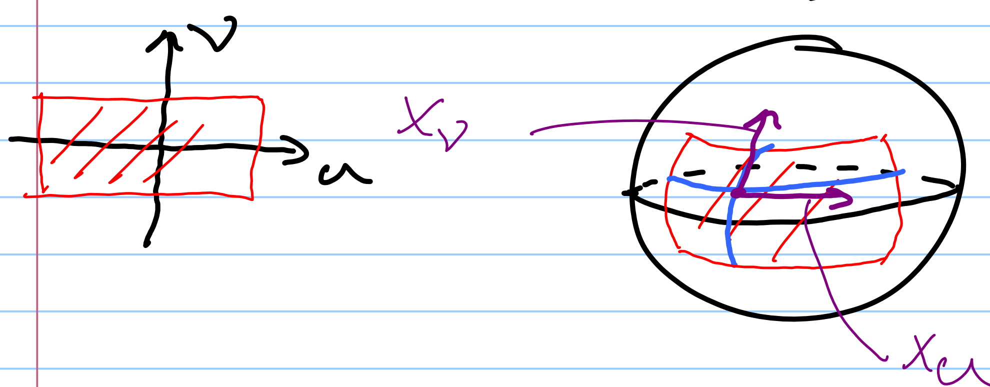
$x_u = \left(\frac{\partial x_1}{\partial u}, \frac{\partial x_2}{\partial u}, \frac{\partial x_3}{\partial u}\right)$ and

$$x_v = \left(\frac{\partial x_1}{\partial v}, \frac{\partial x_2}{\partial v}, \frac{\partial x_3}{\partial v}\right).$$

<u>Example:</u> Let Σ be the sphere centered at origin with radius r. The spherical coordinates defines a coordinate patch

$$x(u,v) = (r\cos v\cos u,\ r\cos v\sin u,\ r\sin v)$$

where $u \in (-\pi, \pi)$, $v \in (-\pi/2, \pi/2)$.



$$x_u = (-r\cos v\sin u,\ r\cos v\cos u,\ 0)$$

$$x_v = (-r\sin v\cos u,\ -r\sin v\sin u,\ r\cos v).$$

<u>Definition:</u> A regular mapping $x: D \to \mathbb{R}^3$ whose image lies in a surface M is called a parametrization of the

region $x(D)$ of M.

(So, a coordinate patch is a one-to-one proper parametrization.

Remark: The spherical parametrization defined above is defined on $\mathbb{R}^2$, $x:\mathbb{R}^2 \to \mathbb{R}^3$, is a parametrization of the sphere. Note that x is <u>not</u> 1-1. Moreover,

$$x_u \times x_v = r^2 \begin{vmatrix} u_1 & u_2 & u_3 \\ -\cos v \sin u & \cos v \cos u & 0 \\ -\sin v \cos u & -\sin v \sin u & \cos v \end{vmatrix}$$

$$= r^2(\cos^2 v \cos u, \cos^2 v \sin u, \cos v \sin v)$$

and thus

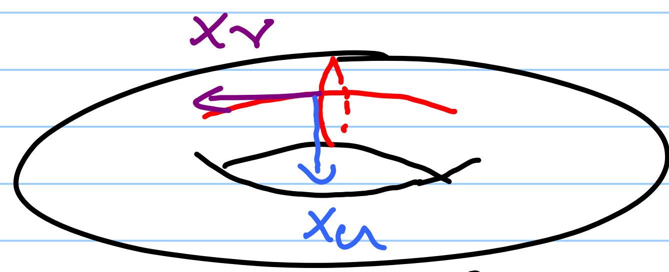
$$\| x_u \times x_v \| = r^2 \cos^2 v \neq 0,$$

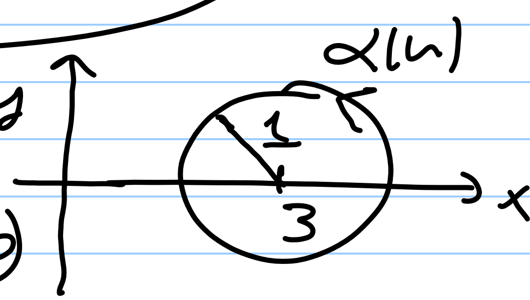
provided that $v \notin (-\pi/2, \pi/2)$.
So, x is regular and hence a parametrization.

Example Surface of a revolution.

Let $\alpha(u) = (g(u), h(u), 0)$ be a curve in the xy-plane. Then as we saw earlier a parametrization is given by $x(u,v) = (g(u), h(u)\cos v, h(u)\sin v)$.

Note that $x_u = (g', h'\cos v, h'\sin v)$ and $x_v = (0, -h\sin v, h\cos v)$.





$\alpha(u) = (3+\cos u, \sin u, 0)$

$g(u) = 3+\cos u$, $h(u) = \sin u$.

Note that

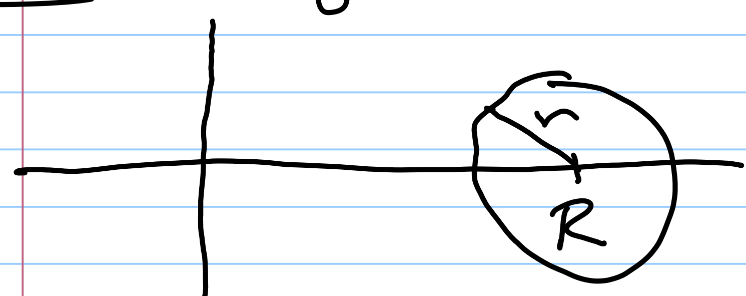
$$x_u \times x_v = \begin{vmatrix} u_1 & u_2 & u_3 \\ g' & h'\cos v & h'\sin v \\ 0 & -h\sin v & h\cos v \end{vmatrix}$$

$$= (hh', -g'h\cos v, -g'h\sin v)$$

and $\|X_u \times X_v\|^2 = h^2(h'^2 + g'^2)$.

So if $\alpha(u) = (h(u), g(u))$ is regular ($\Rightarrow \alpha'(u) = (h', g') \neq (0,0)$) and $\alpha(u)$ does not meet the y-axis then $\|X_u \times X_v\|^2 > 0$ so that $X(u,v)$ is regular. So it is a parametrization.

Ex The general torus



Its parametrization is given by $\alpha(u) = (R + \cos u, \sin u, 0)$ and

$$X(u,v) = ((R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u).$$

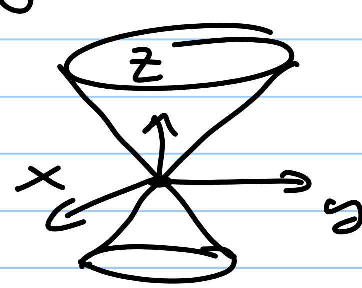
Since $X(u + 2\pi, v) = X(u,v)$ it is

not a surface patch.

<u>Definition</u>: A ruled surface is a surface swept out by a straight line ℓ moving along a curve β. The various positions of the lines generating the surface are called the rulings of the surface. Such a surface always has a ruled parametrization,

$x(u,v) = \beta(u) + v\,\delta(u)$, $\beta(u)$ is called the base curve, δ the director curve.

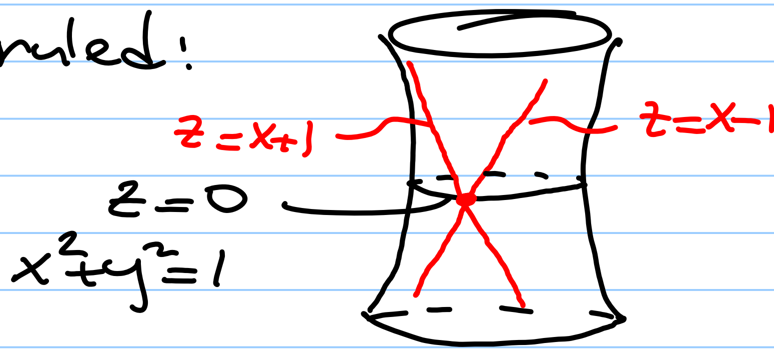
<u>Example</u>: Consider the cone given by $z^2 = x^2 + y^2$ is a ruled surface.



This is a ruled surface.

The surface $z^2 = x^2 + y^2 - 1$

is also ruled:



Note that if $y=0$ then the equation becomes

$z = x^2 - 1 = (x-1)(x+1)$. Hence the surface contains the line

$(z = x-1,\ y=0)$ and

$(z = x+1,\ y=0)$

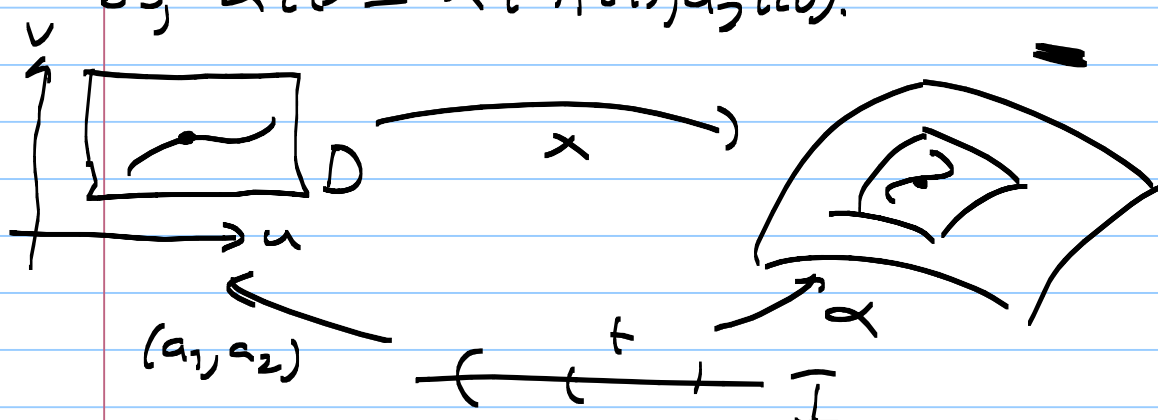
§4.3. Differentiable Functions and Tangent Vectors:

Lemma: Let $\alpha : I \to M$ be a (differentiable) curve so that $\alpha(I)$ lies in the image $x(D)$ of a single patch $x : D \to M$. Then there exist unique differentiable functions a_1, a_2 on I so that $\alpha(t) = x(a_1(t), a_2(t))$, for all $t \in I$.

Proof: Consider the differentiable $x^{-1} \circ \alpha : I \to D \subseteq \mathbb{R}^2$, where

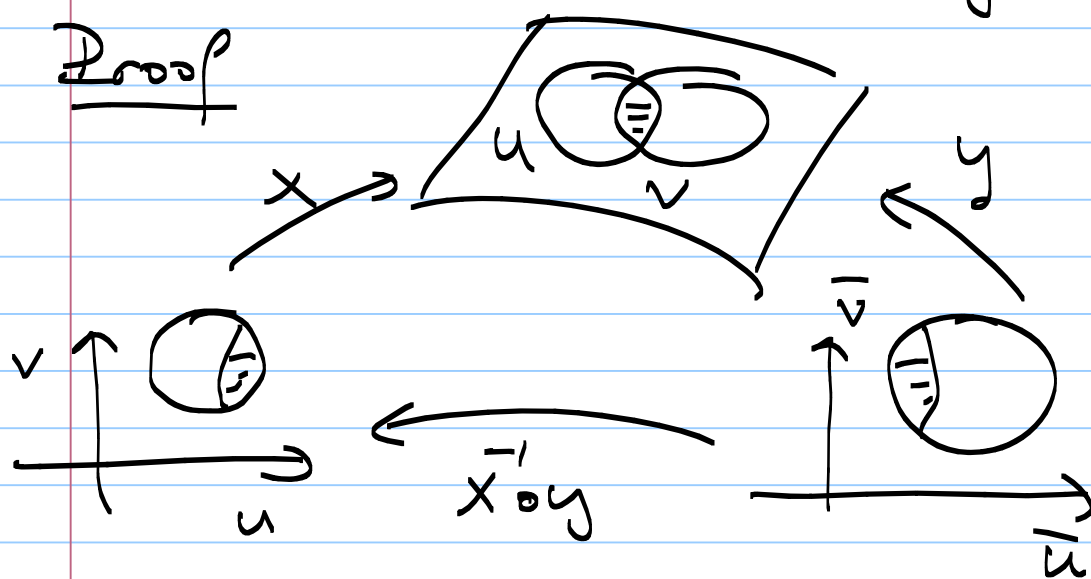
$x^{-1} \circ \alpha(t) = (a_1(t), a_2(t))$ for some functions $a_i : I \to \mathbb{R}$, $i = 1, 2$.

So, $\alpha(t) = x(a_1(t), a_2(t))$. ∎



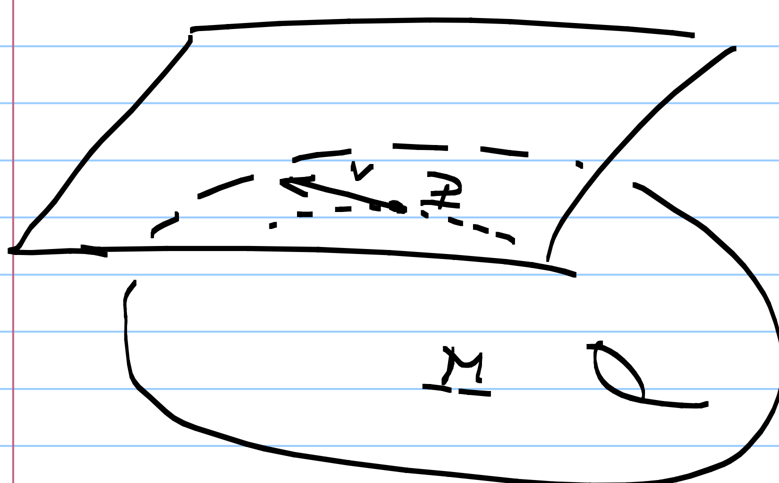
Theorem: If x and y are overlapping patches in M, then there exist unique differentiable functions $\bar{u}$ and $\bar{v}$ such that $y(u,v) = x(\bar{u}(u,v), \bar{v}(u,v))$ for all (u,v) in the domain of $x^{-1}y$.

Proof



$(y^{-1} \circ x)(u,v) = (\bar{u}(u,v), \bar{v}(u,v))$, for some functions $\bar{u}, \bar{v}$ on $x^{-1}(U \cap V)$. Clearly, $\bar{u}, \bar{v}$ are differentiable since x and y are. ∎

Definition: A tangent vector to a surface M at a point $p \in M$ is the velocity vector $\alpha'(t_0)$ of a curve $\alpha: I \longrightarrow M$, where $p = \alpha(t_0)$. The set of all tangent vectors to M at the point p is called the tangent space to M at p, denoted as T_pM.



<u>Lemma</u>: Any tangent vector in T_pM can be written as a linear combination of the vectors $X_u(u_0,v_0)$ and $X_v(u_0,v_0)$, $p = X(u_0,v_0)$.

<u>Proof</u>: Let $\alpha: I \to M$ be a curve with $\alpha(t_0) = p$. Then $\alpha(t) = X(a_1(t), a_2(t))$

for some differentiable functions a_1, a_2 from I to $\mathbb{R}$. So, the tangent vector

$$\alpha'(t) = \frac{d}{dt}\left(x(a_1(t), a_2(t))\right)$$

$$= x_u(a_1(t), a_2(t))\, a_1'(t)$$

$$+ x_v(a_1(t), a_2(t))\, a_2'(t)$$

and thus

$$\alpha'(t_0) = c_1\, x_u(u_0, v_0) + c_2\, x_v(u_0, v_0),$$

where $(u_0, v_0) = (a_1(t_0), a_2(t_0))$

and $c_1 = a_1'(t_0)$, $c_2 = a_2'(t_0)$.

This finishes the proof. ■

<u>Definition</u>: A Euclidean vector field Z on a surface M in $\mathbb{R}^3$ is a function that assigns to each point p of M a tangent $Z(p)$ to $\mathbb{R}^3$ at p.

A normal vector z at a point

p of M is a vector that is perpendicular to T_pM. A normal vector field on M is a Euclidean vector field Z so that $Z(p)$ is normal to T_pM for each $p \in M$.

<u>Lemma</u>: If $M: g=c$ is a surface in $\mathbb{R}^3$, then the gradient $\nabla g = \sum_{i=1}^{3} \frac{\partial g}{\partial x_i} U_i$ is a normal vector field on M.

<u>Proof</u> Let $\alpha: I \longrightarrow M$ be any curve. Then for the tangent vector $\alpha'(t_0) \in T_pM$, $p = \alpha(t_0)$, satisfies

$$0 = \frac{d}{dt}(c)\Big|_{t=t_0} = \frac{d}{dt}\big(g(\alpha(t))\big)\Big|_{t=t_0}$$

$$= \sum_{i=1}^{3} \frac{\partial g}{\partial x_i}(\alpha(t_0))\, \alpha_i'(t_0),$$

$\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$.

Hence, $0 = \nabla g(p) \cdot (\alpha_1'(t_0), \alpha_2'(t_0), \alpha_3'(t_0))$

$$= \nabla g(p) \cdot \alpha'(t_0)$$

$$\Rightarrow \nabla g(p) \perp \alpha'(t_0).$$

Since $\alpha(t)$ is an arbitrary curve we see that $\nabla g(p) \perp T_pM$.

Example Let $\Sigma = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid \sum x_i^2 = r^2\}$ the sphere centered at the origin with radius r. So $\Sigma : g = r^2$, where $g(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$, and thus $T_p\Sigma$ is the plane in $\mathbb{R}^3$ perpendicular to $\nabla g(p)$. So if $p = (p_1, p_2, p_3)$, then

$$T_p\Sigma = \{(v_1, v_2, v_3) \in T_p\mathbb{R}^3 \mid (v_1, v_2, v_3) \cdot \nabla g(p) = 0\}$$

$$= \{(v_1, v_2, v_3) \in T_p\mathbb{R}^3 \mid \sum_{i=1}^{3} v_i p_i = 0\},$$

$$\nabla g(p) = (2p_1, 2p_2, 2p_3).$$

Definition: Let v be a tangent vector to M at p, and let f be be a differentiable real-valued function on M. The derivative $v[f]$ of f with respect to v is the common value of $(d/dt)(f(\alpha))(0)$, for all curves α in M with initial velocity v.

CHAPTER 5: Shape Operators

§5.1. The Shape Operator of $M \subseteq \mathbb{R}^3$

Let Z be a Euclidean vector field on a surface M. Let $p \in M$ and $v \in T_pM$. We define the covariant derivative of Z at p along v as $\nabla_v Z = (Z_\alpha)'(0)$, where $\alpha : I \to M$ is a curve so that $\alpha(0) = p$ and $\alpha'(0) = v$.

__Lemma:__ Let $Z = \sum z_i U_i$. Then $\nabla_v Z = \sum v[z_i] U_i$.

__Proof:__
$$Z_\alpha(t) = Z(\alpha(t))$$
$$= \sum z_i(\alpha(t)) U_i.$$

So,
$$(Z_\alpha)'(0) = \sum_i \frac{d}{dt}(z_i(\alpha(t)))\Big|_{t=0} U_i$$
$$= \sum_i v[z_i] U_i.$$

∎

This lemma also show that the quantity $\nabla_v Z = (Z_\alpha)'(0)$ is independent of the curve α and thus it is well-defined.

Definition: A surface $M \subseteq \mathbb{R}^3$ is called orientable if there is a continuous function $U : M \longrightarrow \mathbb{R}^3$ so that $U(p) \neq 0$ and $U(p) \perp T_pM$ for all $p \in M$.

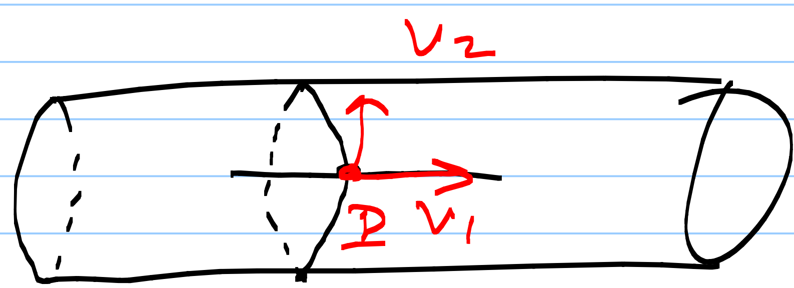
Example: Sphere, torus, $\mathbb{R}^2$ are orientable surfaces. However, the Möbius Band is non-orientable.

Definition: If $p \in M$ is a point, then for each tangent vector

$S_p(v) \doteq -\nabla_v U$, where U is a unit normal vector field defined in a neighborhood p on M.

S_p is called the shape operator of M.

Note that $S_p(v)$ measures how fast the normal vector (hence the tangent plane T_pM) changes as we move from the point p in the direction v. For example if in the cylinder below in the direction v_1 the change is zero and in the direction v_2 it is clearly non-zero.



__Lemma:__ For each $p \in M \subseteq \mathbb{R}^3$, the shape operator is a linear operator $S_p : T_pM \to T_pM$, for all $p \in M$.

Proof: Since U is a unit vector field we have $U(p) \cdot U(p) = 1$ for all $p \in M$. Therefore,

$$0 = v[U \cdot U] = 2(\nabla_v U) \cdot U(p)$$
$$= -2\, S_p(v) \cdot U(p).$$

Since $U(p) \perp T_pM$ we deduce that $S_p(v) \in T_pM$. So S_p is a map $S_p : T_pM \to T_pM$. Linearity follows from the computation

$$S_p(av+bw) = -\nabla_{av+bw} U = -(a\nabla_v U + b\nabla_w U)$$
$$= a\, S_p(v) + b\, S_p(w).$$

∎

Remark: The map $U : M \to S^2$, $p \mapsto U(p)$ is called the Gauss map of M. Note that we can identify T_pM with $T_{U(p)} S^2$ because both are planes in $\mathbb{R}^3$

having $U(p)$ as their normal vectors. Hence the derivative of U at p is a linear map

$$U_*(p) : T_p M \to T_{U(p)} S^2$$

given by

$$U_*(p)(v) = \frac{d}{dt}(U(\alpha(t)))\Big|_{t=0},$$

where α is a curve on M with $\alpha(0) = p$ and $\alpha'(0) = v$.

Hence, by definition

$$U_*(p)(v) = \nabla_v U = -S_p(v).$$

__Examples__ 1) $\Sigma = \{(x_1, x_2, x_3) \mid \Sigma x_i^2 = r^2\}$ a sphere of radius r. Then the (outer) unit normal vector

$$U(p) = \frac{1}{r}\sum x_i U_i, \quad p = (x_1, x_2, x_3).$$

Then $\nabla_v U = \frac{1}{r}\sum v[x_i]\, U_i(p)$

$\Rightarrow \nabla_v U = \frac{1}{r} \sum v_i U_i(p) = \frac{v}{r}$, where

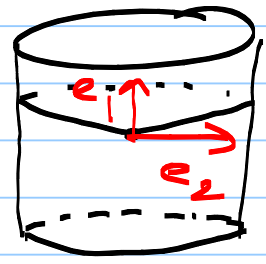
$v = (v_1, v_2, v_3)$. So $S_p(v) = -\nabla_v U = -\frac{v}{r}$.

Hence, the operator $S_p(v)$ on $T_p\Sigma$ is given by scalar multiplication $-1/r$.

2) Let P be plane in $\mathbb{R}^3$. Then the unit normal function U on P is constant and thus $S_p(v) = -\nabla_v U = 0$.

3) Let C be the circular cylinder in $\mathbb{R}^3$ given by $x^2+y^2=1$. Then if $p=(x,y,z)$, $U(p) = U(x,y,z) = \frac{1}{r}(xU_1 + yU_2)$.

Let $e_1 = (0,0,1)$ and $e_2 = (-y,x,0)$.

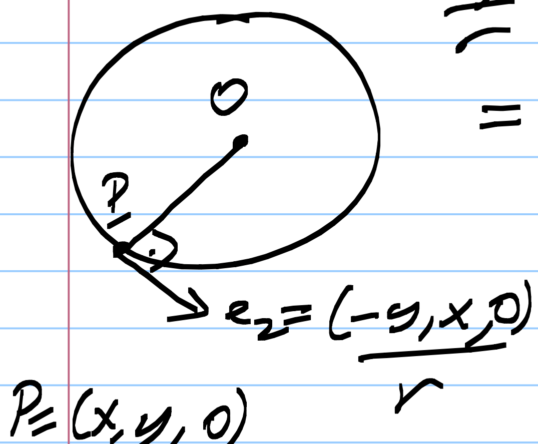


Clearly, $T_pC = \mathrm{span}\{e_1, e_2\}$.

Then $\nabla_{e_1} U(p) = \nabla_{U_3}\left(x\frac{U_1}{r} + y\frac{U_2}{r}\right) = 0$.

On the other hand,

$$\nabla_{e_2} U(p) = \nabla_{(-y,x,0)} (xU_1 + yU_2)\frac{1}{r}$$
$$= -\frac{y}{r}\frac{\partial x}{\partial x_1} U_1 + \frac{x}{r}\frac{\partial y}{\partial y} U_2$$
$$= -\frac{y}{r}U_1 + \frac{x}{r}U_2 = \frac{e_2}{r}$$



Hence, $S_p(e_2) = -\frac{e_2}{r}$.

4) The Saddle surface $M: z = xy$.

$(u,v) \mapsto (u,v,uv)$ is a surface patch and thus $x_u = (1,0,v)$ and $x_v = (0,1,u)$ spans T_pM, $p = (u,v,uv)$.

The normal vector at p is

$$x_u \times x_v = \begin{vmatrix} U_1 & U_2 & U_3 \\ 1 & 0 & v \\ 0 & 1 & u \end{vmatrix}$$
$$= (-v,-u,1)$$ and thus the unit normal vector field is

$$U(p) = \frac{(-v,-u,1)}{\sqrt{1+u^2+v^2}}.$$ Let $p=(0,0,0)$ $(u=0, v=0)$. Then $x_u = (1,0,0) = U_1$

$x_v = (0,1,0) = U_2$

<u>Exercise:</u> Show that

$S_p(aU_1 + bU_2) = bU_1 + aU_2$.

We finish the section with a lemma which will be proved later:

<u>Lemma:</u> For any point $p \in M$ the shape operator

$S: T_pM \to T_pM$ is symmetric,

that is

$$S(v) \cdot w = S(w) \cdot v$$

for all $v, w \in T_pM$.

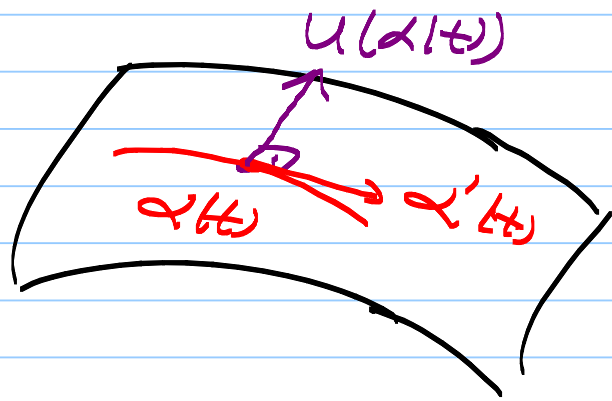
§5.2. Normal Curvature:

Let M be a surface and U is an orientation on M (i.e. a choice of unit normal vector field).

<u>Lemma:</u> If α is a curve in $M \subseteq \mathbb{R}^3$, then $\alpha'' \cdot U = S(\alpha') \cdot \alpha'$.

<u>Proof:</u> Since α is a curve in M its velocity vector is always tangent to M, i.e., $\alpha'(t) \in T_{\alpha(t)}M$ for all t. In particular,

$$\alpha'(t) \cdot U(\alpha(t)) = 0, \text{ for all}.$$



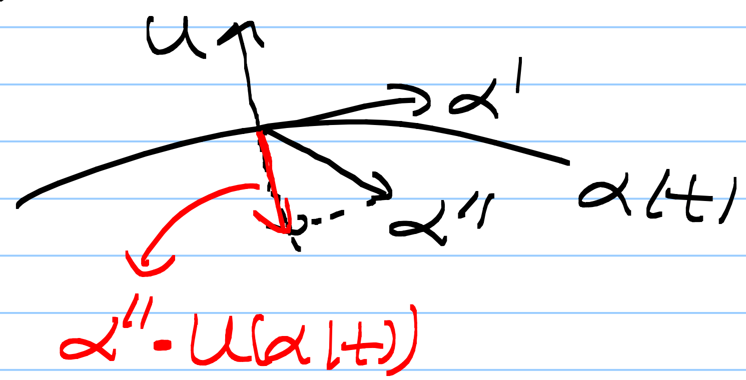
Taking derivative of the above equation with respect to t we obtain

$$\alpha''(t)\cdot U(\alpha(t)) + \alpha'(t)\cdot \left(U(\alpha(t))\right)' = 0.$$

Since $S(\alpha'(t)) = -\left(U(\alpha(t))\right)'$ we obtain

$$S(\alpha'(t))\cdot\alpha'(t) = \alpha''(t)\cdot U(\alpha(t)).$$

$\alpha''(t)\cdot U$ is the normal component of the acceleration vector $\alpha''(t)$.



Hence, this component is determined by the velocity vector $\alpha'(t)$ and the shape operator S.

Definition: Let u be a unit vector tangent to $M\subseteq\mathbb{R}^3$ at a point p.

Then the number $k(u) = S(u) \cdot u$ is called the <u>normal curvature</u> of M in the u direction.

<u>Remark</u>: 1) $k(u) = S(u) \cdot u$

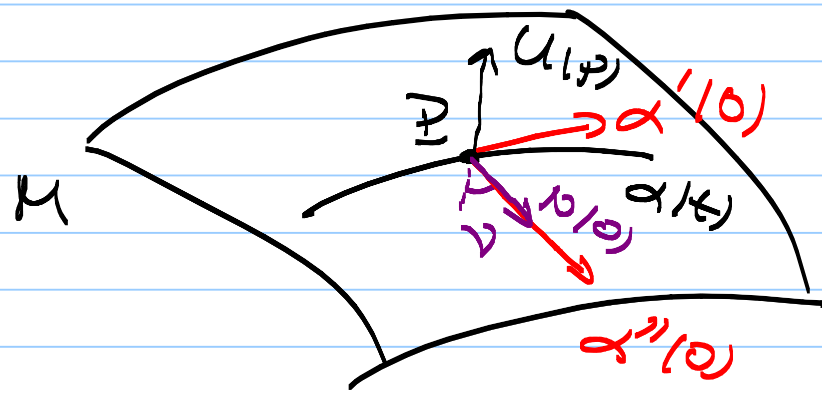
$$= (-S(u)) \cdot (-u)$$
$$= S(-u) \cdot (-u)$$
$$= k(-u),$$

because S is a linear operator.

2) Let $u = \alpha'(0)$ for some curve in M. Then

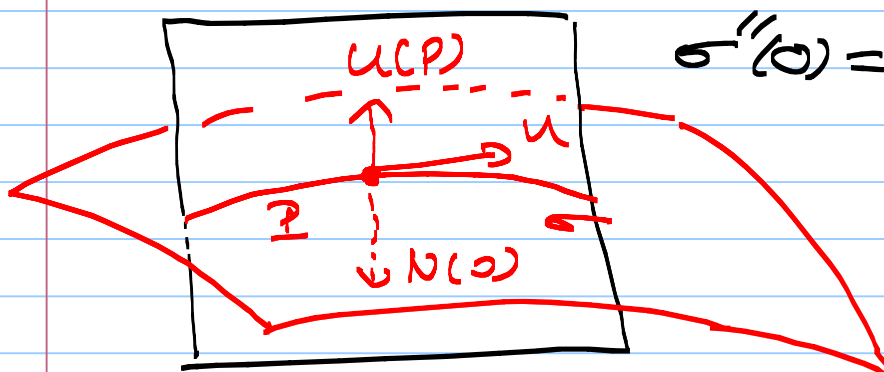
$$k(u) = S(u) \cdot u$$
$$= S(\alpha'(0)) \cdot \alpha'(0)$$
$$= \alpha''(0) \cdot S(\alpha'(0))$$
$$= \kappa(0)\, N(0) \cdot U(p)$$

$= \kappa(0) \cos\vartheta$, where κ is the curvature of the curve $\alpha(t)$ and $p = \alpha(0)$, and ϑ is the angle between the principal normal vector N of α and the normal vector $U(p)$ to M at p.



3) Let $u \in T_pM$ be any unit vector and P be the plane through p containing u and $U(p)$. The intersection of the plane P with the surface M is a curve on M passing through p. Let σ be a unit speed parametrization of this curve. Since σ lies in the plane P

also the principal normal N
to α must be $\pm U(p)$.



$\sigma''(0) = \kappa(0)\, N(0)$

So, $k(u) = \sigma''(0) \cdot U(p)$

$$= \kappa_\sigma(0)\, N(0) \cdot U(p)$$

$$= \pm \kappa_\sigma(0), \text{ because}$$

$N(0) \cdot U(p) = \pm 1$.

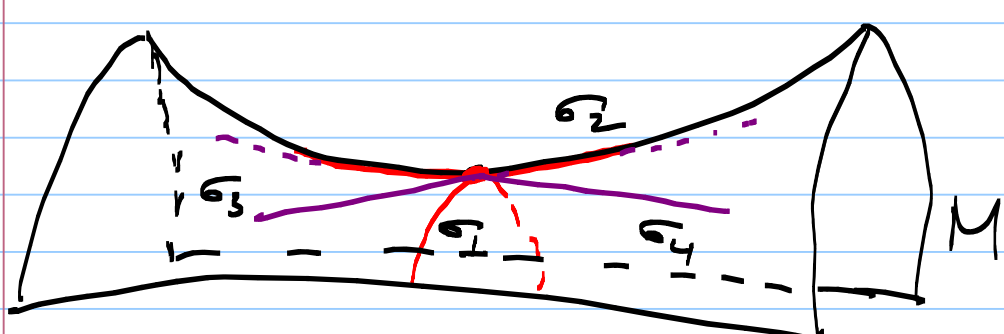
Some observations:

1) If $k(u) > 0$, then $N(0) = U(p)$, so the normal section σ is bending toward $U(p)$ at p. In other words, in the direction u the surface M is bending toward $U(p)$.

2) If $K(p)<0$, then $N(0)=-U(p)$, so the normal section σ is bending away from $U(p)$ at p. Thus in the direction M is bending away from $U(p)$.

3) If $K(p)=0$, then $k_\sigma(0)=0$ and $N(0)$ is undefined. Hence, the normal section σ is <u>not</u> turning at $\sigma(0)=p$.

<u>Example:</u> $z=xy$



1) $\sigma_1 = M \cap P_1$, $P_1: y=-x$

$\sigma_1: z=-x^2,\ y=-x$

So σ_1 is a parabola in the plane $y=-x$. Since the parabola bends downward the normal curvature is negative.

ii) $\sigma_2 = M \cap P_2$, $P_2 : y = x$

$\sigma_2 : z = x^2, \; y = x$

This time σ_2 is a parabola in the plane $y=x$ bending upward and thus the normal curvature is positive.

iii) σ_3 and σ_4 are lines given by $(z=0, x=0)$ and $(z=0, y=0)$. Hence, the corresponding normal curvatures in these directions are zero.

<u>Definition</u>: Let p be a point of $M \subseteq \mathbb{R}^3$. The maximum and

minimum values of the normal curvature $k(u)$ of M at p are called the principal curvatures of M at p, and are denoted k_1 and k_2. The directions in which these extreme values occur are called principal vectors of M at p.

<u>Definition</u>: A point p of M is umbilic provided that the normal curvature $k(u)$ is constant on all unit tangent vectors u at p.

<u>Example</u>: If Σ is a sphere of radius r we have seen that all the normal curvature

are $-1/r$.

<u>Theorem</u>: 1) If p is a umbilic of $M \subseteq \mathbb{R}^3$, then the shape operator S at p is just scalar multiplication by $k = k_1 = k_2$.

2) If p is a nonumbilic point, $k_1 \neq k_2$, then there are exactly two principal directions, and these are orthogonal. Furthermore if e_1 and e_2 are principal vectors in these directions, then $S(e_1) = k_1 e_1$ and $S(e_2) = k_2 e_2$.

<u>Remark</u>: We'll see later that principal curvatures and principal direction are just the eigenvalues and eigenvectors of the linear operator S.

Proof: By the last lemma of the previous section the shape operator S is symmetric.

$$S_p(v)\cdot w = S_p(w)\cdot v,$$

for all $v, w \in T_pM$.

However, any symmetric linear operator is diagonalizable with real eigenvalues. Moreover the eigenvectors are orthogonal. Let $\lambda_1, \lambda_2 \in \mathbb{R}$ be the eigenvalues and e_1, e_2 be associated eigenvectors. Hence,

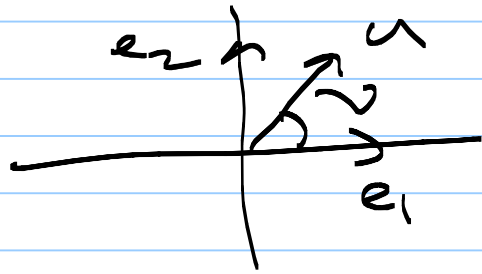
$S(e_1) = \lambda_1 e_1$ and $S(e_2) = \lambda_2 e_2$.

We may assume that $\{e_1, e_2\}$ is an orthonormal basis.

Then for any unit vector $u \in T_pM$ we have

$u = c e_1 + s e_2$, where

$c = \cos\vartheta$, $s = \sin\vartheta$



Then $k(u) = S(u)\cdot u$

$$= S(ce_1 + se_2)\cdot(ce_1 + se_2)$$

$$= (c\lambda_1 e_1 + s\lambda_2 e_2)\cdot(ce_1 + se_2)$$

$$= \lambda_1 c^2 + \lambda_2 s^2.$$

Assume $\lambda_1 \geq \lambda_2$.

If $\lambda_1 = \lambda_2$ then the linear operator S is just multiplication by the common eigenvalue $\lambda_1 = \lambda_2$.

In particular, $k_1 = k_2 = \lambda_1 = \lambda_2$, and $k(u) = k_1 c^2 + k_2 s^2$

$$= k_1(c^2 + s^2)$$

$$= k_1.$$

If $\lambda_1 > \lambda_2$ then
$k(u) = \lambda_1 c^2 + \lambda_2 s^2$ takes its maximum value when $c = \pm 1$ and $s = 0$ and takes its minimum value when $c = 0$ and $s = \pm 1$.
In particular, $k_1 = \lambda_1$, $k_2 = \lambda_2$ are the principal curvatures and the principal directions e_1 and e_2 are orthogonal. ■

Corollary Let k_1, k_2 and e_1, e_2 be the principal curvatures and vectors of $M \subseteq \mathbb{R}^3$ at p. Then if $u = \cos\nu \, e_1 + \sin\nu \, e_2$, the normal curvature of M in the u direction is
$k(u) = k_1 \cos^2\nu + k_2 \sin^2\nu$.

§5.3. Gaussian Curvature:

Definition: Let $p \in M \subseteq \mathbb{R}^3$ and $S_p : T_pM \to T_pM$ be the shape operator at $p \in M$. The determinant of S_p is called the Gaussian curvature of the surface at p, and the trace of S_p is called the scalar curvature of M at p.

Notation: $K(p) = \det(S_p)$,
$H(p) = \operatorname{tr}(S_p)$.

Lemma: If k_1 and k_2 are the principal curvatures at $p \in M$ then $K(p) = k_1 k_2$ and
$H(p) = (k_1 + k_2)/2$.

Proof If e_1 and e_2 are the principal directions at

$p \in M$ then we know that $\beta = \{e_1, e_2\}$ is an orthonormal basis for T_pM. We know that the matrix representation of S_p in the basis β is

$$[S_p]_\beta = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}.$$

So, $K = \det S_p = \det\left(\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}\right)$
$= k_1 k_2$ and

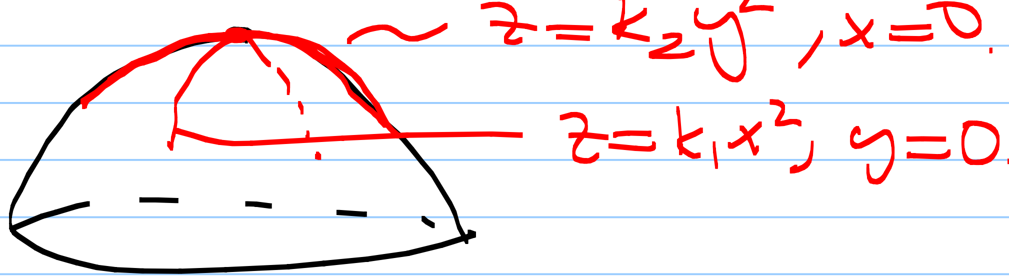
$$H = \frac{\operatorname{tr} S_p}{2} = \operatorname{tr}\left(\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}\right)$$
$$= \frac{k_1 + k_2}{2}.$$

<u>Remark</u>: Recall that replacing the unit normal field U to M by $-U$ replaces S_p with $-S_p$. Clearly, $\det(-S_p) = \det(S_p)$ and $\operatorname{tr}(-S_p) = -\operatorname{tr}(S_p)$.

Hence, the Gaussian curvature is unaffected but the scalar curvature reverses its sign, if we replace U by $-U$.

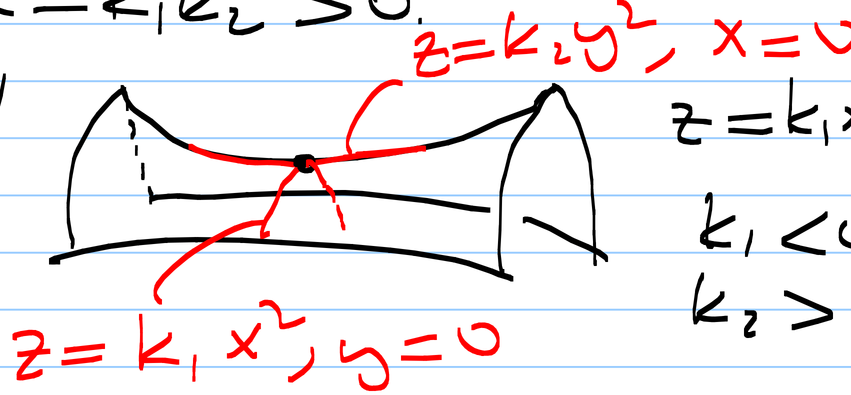
<u>Examples)</u> $\quad M: z = k_1 x^2 + k_2 y^2$.

1) $k_1 < 0, \ k_2 < 0$



$K = k_1 k_2 > 0$.

2)



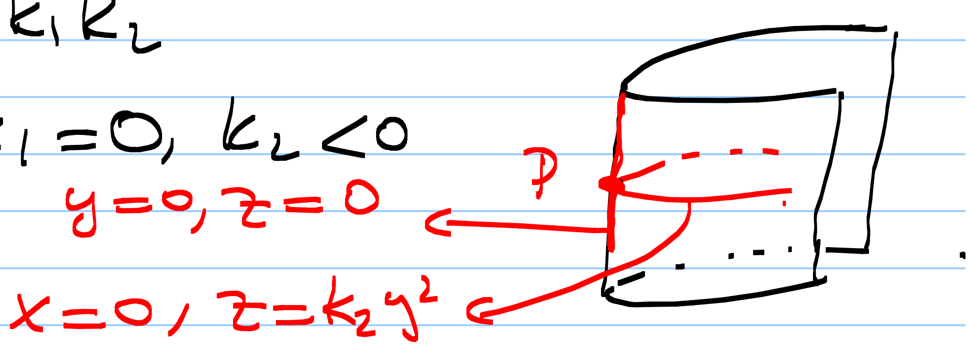
$z = k_1 x^2 + k_2 y^2$

$k_1 < 0$

$k_2 > 0$

$K = k_1 k_2$

3) $k_1 = 0, \ k_2 < 0$



Lemma: If v and w are linearly independent vectors in T_pM, $M \subseteq \mathbb{R}^3$, then

$$S(v) \times S(w) = K(p)\, v \times w$$

and

$$S(v) \times w + v \times S(w) = 2H(p)\, v \times w.$$

Proof: By assumption $\mathcal{B} = \{v, w\}$ is a basis for T_pM. Let $A = [S_p]_{\mathcal{B}} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$. So

$$S_p(v) = av + bw, \quad S_p(w) = cv + dw.$$

Now,

$$S_p(v) \times S_p(w) = (av + bw) \times (cv + dw)$$
$$= (ad - bc)\, v \times w, \text{ since}$$

$$v \times v = 0 = w \times w \quad \text{and}$$

$$v \times w = -w \times v.$$

Similarly, by direct calculation

$S(v) \times w + v \times S(w)$

$= (av + bw) \times w + v \times (cv + dw)$

$= a(v \times w) + d(v \times w)$

$= (a+d)\, v \times w$

$= 2H(p)\, v \times w$

$\blacksquare$

Since $k_1 k_2 = K$ and $k_1 + k_2 = 2H$ we deduce that

$$k_1, k_2 = H \pm \sqrt{H^2 - K}.$$

Definition: A surface M in $\mathbb{R}^3$ is flat if $K(p) = 0$, for all $p \in M$ and is minimal if $H(p) = 0$, for all $p \in M$.

§5.4. Computational Techniques:

$M \subseteq \mathbb{R}^3$ surface

$x: D \to M$ a surface patch.

We've have defined before the following quantities

$E = x_u \cdot x_u, \quad F = x_u \cdot x_v, \quad G = x_v \cdot x_v$

Remark: 1) $F = x_u \cdot x_v = \|x_u\| \|x_v\| \cos\vartheta$

$\Rightarrow F = \sqrt{EG}\cos\vartheta$, where ϑ is the angle between x_u and x_v.

2) Also note that

$$\|x_u \times x_v\|^2 = \|x_u\|^2 \|x_v\|^2 - (x_u \cdot x_v)^2 = EG - F^2.$$

3) If $v = v_1 x_u + v_2 x_v$ and

$w = w_1 x_u + w_2 x_v$, then

$$v \cdot w = (v_1 x_u + v_2 x_v) \cdot (w_1 x_u + w_2 x_v) = v_1 w_1 E + v_2 w_2 G + (v_1 w_2 + v_2 w_1) F.$$

The normal vector on M determined by the surface patch is

$$U = x_u \times x_v / \|x_u \times x_v\|.$$

Let $x(u,v) = (x_1(u,v), x_2(u,v), x_3(u,v))$ be the coordinate patch. Then similar to x_u and x_v we define

$$x_{uu} = \left(\frac{\partial^2 x_1}{\partial u^2}, \frac{\partial^2 x_2}{\partial u^2}, \frac{\partial^2 x_3}{\partial u^2}\right),$$

$$x_{uv} = \left(\frac{\partial^2 x_1}{\partial u \partial v}, \frac{\partial^2 x_2}{\partial u \partial v}, \frac{\partial^2 x_3}{\partial u \partial v}\right) \text{ and}$$

$$x_{vv} = \left(\frac{\partial^2 x_1}{\partial v^2}, \frac{\partial^2 x_2}{\partial v^2}, \frac{\partial^2 x_3}{\partial v^2}\right).$$

Also we define,

$$L = S(x_u) \cdot x_u$$

$$M = S(x_u) \cdot x_v = S(x_v) \cdot x_u$$

$$N = S(x_v) \cdot x_v$$

__Corollary__ If x is a surface patch in $M \subseteq \mathbb{R}^3$, then

$$K(x) = \frac{LN - M^2}{EG - F^2}, \quad H(x) = \frac{GL + EN - 2FM}{2(EG - F^2)}$$

__Proof__: In the last lecture we've seen that for any two tangent vectors v, w in T_pM we have:

(1) $S(v) \times S(w) = K(p)\, v \times w$ and

(2) $S(v) \times w + v \times S(w) = 2H(p)\, v \times w$.

Now consider the so called "Lagrange Identity": for any vectors x, y, v, w in $\mathbb{R}^3$ we have

$$(x \times y) \cdot (v \times w) = \begin{vmatrix} x \cdot v & x \cdot w \\ y \cdot v & y \cdot w \end{vmatrix}.$$

(Exercise 6 of §6.3)

So if we take dot product of the equations (1) and (2) with the normal vector $v\times w$ we get

$$\begin{vmatrix} S(v)\cdot v & S(v)\cdot w \\ S(w)\cdot v & S(w)\cdot w \end{vmatrix} = K(p)\begin{vmatrix} v\cdot v & v\cdot w \\ w\cdot v & w\cdot w \end{vmatrix}$$

$$\Rightarrow K(p) = \frac{\begin{vmatrix} S(v)\cdot v & S(w)\cdot w \\ S(w)\cdot v & S(w)\cdot w \end{vmatrix}}{\begin{vmatrix} v\cdot v & v\cdot w \\ w\cdot v & w\cdot w \end{vmatrix}} = \frac{LN-M^2}{EG-F^2}$$

and similarly,

$$H(p) = \frac{GL+EN-2FM}{2(EG-F^2)}.$$

This finishes the proof. ∎

<u>Remark:</u> Since $U \perp x_u$ we have $U\cdot x_u = 0$. So

$$0 = \frac{\partial}{\partial x_v}(U\cdot x_u) = U_v\cdot x_u + U\cdot x_{uv}.$$

Since $S_p(v) \doteq -\nabla_v U = -U_v$

we get $S(x_v) = -\nabla_{x_v} U = -U_v$

So, $S(x_v)\cdot x_u = -U_v \cdot x_u = -(-U\cdot x_{uv})$

$\Rightarrow S(x_v)\cdot x_u = U\cdot x_{uv}$.

Similarly, $S(x_u)\cdot x_v = U\cdot x_{vu}$

$$= U\cdot x_{uv}$$

$$= S(x_v)\cdot x_u$$

implying that S is symmetric.

<u>Lemma:</u> If x is a patch in $M \subseteq \mathbb{R}^3$, then

$L = S(x_u)\cdot x_u = U\cdot x_{uu}$

$M = S(x_u)\cdot x_v = U\cdot x_{uv}$

$N = S(x_v)\cdot x_v = U\cdot x_{vv}$

<u>Proof</u> The first and the third formulas follow from the first lemma of Section 5.2, that states "$\alpha''\cdot U = S(\alpha')\cdot \alpha'$".

The middle one is done just above. ∎

Example: 1) Helicoid

$x(u,v) = (u\cos v,\ u\sin v,\ bv)$, $b \neq 0$

$x_u = (\cos v,\ \sin v,\ 0)$

$x_v = (-u\sin v,\ u\cos v,\ b)$

$E = x_u \cdot x_u = 1$, $F = x_u \cdot x_v = 0$ and

$G = x_v \cdot x_v = b^2 + u^2$.

$$x_u \times x_v = \begin{vmatrix} \hat{i} & \hat{j} & k \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & b \end{vmatrix}$$

$$= (b\sin v, -b\cos v, u).$$

$$W = \|x_u \times x_v\| = \sqrt{EG - F^2} = \sqrt{b^2+u^2}$$

$$U = \frac{x_u \times x_v}{\|x_u \times x_v\|} = \frac{(b\sin v, -b\cos v, u)}{\sqrt{b^2+u^2}}$$

$x_{uu} = 0$, $x_{uv} = (-\sin v, \cos v, 0)$

$x_{vv} = (-u\cos v, -u\sin v, 0)$.

Hence,

$$L = x_{uu} \cdot U = x_{uu} \cdot \left(\frac{x_u \times x_v}{W}\right) = 0$$

$$M = x_{uv} \cdot U = x_{uv} \cdot \left(\frac{x_u \times x_v}{W}\right) = \frac{-b}{W}$$

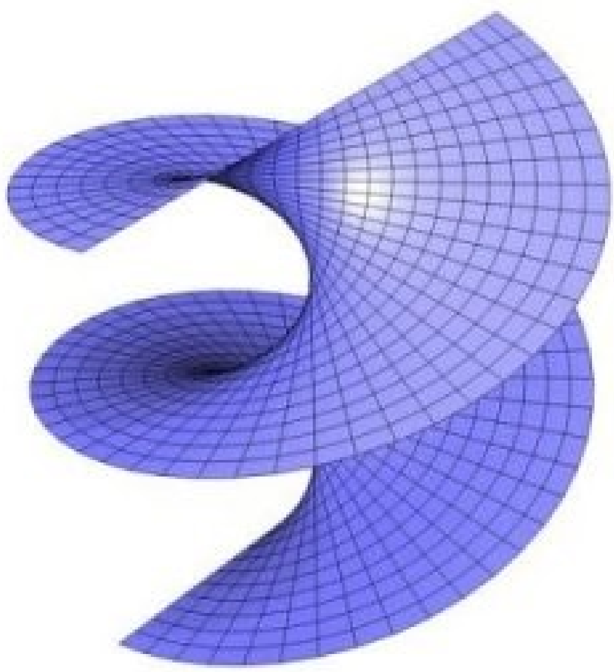
$$N = x_{vv} \cdot \left(\frac{x_u \times x_v}{W} \right) = 0.$$

Finally,

$$K = \frac{LN - M^2}{EG - F^2} = \frac{-(b/W)^2}{W^2} = \frac{-b^2}{W^4}$$

$$= \frac{-b^2}{(b^2+u^2)^2}, \quad \text{and}$$

$$H = \frac{GL + EN - 2FM}{2(EG - F^2)} = 0.$$



2) The Saddle Surface:

$M: z = xy$

$x(u,v) = (u, v, uv).$

$x_u = (1, 0, v)$, $x_v = (0, 1, u)$

$E = x_u \cdot x_u = 1+v^2$, $F = x_u \cdot x_v = uv$

$G = x_v \cdot x_v = 1+u^2.$

$$U = \frac{x_u \times x_v}{\|x_u \times x_v\|} = \frac{(-v, -u, 1)}{\sqrt{1+u^2+v^2}}$$

$x_{uu} = 0$, $x_{uv} = (0,0,1)$, $x_{vv} = 0.$

$$L = x_{uu} \cdot \frac{x_u \times x_v}{\|x_u \times x_v\|} = 0$$

$$M = x_{uv} \cdot \frac{x_u \times x_v}{\|x_u \times x_v\|} = \frac{1}{\sqrt{1+u^2+v^2}} = \frac{1}{W}$$

$$N = x_{vv} \cdot \frac{x_u \times x_v}{\|x_u \times x_v\|} = 0.$$

$$K = \frac{LN - M^2}{EG - F^2} = \frac{-1}{(1+u^2+v^2)^2} \quad \text{and}$$

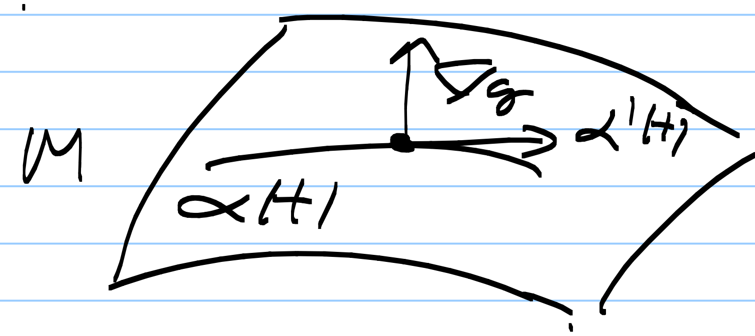
$$H = \frac{GL + EN - 2FM}{2(EG - F^2)} = \frac{-uv}{(1+u^2+v^2)^{3/2}}.$$

§5.5. The Implicit Case:

Let $M \subseteq \mathbb{R}^3$ be a surface described by a single equation $g=0$ for some smooth function $g: \mathbb{R}^3 \to \mathbb{R}$, for which $0 \in \mathbb{R}$ is a regular value. In other words, the derivative map $g_*: T_p\mathbb{R}^3 \to T_0\mathbb{R} \simeq \mathbb{R}$ is onto for any $p \in \mathbb{R}$. Hence, the gradient $\nabla g = \sum \frac{\partial g}{\partial x_i}$ is never zero on M. Clearly, if $\alpha: I \to M$ is a curve on M then $\alpha'(t) \perp \nabla g(\alpha(t))$ for all $t \in I$, because $g(\alpha(t)) = 0$, for all $t \in I$ and thus (taking derivative)

$$\nabla g(\alpha(t)) \cdot \alpha'(t) = 0.$$

In other words, ∇g is a non vanishing normal field on M.



Let $Z \doteq \nabla g$ and $U = \frac{Z}{\|Z\|}$ be the unit normal field determined by Z.

Let S be the shape operator on M corresponding to U:

$$S_q(V) = -\nabla_V U, \text{ for } V \in T_pM.$$

Writing $Z = \sum_{i=1}^{3} z_i U_i$, we know that

$$\nabla_V Z = \sum_{i=1}^{3} V[z_i] U_i, \text{ and}$$

$$\nabla_V U = \nabla_V \left(\frac{Z}{\|Z\|}\right) = \frac{\nabla_V Z}{\|Z\|} + V\left(\frac{1}{\|Z\|}\right) Z.$$

The vector $V\left(\frac{1}{\|Z\|}\right)Z$ is normal to the surface M and let's denote it by $-N_V$. Thus

$$S(V) = -\nabla_V U = -\frac{\nabla_V Z}{\|Z\|} + N_V.$$

<u>Remark</u>: 1) If $W \in T_pM$ then N_W is normal to M and thus parallel to N_V. Hence,

$$N_V \times N_W = 0.$$

2) Also if Y is a (tangent) vector field on M then $Y \times N_V$ is clearly perpendicular to N_V and thus tangent to M.

Now we state the following

<u>Lemma</u>: Let Z be a nonvanishing normal field on M. If V and W are tangent vector fields

such that $V \times W = Z$, then

$$K = \frac{Z \cdot (\nabla_V Z \times \nabla_W Z)}{\|Z\|^4} \quad \text{and}$$

$$H = -Z \cdot \frac{\nabla_V Z \times W + V \times \nabla_W Z}{2\|Z\|^3}.$$

<u>Proof</u>: We know from (Lemma 3.4) that

(1) $S(V) \times S(W) = K(p)\, V \times W$ and

(2) $S(V) \times W + V \times S(W) = 2H(p)\, V \times W$.

Now by the above computations

$$S(V) = -\frac{\nabla_V Z}{\|Z\|} + N_V \quad \text{and}$$

$$S(W) = -\frac{\nabla_W Z}{\|Z\|} + N_W.$$

By assumption $V \times W = Z$ and taking dot product with Z we get

$$K(p) = \frac{Z \cdot (S(V) \times S(W))}{Z \cdot Z}$$
$$= \frac{Z \cdot (\nabla_V Z \times \nabla_W Z)}{\|Z\|^2 \|Z\|^2},$$

because

$$S(V) \times S(W) = \frac{\nabla_V Z \times \nabla_W Z}{\|Z\|^2} - \frac{\nabla_V Z \times N_W}{\|Z\|} - \frac{N_V \times \nabla_W Z}{\|Z\|} + \underbrace{N_V \times N_W}_{=0}$$

so that the second and the third terms are perpendicular to Z since N_W and N_V are parallel to Z.

Hence, $K(p) = \dfrac{Z \cdot (\nabla_V Z \times \nabla_W Z)}{\|Z\|^4}$.

The second identity follows from (2) in a similar fashion. ∎

Example: Let $M: g = \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} = 1$ be an ellipsoid.

Now $Z = \frac{1}{2}\nabla g = \sum_{i=1}^{3} \frac{x_i}{a_i^2} U_i$.

Let $V = \sum v_i U_i$ be a tangent field on M. Then

$$\nabla_V Z = \sum_{i=1}^{3} \frac{V[x_i]}{a_i^2} U_i = \sum_{i=1}^{3} \frac{v_i}{a_i^2} U_i,$$

because $V[x_i] = dx_i(V) = v_i$.

Now,

$$Z \cdot (\nabla_V Z \times \nabla_W Z) = \begin{vmatrix} \frac{x_1}{a_1^2} & \frac{x_2}{a_2^2} & \frac{x_3}{a_3^2} \\ \frac{v_1}{a_1^2} & \frac{v_2}{a_2^2} & \frac{v_3}{a_3^2} \\ \frac{w_1}{a_1^2} & \frac{w_2}{a_2^2} & \frac{w_3}{a_3^2} \end{vmatrix}$$

$$= \frac{1}{a_1^2 a_2^2 a_3^2} X \cdot (V \times W),$$

where $X = \sum x_i U_i$. If V and W are chosen so that $V \times W = Z$, then

$$X \cdot (V \times W) = X \cdot Z = \sum_{i=1}^{3} \frac{x_i^2}{a_i^2} = 1.$$

Now by the previous lemma

$$K = \frac{1}{a_1^2 a_2^2 a_3^2 \|Z\|^4}, \quad \|Z\|^4 = \left(\sum_{i=1}^{3} \frac{x_i^2}{a_i^4} \right)^2.$$

Finally, note that if $a_1 = a_2 = a_3 = r$, i.e., the ellipsoid is a sphere then

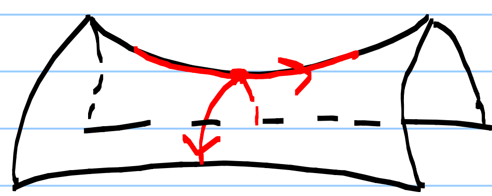
$$\|Z\|^4 = \Bigg(\frac{1}{r^2} \underbrace{\sum_{i=1}^{3} \frac{x_i^2}{r^2}}_{= 1} \Bigg)^2 = \frac{1}{r^4}$$

and thus $K = \dfrac{1}{r^6 \left(1/r^4 \right)} = \dfrac{1}{r^2}$.

§5.6. Special Curves in a Surface

Definition: A regular curve α in $M \subseteq \mathbb{R}^3$ is a principal curve provided that the velocity α' of α always points in a principal direction.

Remark: Principal curves moves in directions in which the surface bends in extreme values. Through a non umbolic points, neglecting parametrizations, there are exactly two principal curves, which are orthogonal.



On the other hand, at an umbolic point every direction is principal.

Lemma: Let α be a regular curve in $M \subseteq \mathbb{R}^3$, and let U be a unit normal field along α. Then

1) The curve α is principal if and only if U' and α' are collinear at each point.

2) If α is a principal curve, then the principal curvature along α' is $\alpha'' \cdot U / (\alpha' \cdot \alpha')$.

<u>Proof</u>: <u>Claim</u>: $S(\alpha') = -U'$.

<u>Proof</u>: We know that $S_p(v) = -D_p U(v)$ and thus
$$S_p(\alpha') = -\frac{d}{dt}(U(\alpha(t)))$$
$$= -U'(t).$$

Hence, U' and α' are collinear if and only if $S(\alpha')$ and α' are collinear. But this means that $S(\alpha') = \lambda \alpha'$ and thus α' points in the principal direction, that is α is a principal curve.

2) If α is a principal curve, $\alpha'/\|\alpha'\|$ is always a principal direction. Hence, if k_i is the principal curvature corresponding to α, then $S(\alpha'/\|\alpha'\|) = k_i \frac{\alpha'}{\|\alpha'\|}$. Thus

$$k_i = \frac{\alpha'}{\|\alpha'\|} \cdot \frac{\alpha'}{\|\alpha'\|} = S(\alpha'/\|\alpha'\|) \cdot \frac{\alpha'}{\|\alpha'\|}$$

$$= \frac{S(\alpha') \cdot \alpha'}{\|\alpha'\|^2}$$

$$= \frac{\alpha'' \cdot U}{\alpha' \cdot \alpha'}.$$

$\blacksquare$

<u>Lemma</u>: Let α be a curve cut from a surface $M \subseteq \mathbb{R}^3$ by a plane. If the angle between M and P is constant along α, then α is a principal curve in M.

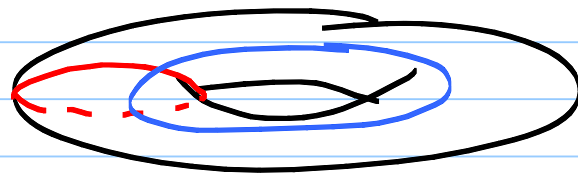
Proof Let U and V be unit normals to M and P, respectively. Then $V' = 0$ since P is plane and hence, V is a constant function. By assumption $U \cdot V = \text{constant}$ and thus $0 = (U \cdot V)'$.

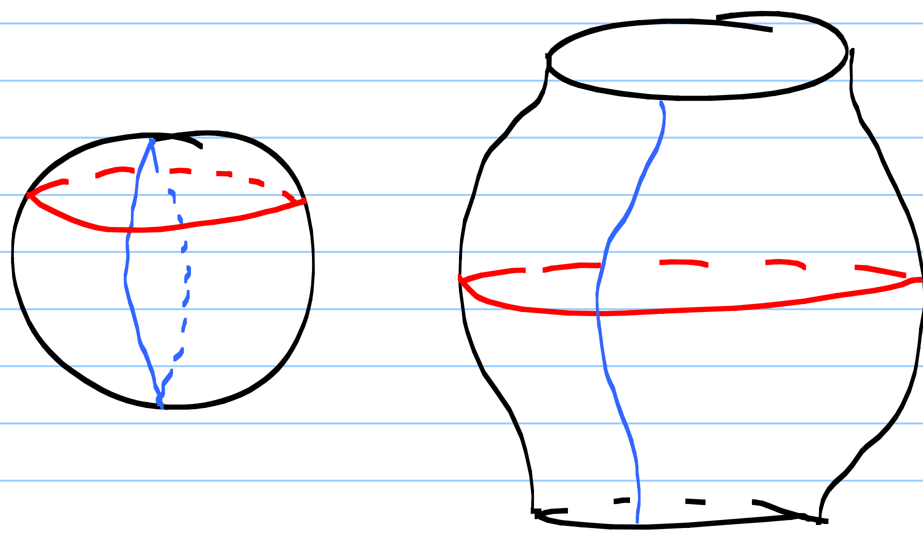
$\Rightarrow U' \cdot V + U \cdot V' = 0 \Rightarrow U' \cdot V = 0$

Since U is a unit field U' is orthogonal to U. Hence, U' is orthogonal to both P and U. α' is also orthogonal to P and U and thus α' and U' are in the same direction. Hence, by the principal lemma, α is principal. Of course, this argument is valid if we assume U and V are linearly independent. If U

and V are linearly dependent then $U = \pm V$. However, since $V' = 0$ we get $U' = 0$ so that α is clearly principal in this case.

<u>Example</u> Let M be a surface of revolution. Then meridians and parallels are principal curves.





To get a meridian take a plane P containing the axis of rotation and intersect with M. To get a parallel take a plane P whose normal is the axis of rotation.

Directions tangent to $M \subseteq \mathbb{R}^3$ in which the normal curvature is zero are called asymptotic directions. So a tangent vector v is asymptotic if $k(v) = S(v) \cdot v = 0$.

<u>Lemma</u>: Let $p \in M \subseteq \mathbb{R}^3$ a point.

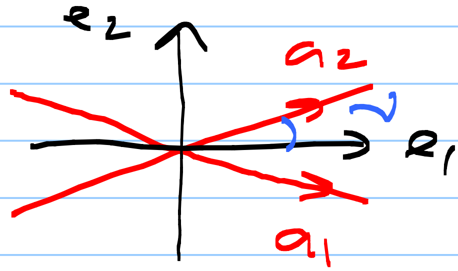
1) If $K(p) > 0$, then there are no asymptotic directions at p.

2) If $K(p) < 0$, then there are exactly two asymptotic directions at p and they are bisected by the principal direction at angle ϑ such that

$$\tan^2 \vartheta = \frac{-k_1(p)}{k_2(p)}.$$

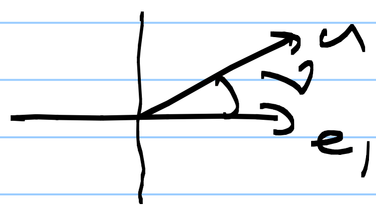
3) If $K(p) = 0$, then every direction is asymptotic if p is a planar point; otherwise there is exactly

one asymptotic direction and it is also principal.



<u>Proof</u>: Recall Euler's formula

$$k(u) = k_1(p)\cos^2 v + k_2(p)\sin^2 v$$



1) $K(p) > 0$, then $k_1(p)\,k_2(p) > 0$ and thus $k(u)$ is never zero.

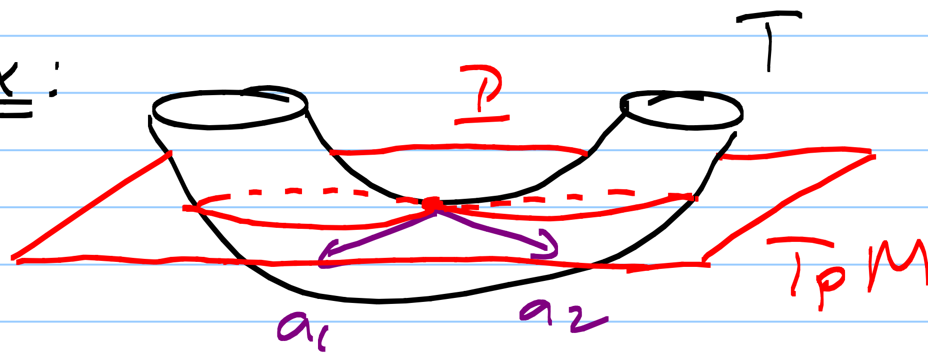
2) If $K(p) < 0$ then $k_1(p)\,k_2(p) < 0$. So $k(u) = 0$ when $\tan^2 v = -k_1/k_2$ (>0).

3) If $K(p) = 0$, then either both $k_1 = k_2 = 0$ ($\Leftrightarrow$ p is planar) so that all directions are asymptotic.

If only $k_2 = 0$ then $k(u) = k_1(p)\cos^2 v$

and thus the asymptotic direction is $u = e_2$. ∎

Ex:



Definition: A regular curve α in $M \subseteq \mathbb{R}^3$ is an asymptotic curve provided its velocity α' always points in an asymptotic direction.

Thus α is asymptotic if and only if $k(\alpha') = S(\alpha') \cdot \alpha' = 0$.

Since $S(\alpha') = -U'$, α is asymptotic if $U' \cdot \alpha' = 0$.

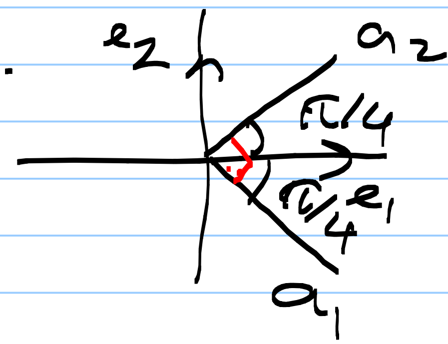
Remark: If α is asymptotic. Since α is tangent to M $\alpha' \cdot U = 0$. So taking derivative

$$0=(\alpha'\cdot u)'=\underbrace{\alpha'\cdot u'}_{=0}+\alpha''\cdot u$$

$$\Leftrightarrow \alpha''\cdot u=0.$$

An Application to Minimal and Flat Surfaces:

Recall that a surface is minimal if $H(p) = k_1(p) + k_2(p) = 0$ for all point p of the surface. Hence by the previous lemma either $k_i(p) = 0$ for $i = 1, 2$ or $k_1(p) = -k_2(p) \neq 0$ ($K(p) < 0$) and there are exactly two asymptotic directions, with $\nu = \pm \pi/2$.



In particular, the asymptotic directions a_1 and a_2 are orthogonal either. Thus a surface with $K < 0$ is minimal if and only if through every point there are exactly two asymptotic curves passes and they are orthogonal.

Recall that for the Helicoid we had $K<0$ and $H=0$ at all points and thus the Helicoid is a minimal surface.

Recall also that the Saddle surface had also that $K<0$. The lemma below shows that this is always the case for any ruled surface (such as Helicoid and Saddle Surface).

<u>Lemma</u>: A ruled surface M has Gaussian curvature $K \leq 0$. Furthermore $K=0$ if and only if the unit normal U is parallel along each ruling of M (so all

the point p on a ruling have the same Euclidean tangent plane T_pM.)

Proof: A ruling in a ruled surface is a straight line $t \mapsto p + tq$ is clearly asymptotic because its acceleration is zero and thus tangent to M. Hence, $K(p) \leq 0$ for all $p \in M$ (because $k(u) = 0$ means that $K = k_1 k_2 \leq 0$).

Now let $\alpha(t) = p + tq$ be an arbitrary ruling in M. If U is parallel along α, then $S(\alpha') = -U'$ $= 0 = 0 \cdot \alpha'$. Thus α is a principal curve with principal curvature $k(\alpha') = 0$. Hence,

$K = k_1 k_2 = 0$.

Conversely, if $K = 0$ we see

from the previous lemma that the asymptotic directions (and thus curves) are also principal. Therefore, each ruling is principal ($S(\alpha') = k(\alpha')\alpha'$) as well as asymptotic ($k(\alpha')=0$); hence

$$U' = -S(\alpha') = 0 \text{ and}$$

U is parallel along each ruling of M. ∎

<u>Definition</u>: A curve α in $M \subseteq \mathbb{R}^3$ is a geodesic of M provided its acceleration α'' is always normal to M.

<u>Proposition</u>: A geodesic has constant speed.

<u>Proof</u>: $\frac{d}{dt}(\|\alpha'\|^2) = \frac{d}{dt}(\alpha' \cdot \alpha')$

$= 2\alpha' \cdot \alpha'' = 0$ because, α'' is normal to the surface. Hence, $\|\alpha'\|$ is constant. ∎

<u>Proposition</u>: Any line in a surface is a geodesic.

<u>Proof</u> Any line $\alpha(t) = p + tq$ has $\alpha'' = 0$ and thus α'' is normal to M. Hence, α is a geodesic. ∎

<u>Examples</u> 1) Let P be a plane in $\mathbb{R}^3$. If α is any curve in P, then $\alpha' \cdot U = 0$ and hence, $\alpha'' \cdot U + \alpha \cdot U' = 0$. However, $U' = 0$ (since the surface is a plane) and thus $\alpha'' \cdot U = 0$. Finally, since α is a geodesic

$\alpha'' = \lambda u$ and thus $0 = \lambda u \cdot u = \lambda$

$\Rightarrow \alpha'' = 0$. So $\alpha' = q$ is a constant and thus $\alpha = pt + q$, i.e., α is a line. So a curve in a plane is a geodesic if and only if α is line.

2) Let $\Sigma \subseteq \mathbb{R}^3$ be a sphere, and P be a plane through the center of the sphere. Let α be a unit speed parametrization of that the circle, the intersection of Σ with P. Since α is a circle in P its acceleration vector points to the center. Hence α'' is collinear to the line joining $\alpha(t)$ to the center of the circle (and thus the sphere).

Hence $\alpha(t)$ is a geodesic in Σ.

<u>Remark</u>: It is known that great circles on a sphere are the only geodesics of the sphere.

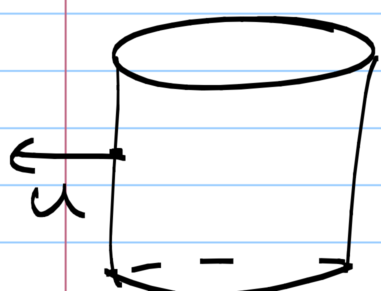
3) <u>Cylinders</u>: $M \subseteq \mathbb{R}^3$, $x^2+y^2=r^2$.

<u>Claim</u>: Any geodesic on M has the form

$$\alpha(t) = (r\cos(at+b), r\sin(at+b), ct+d).$$

<u>Proof</u>: Let $\alpha(t) = (r\cos\vartheta(t), r\sin\vartheta(t), h(t))$ be a geodesic on M.

Since $\alpha''(t) \parallel u = (*, *, 0)$, we see



that $h''(t) = 0$. Hence, $h(t) = ct+d$ for some $c, d \in \mathbb{R}$. Also the speed of any geodesic is constant and thus $\|\alpha'\| = \sqrt{(r\vartheta')^2 + c^2}$ is constant. Hence, ϑ' is constant

So $v(t) = at + b$ for some $a, b \in \mathbb{R}$.

<u>Proposition</u>: Let P be a plane orthogonal a surface M at any point of intersection curve, say α. Then (assuming its unit speed) the curve α is a geodesic.

<u>Proof</u>: Since α is a unit speed curve in P, $\alpha' \perp \alpha''$, where both curves lie in P. However, U is P and $U \perp \alpha'$. Thus U and α'' are collinear. Hence, α'' is normal to the surface at all points. This finishes the proof.

<u>Example</u>: In a surface of revolution Σ a plane contain

Ing the rotation axis is normal to the surface. Hence, any meridian is a geodesic.

Summary

Principal Curves	$k(\alpha') = k_1$ or k_2	$S(\alpha') \parallel \alpha'$
Asymptotic Curves	$k(\alpha') = 0$	$S'(\alpha) \perp \alpha'$. $\alpha'' \in T_pM$
Geodesics		$\alpha'' \perp T_pM$.

§2.7. Connection Forms:

Lemma: Let E_1, E_2, E_3 be a frame field on $\mathbb{R}^3$. For each tangent vector v to $\mathbb{R}^3$ at the point p, let $\omega_{ij}(v) = (\nabla_v E_i) \cdot E_j(p)$, for $i, j = 1, 2, 3$. Then each ω_{ij} is a 1-form and $\omega_{ij} = -\omega_{ji}$. These 1-forms are called the connection forms of the frame field E_1, E_2, E_3.

Proof: $\omega_{ij}(av+bw) = (\nabla_{av+bw} E_i) \cdot E_j(p)$

$$= (a\nabla_v E_i + b\nabla_w E_i) \cdot E_j(p)$$

$$= a(\nabla_v E_i) \cdot E_j(p) + b(\nabla_w E_i) \cdot E_j(p)$$

$= a\,\omega_{ij}(v) + b\,\omega_{ij}(w)$, hence ω_{ij} is a 1-form.

For the second statement, consider

$$0 = v(\delta_{ij}) = v(E_i \cdot E_j)$$

$$= \nabla_v E_i \cdot E_j(p) + E_i(p) \cdot \nabla_v E_j$$

$$= \omega_{ij}(v) + \omega_{ji}(v).$$

Hence, $\omega_{ij}(V) = -\omega_{ji}(V)$. ■

As a consequence we get

Theorem: For any vector field V on $\mathbb{R}^3$ we have

$$\nabla_V E_i = \sum_j \omega_{ij}(V) E_j .$$

Proof. By definition

$\omega_{ij}(V) = (\nabla_V E_i) \cdot E_j(p)$. Since the E_1, E_2, E_3 is a frame the result follows. ■

Remark:

Since $\omega_{ij} = -\omega_{ji}$, we have $\omega_{ii} = 0$.

So the matrix

$$\omega = \begin{bmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \end{bmatrix} = \begin{bmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{12} & -\omega_{23} & 0 \end{bmatrix}.$$

Given a frame field E_1, E_2, E_3 on $\mathbb{R}^3$ write $E_i = \sum_{j=1}^{3} a_{ij} U_j$, where $a_{ij} = E_i \cdot U_j$. The matrix

$A = (a_{ij})$ is called the attitude matrix of the field E_1, E_2, E_3.

Define dA as $dA = (da_{ij})$, a matrix of 1-forms.

<u>Theorem:</u> Assume the above setup. Then $\omega = dA \cdot A^t$ (matrix multiplication) or equivalently

$$\omega_{ij} = \sum_k a_{jk}\, da_{ik} \quad \forall i,j.$$

<u>Proof:</u> $E_i = \sum a_{ik} U_k$

$$\omega_{ij}(v) = (\nabla_v E_i) \cdot E_j(p)$$

$$= \left(\nabla_v \sum_k a_{ik} U_k\right) \cdot \left(\sum_\ell a_{j\ell} U_\ell\right)$$

$$= \left(\sum_k da_{ik}(v)\, U_k + \sum_k a_{ik} \nabla_v U_k\right) \cdot \left(\sum_\ell a_{j\ell} U_\ell\right)$$

$$= \sum_{k,\ell} da_{ik}(v)\, a_{j\ell} \overbrace{U_k \cdot U_\ell}^{= \delta_{k\ell}} + \sum_{k,\ell} a_{ik} a_{j\ell} \underbrace{(\nabla_v U_k)}_{= 0} \cdot U_\ell$$

$$= \sum_k a_{jk}\, da_{ik}(v).$$

So $\omega_{ij} = \sum_k a_{jk}\, da_{ik}$.

§2.8. The Structural Equations:

Definition: For a frame field E_1, E_2, E_3 on $\mathbb{R}^3$, we define the dual 1-forms $\theta_1, \theta_2, \theta_3$ as

$\theta_i(v) = v \cdot E_i(p)$, for any vector $v \in T_p\mathbb{R}^3$.

Example: Consider the natural frame U_1, U_2, U_3. Then

$\theta_i(v) = v \cdot U_i(p) = v_i$, $v = (v_1, v_2, v_3)$.

So $\theta_i = dx_i$, $x_i : \mathbb{R}^3 \to \mathbb{R}$

$x_i(v) = v_i$.

Lemma: Let E_i and θ_i, $i = 1, 2, 3$, be as above. Then any 1-form ϕ on $\mathbb{R}^3$ has a unique expression

$$\phi = \sum_{i=1}^{3} \phi(E_i)\,\theta_i.$$

Proof $\left(\sum \phi(E_i)\theta_i\right)(V) = \sum \phi(E_i)\,\theta_i(V)$

$= \phi\left(\sum \theta_i(V) E_i\right) = \phi(V)$, for any vector field V. So the result follows. ∎

Recall the matrix $A = (a_{ij})$ is defined by the equation

$$E_i = \sum a_{ij} U_j.$$

Hence, $\theta_i(v) = v \cdot E_i(p)$

$$= v \cdot \left(\sum a_{ij} U_j(p)\right)$$

$$= \sum a_{ij} \left(v \cdot U_j(p)\right)$$

$$= \sum a_{ij}\, dx_j(v),$$

and thus $\theta_i = \sum a_{ij}\, dx_j$.

Since $a_{ij} = E_i \cdot U_j = \theta_i(U_j)$

and thus $\theta_i = \sum_j \theta_i(U_j)\, dx_j$.

Theorem (Cartan Structural Equations)
Let E_i, θ_i and ω_{ij} be as above. Then we have

1) the first structural equations:
$$d\theta_i = \sum_j \omega_{ij} \wedge \theta_j$$

2) the second structural equations:
$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj}.$$

Proof: 1) The equation obtained above $\theta_i = \sum_j a_{ij}\, dx_j$ can be written as $\theta = A\, d\xi$, where
$$\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} \text{ and } d\xi = \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix}.$$

Now,
$$d\theta = d(A\, d\xi) = dA\, d\xi - A\overbrace{d^2\xi}^{=0}$$
$$= dA\, d\xi$$

$$= dA \cdot (A^t \cdot A) \cdot d\xi$$
$$= (dA \cdot A^t)(A\, d\xi)$$
$$= \omega\, \theta, \text{ because}$$

the attitude matrix A is orthogonal so that $A^t A = Id$.

Recall that $E_i = a_{ij} U_j$ so that A is a base change matrix from an orthogonal frame U_i to another orthogonal frame E_i.

2) $d\omega = d(dA\, A^t)$
$$= -dA\, dA^t$$
$$= -(dA\, A^t)(A\, dA^t)$$
$$= -\omega\, (dA\, A^t)^t$$
$$= -\omega\, \omega^t$$
$$= \omega\omega, \text{ because } \omega^t = -\omega.$$

Here, $\omega\omega$ we mean matrix multiplication of 1-forms we entries are multiplied via wedge products.

Example: Consider the spherical coordinates given by

$$x_1 = \rho \cos\varphi \cos\vartheta$$
$$x_2 = \rho \sin\varphi \cos\vartheta \qquad \begin{pmatrix} \rho & \varphi & \vartheta \\ 1 & 2 & 3 \end{pmatrix}$$
$$x_3 = \rho \sin\vartheta$$

Since $\theta_i = \sum a_{ij}\, d\alpha_j$, where

$$a_{ij} = E_i \cdot u_j.$$

$$E_1 = \frac{(\cos\varphi\cos\vartheta, \sin\varphi\cos\vartheta, \sin\vartheta)}{1}$$

$$= (\overset{a_{11}}{\cos\varphi\cos\vartheta}, \overset{a_{12}}{\sin\varphi\cos\vartheta}, \overset{a_{13}}{\sin\vartheta})$$

$$E_2 = \frac{(-\rho\sin\varphi\cos\vartheta, \rho\cos\varphi\cos\vartheta, 0)}{\rho\cos\vartheta}$$

$$= (\overset{a_{21}}{-\rho\sin\varphi}, \overset{a_{22}}{\rho\cos\varphi}, \overset{a_{23}}{0})$$

$$E_3 = \frac{(-\rho\cos\varphi\sin\vartheta, -\rho\sin\varphi\sin\vartheta, \rho\cos\vartheta)}{\rho}$$

$$= (\overset{a_{31}}{-\cos\varphi\sin\vartheta}, \overset{a_{32}}{-\rho\sin\varphi\sin\vartheta}, \overset{a_{33}}{\cos\vartheta})$$

So, $\theta_1 = \sum a_{1j}\, dx_j$

$$= a_{11}\,dx_1 + a_{12}\,dx_2 + a_{13}\,dx_3$$

$$= (\cos\varphi\cos\vartheta)(\cos\varphi\cos\vartheta\, d\rho -$$

$$\rho\sin\varphi\cos\vartheta\, d\varphi - \rho\cos\varphi\sin\vartheta\, d\vartheta)$$

$$+ (\sin\varphi\cos\vartheta)(\sin\varphi\cos\vartheta\, d\rho -$$

$\rho \cos\varphi \cos\vartheta \, d\varphi - \rho \sin\varphi \sin\vartheta \, d\vartheta)$
$+ \sin\vartheta \, (\sin\vartheta \, d\rho + \rho \cos\vartheta \, d\vartheta)$

So, $\theta_1 = d\rho + 0 + 0 = d\rho$.
Similarly,

$$\theta_2 = \rho \cos\varphi \, d\vartheta \text{ and } \theta_3 = \rho \, d\varphi.$$

As we've computed in a theorem that $\omega = dA \, A^t$ or equivalently $\omega_{ij} = \sum_k a_{jk} \, da_{ik}$, we obtain

$\omega_{12} = \cos\varphi \, d\vartheta$, $\omega_{13} = d\varphi$ and $\omega_{23} = \sin\varphi \, d\vartheta$.

Now from the first structural equation $d\theta_i = \sum_j \omega_{ij} \wedge \theta_j$.
So, say

$$d\theta_3 = \omega_{31} \wedge \theta_1 + \omega_{32} \wedge \theta_2$$
$$= d\rho \wedge d\varphi,$$ which is really the case since $\theta_3 = \rho \, d\varphi$.
Let's compute also $d\omega_{ij}$.

From the second structural equation

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj}.$$

Using the skew-symmetry of ω_{ij} we obtain:

$$d\omega_{12} = \sum_{k=1}^{3} \omega_{1k} \wedge \omega_{k2} = \omega_{13} \wedge \omega_{32}$$

$$= d\varphi \wedge (-\sin\varphi \, d\vartheta)$$

$$= -\sin\varphi \, d\varphi \wedge d\vartheta, \text{ which}$$

is really $d\omega_{12} = d(\cos\varphi \, d\vartheta)$.

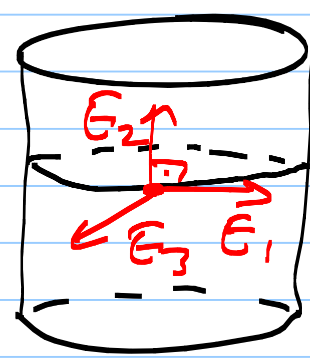
CHAPTER 6: Geometry of Surfaces in $\mathbb{R}^3$

§6.1. The Fundamental Equations:

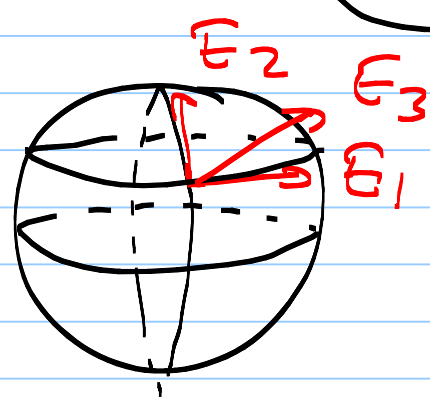
Definition: An adapted frame field E_1, E_2, E_3 on a region $\mathcal{O}$ in $M \subseteq \mathbb{R}^3$ is a Euclidean frame such that E_3 is always normal to M. (So E_1 and E_2 are tangent to M).

Lemma: There is an adapted frame field on a region $\mathcal{O}$ in M if and only if $\mathcal{O}$ is orientable.

Examples 1)



2)



Definition: The 1-forms ω_{ij} defined by the equation

$$\nabla_v E_i = \sum_{j=1}^{3} \omega_{ij}(v)\, E_j(p)$$

are called connection 1-forms on M.

Lemma: $S(v) = \omega_{13}(v) E_1(p) + \omega_{23}(v) E_2(p)$

Proof By definition $S(v) = -\nabla_v E_3$.

Hence, $S(v) = -\nabla_v E_3 = -\omega_{31}(v) E_1(p) - \omega_{32}(v) E_2(p) = \omega_{13}(v) E_1(p) + \omega_{23}(v) E_2(p)$

because $\omega_{ij} = -\omega_{ji}$ and $\omega_{ii} = 0$.

$E_i \cdot E_k = \delta_{ik} \Rightarrow 0 = \nabla_v (E_i \cdot E_k) = 0$

$\nabla_v E_i \cdot E_k + \nabla_v E_k \cdot E_i = 0$

$\left(\sum_j \omega_{ij}(v) E_j\right) \cdot E_k + \left(\sum_j \omega_{kj}(v) E_j\right) \cdot E_i = 0$

$\Rightarrow \omega_{ik}(v) + \omega_{ki}(v) = 0$

Define: Given an adapted frame E_1, E_2, E_3 for a region $\mathcal{U}$ in M we define dual 1-forms $\theta_1, \theta_2, \theta_3$ by $\theta_i(v) \doteq v \cdot E_i(p)$.

Note that if v is a tangent vector field then $\theta_3(v) = v \cdot E_3(p) = 0$. Thus θ_3 is identically zero on M.

Example: Let Σ be a sphere of radius r and consider spherical coordinates on Σ. By shifting the indices of the example we studied in §2.8 by $1 \to 3$, $2 \to 1$, $3 \to 2$ we obtain

$\theta_1 = r \cos\varphi \, d\vartheta$, $\theta_2 = r\, d\varphi$

$\omega_{12} = \sin\varphi \, d\vartheta$, $\omega_{13} = -\cos\vartheta \, d\vartheta$

and $\omega_{23} = -d\varphi$.

<u>Theorem</u>: If E_1, E_2, E_3 is an adapted frame for M then we have:

1) $\begin{cases} d\theta_1 = \omega_{12} \wedge \theta_2 \\ d\theta_2 = \omega_{21} \wedge \theta_1 \end{cases}$ First structural equation

2) $\omega_{31} \wedge \theta_1 + \omega_{32} \wedge \theta_2 = 0$

Symmetry equation

3) $d\omega_{12} = \omega_{13} \wedge \omega_{32}$ Gauss Equation

4) $\begin{cases} d\omega_{13} = \omega_{12} \wedge \omega_{23} \\ d\omega_{23} = \omega_{21} \wedge \omega_{13} \end{cases}$ Codazzi Equation

<u>Proof</u>: Recall Cartan Structural Equations.

1) $d\theta_i = \sum_j \omega_{ij} \wedge \theta_j$, and

2) $d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj}$.

So, for example

$d\theta_1 = \omega_{12} \wedge \theta_2$ and $d\omega_{13} = \omega_{12} \wedge \omega_{23}$.

§6.2. Form Computations:

Let $M \subseteq \mathbb{R}^3$ be a surface, E_1, E_2, E_3 an adapted frame on M and $\theta_1, \theta_2, \theta_3$ the dual 1-form:

$$\theta_i(E_j) = \delta_{ij}, \text{ for all } i, j.$$

Lemma: (The Basis Formulas)

If ϕ is a 1-form on M and ψ is a 2-form on M, then

1) $\phi = \phi(E_1)\theta_1 + \phi(E_2)\theta_2$,

2) $\psi = \psi(E_1, E_2)\theta_1 \wedge \theta_2$.

Proof: Since E_1, E_2 form a basis for T_pM at any $p \in M$ it is enough to check the identities on these vectors.

1) $(\phi(E_1)\theta_1 + \phi(E_2)\theta_2)(E_1) = \phi(E_1)\theta_1(E_1)$
$= \phi(E_1)$

and similarly,

$(\phi(E_1)\theta_1 + \phi(E_2)\theta_2)(E_2) = \phi(E_2)$ and thus, $\phi = \phi(E_1)\theta_1 + \phi(E_2)\theta_2$.

2) $\mu(E_1, E_2)\theta_1 \wedge \theta_2 (E_1, E_2)$

$$= \mu(E_1, E_2) \det \begin{bmatrix} \theta_1(E_1) & \theta_1(E_2) \\ \theta_2(E_1) & \theta_2(E_2) \end{bmatrix}$$

$$= \mu(E_1, E_2) \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \mu(E_1, E_2).$$

Hence, $\mu = \mu(E_1, E_2)\theta_1 \wedge \theta_2$. ■

<u>Lemma</u>: 1) $\omega_{13} \wedge \omega_{23} = K\, \theta_1 \wedge \theta_2$

2) $\omega_{13} \wedge \theta_1 + \theta_1 \wedge \omega_{23} = 2H\, \theta_1 \wedge \theta_2$.

<u>Proof</u>: By definition

$S(E_1) = -\nabla_{E_1} E_3 \quad (E_3 = U)$

$\quad = -\omega_{31}(E_1)E_1 - \omega_{32}(E_1)E_2$

and

$S(E_2) = -\nabla_{E_2} E_3$

$\quad = -\omega_{31}(E_2)E_1 - \omega_{32}(E_2)E_2$.

Thus the matrix representation of the shape operator in the basis $\{E_1, E_2\}$ of T_pM becomes

$$\begin{pmatrix} \omega_{13}(E_1) & \omega_{13}(E_2) \\ \omega_{23}(E_1) & \omega_{23}(E_2) \end{pmatrix}.$$

For the part (1) of the lemma

$$K\,\theta_1\wedge\theta_2(E_1, E_2) = K = \det(S)$$

$$= \omega_{13}(E_1)\,\omega_{23}(E_2) - \omega_{13}(E_2)\,\omega_{23}(E_1)$$

$$= \omega_{13}\wedge\omega_{23}(E_1, E_2).$$

Hence, $\omega_{13}\wedge\omega_{23} = K\,\theta_1\wedge\theta_2$.

The second statement can be handled similarly.

By the second structural equation we have $d\omega_{12} = -\omega_{13}\wedge\omega_{23}$ (Gauss Equation) and thus

<u>Corollary</u> $d\omega_{12} = -K\,\theta_1\wedge\theta_2$.

<u>Remark:</u> In the previous section (§ 6.1) we had computed θ_i and ω_{ij} for spherical coordinates on a sphere of radius r. Hence,

$$\theta_1 \wedge \theta_2 = r^2 \cos\varphi \, dv \wedge d\varphi$$
$$= -r^2 \cos\varphi \, d\varphi \wedge dv, \text{ and}$$

$$d\omega_{12} = d(\sin\varphi \, dv)$$
$$= \cos\varphi \, d\varphi \wedge dv.$$

Hence, by the above Corollary we see that $K = 1/r^2$, as expected.

<u>Definition</u>: A principal frame on a surface M is an adapted frame E_1, E_2, E_3 so that E_1 and E_2 are principal vectors.

<u>Lemma</u>: If p is a nonumbilic point of $M \subseteq \mathbb{R}^3$, then there exists a principal frame in a neighborhood of p in M.

<u>Proof</u>: By hypothesis $k_1(p) \neq k_2(p)$

and thus since k_i's are continuous functions $k_1(p) \neq k_2(p)$ in a neighborhood U of p. Principal directions are unit vectors along eigenvectors of the shape operator. Let S have matrix representation in some adapted frame $\{F_1, F_2, F_3\}$ as

$$S = (S_{ij}) = \begin{pmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{pmatrix}$$

$$\det(S - \lambda I) = \lambda^2 - 2H\lambda + K = 0$$

$$\lambda_{1,2} = \frac{2H \pm \sqrt{4H^2 - 4K}}{2} = H \pm \sqrt{H^2 - K}.$$

$$k_1 = H - \sqrt{H^2 - K}, \quad k_2 = H + \sqrt{H^2 - K}.$$

For the eigenvector corresponding to k_1 we have the linear equation $(S_{11} - k_1)a + S_{12} b = 0$. So $(a, b) = (S_{12}, k_1 - S_{11})$ and thus $S_{12} F_1 + (k_1 - S_{11}) F_2$ is an eigenvector. So we may let

$$E_1 = \frac{S_{12}F_1 + (k_1 - S_{11})F_2}{\| S_{12}F_1 + (k_1 - S_{11})F_2 \|}.$$

Similarly, $E_2 = \dfrac{(k_2 - S_{22})F_1 + S_{12}F_2}{\| (k_2 - S_{22})F_1 + S_{12}F_2 \|}$.

Finally, let $E_3 = E_1 \times E_2$. Hence, E_1, E_2, E_3 is the desired "principal frame". $\blacksquare$

Now if E_1, E_2, E_3 is a principal frame $S(E_1) = k_1 E_1$ and $S(E_2) = k_2 E_2$. However, by a Corollary from §6.1 we had $S(V) = \omega_{13}(V)E_1 + \omega_{23}(V)E_2$ and hence, $\omega_{13}(E_1) = k_1$, $\omega_{13}(E_2) = 0$, and $\omega_{23}(E_1) = 0$, $\omega_{23}(E_2) = k_2$.

So, $\omega_{13} = k\theta_1$ and $\omega_{23} = k_2\theta_2$.

We'll finish this section with the following version of Codazzi equations.

Theorem: If E_1, E_2, E_3 is a principal frame field on $M \subseteq \mathbb{R}^3$, then
$$E_1[k_2] = (k_1 - k_2)\,\omega_{12}(E_2)$$
$$E_2[k_1] = (k_1 - k_2)\,\omega_{12}(E_1).$$

Proof: From the Codazzi Equations
$$d\omega_{13} = \omega_{12} \wedge \omega_{23} \text{ and}$$
$$d\omega_{23} = \omega_{21} \wedge \omega_{13}.$$
By the line above the theorem
$\omega_{13} = k_1\theta_1$ and $\omega_{23} = k\theta_2$ and hence,
$$d(k_1\theta_1) = d\omega_{13} = \omega_{12} \wedge k\theta_2$$
$$\Rightarrow dk_1 \wedge \theta_1 + k_1\, d\theta_1 = k_2\,\omega_{12} \wedge \theta_2.$$
However, by structural equations $d\theta_1 = \omega_{12} \wedge \theta_2$ and thus we get
$$dk_1 \wedge \theta_1 = (k_2 - k_1)\,\omega_{12} \wedge \theta_2.$$

Now compute the above two forms on the pair of vectors E_1, E_2:

$$(dk_1 \wedge \theta_1)(E_1, E_2) = -dk_1(E_2)$$

$$(k_2 - k_1)\omega_{12} \wedge \theta_2 \, (E_1, E_2) = (k_2 - k_1)\omega_{12}(E_1)$$

and thus

$$E_2[k_1] = dk_1(E_2) = (k_1 - k_2)\omega_{12}(E_1),$$

the first desired equality.

The second one is obtained similarly. ■

§6.3. Some Global Theorems:

Theorem: If the shape operator is identically zero, then M is part of a plane in $\mathbb{R}^3$, provided M is connected.

Proof: $S = 0$ implies that for any unit normal vector field U

$0 = S_v(U) = -U'$, so that U is parallel (constant) along any vector v.

Choose any point $p \in M$. For any other point q, choose a path $\alpha(t)$ so that $\alpha(0) = p$ and $\alpha(1) = q$.

Consider the function

$$f(t) = (\alpha(t) - p) \cdot U.$$

Now, $f'(t) = \alpha'(t) \cdot U + \alpha(t) \cdot \overbrace{U'}^{\overset{0}{\shortparallel}}$

$= \alpha'(t) \cdot U = 0$ for all t, and

$f(0) = (p - p) \cdot U = 0$. Hence, $f(t)$ is identically zero.

In particular,

$$0 = f(1) = (p-q)\cdot U, \text{ and}$$

thus q lies in the plane Π containing p, whose normal is U. Since, $q \in M$ is an arbitrary point we see that $M \subseteq \Pi$!

A surface M is called all-umbilic if the every point of M is umbilic.

<u>Lemma</u>: If M is a connected all-umbilic then M has constant Gaussian curvature $K \geq 0$.

<u>Proof</u>: Let E_1, E_2, E_3 be a frame on M so that E_1, E_2 are tangent to M (and hence $E_3 \perp M$). So at any point $p \in M$,

$k_1(p) = k_2(p) = k(p)$ for some function k. By a theorem of previous section

$E_1[k_2] = (k_1 - k_2)\,\omega_{12}(E_2)$,

$E_2[k_1] = (k_1 - k_2)\,\omega_{12}(E_1)$ and thus

$E_1[k_2] = 0$ and $E_2[k_1] = 0$.

(Since M is all umbilic all directions are principal).

Hence, $dk[E_1] = dk[E_2] = 0$.

So $dk = 0$ on $\mathcal{O}$. $\Rightarrow K = k_1 k_2 = k^2$ and $dK = 2k\,dk = 0$ on $\mathcal{O}$. Hence $dK = 0$ on all of $\mathcal{O}$, i.e., K is constant.

Theorem: If $M \subset \mathbb{R}^3$ is all umbilic and $K > 0$, then M is a part of a sphere of radius $1/\sqrt{K}$.

Proof: Let $p \in M$ and consider the point $c = p + \frac{1}{k(p)} E_3(p)$, where E_1, E_2, E_3 is an adapted frame for M. Let $q \in M$ be any other point and choose a curve $\alpha : [0,1] \longrightarrow M$ so that $\alpha(0) = p$ and $\alpha(1) = q$. Now let γ be the curve, $\gamma(t) = \alpha(t) + \frac{1}{k(\alpha(t))} E_3(\gamma(t))$,

where $k(p) = k_1(p) = k_2(p)$. By the previous lemma $K(p) = k_1(p) k_2(p)$ $= k^2(p)$ is a constant function.

Thus $\gamma'(t) = \alpha'(t) + \frac{1}{k} E_3'$.

However, $E_3' = U' = -S(\alpha') = -k\alpha'$ so that S is a scalar function, because k is constant. Thus $\gamma' = \alpha' + 1/k(-k\alpha') = 0$, so that

γ is a constant function. Thus, $c = \gamma(0) = \gamma(1) = q + \frac{1}{k} E_3(q)$ and hence, $d(c,q) = 1/|k|$ for every point q of M. Since $K = k^2$, $d(c,q) = 1/\sqrt{K}$ for all $q \in M$. Hence, M is a part of a sphere of center c with radius $1/\sqrt{K}$. ∎

Finally, the three results above imply the following

<u>Corollary</u> A surface M is all-umbilic if and only if M is a part of a plane of a sphere.

In particular, if M is compact surface then M is a sphere.

<u>Theorem</u>: On every compact surface M in $\mathbb{R}^3$ there is a point at

which the Gaussian curvature K is positive.

Proof: Let $f: \mathbb{R}^3 \to \mathbb{R}$ be the function $f(p) \doteq \|p\|^2$, the square of the distance of p to the origin. Since f is continuous and M is compact there is a point $m \in M$ so that $f(m)$ is the maximum value of f on M.

Claim: Let $r = \|m\| > 0$. Then

$K(m) \geq 1/r^2 > 0.$

Note that the claim proves the theorem.

Proof of the claim: Let $u \in T_m M$ be a unit vector and choose a unit speed curve α on M with $\alpha(0) = m$ and $\alpha'(0) = u$.

Thus the function $f(\alpha(t))$ has its maximum at $t=0$. Thus $\frac{d}{dt}(f(\alpha(t)))(0)=0$ and $\frac{d^2}{dt^2}(f(\alpha(t)))(0) \leq 0$.

$$0=(f(\alpha(t)))'(0)=(\alpha(t)\cdot\alpha(t))'(0)$$
$$=2\alpha(0)\cdot\alpha'(0)=2m\cdot u$$

However, u is an arbitrary unit vector in T_pM we see that m is normal to M at m.

Now, $\frac{d^2(f\alpha)}{dt^2}=2\alpha'\cdot\alpha'+2\alpha\cdot\alpha''$

and hence $2\alpha'\cdot\alpha'+2\alpha\cdot\alpha'' \leq 0$, at $t=0$. So

$$u\cdot u+m\cdot\alpha''(0)\leq 0.$$

$$1+m\cdot\alpha''(0)\leq 0 \Rightarrow m\cdot\alpha''(0)\leq -1.$$

Since $\|m\|=r$, $\frac{m}{r}$ is a unit normal to M at the point m. We know that

$k(u) = \frac{m}{r} \cdot \alpha''(0) \leq -1/r$.

Again, since $u \in T_m M$ is arbitrary $k(u) \leq -1/r$ for all $u \in T_m M$ so that

$K(m) = k_1(u_1) k_2(u_2) \geq \frac{1}{r^2}$, where $k_i(u_i)$, $i = 1, 2$, are the principal curvatures at m.

<u>Corollary</u> There is no compact surface in $\mathbb{R}^3$ with $K \leq 0$.

<u>Lemma</u> (Hilbert) Let m be a point of $M \subseteq \mathbb{R}^3$ such that

1) k_1 has a local maximum at m,
2) k_2 has a local minimum at m,
3) $k_1(m) > k_2(m)$.

Then $K(m) \leq 0$.

Proof: Since $k_1(m) > k_2(m)$, m is not umbilic and thus by a lemma proved previously there is a principal frame field E_1, E_2, E_3 on a neighborhood of m in M.

Fact: Let f be a function on M so that f has a maximum (minimum) at a point p. If V is a vector at p, then

$V[f] = 0$ and $VV[f] \leq 0$

$(VV[f] \geq 0)$.

So by this fact

$E_1[k_2] = E_2[k_1] = 0$ at m

and $E_1E_1[k_2] \geq 0$ and

$E_2E_2[k_1] \leq 0$ at m.

Now from the Codazzi equation which says that

$E_1[k_2] = (k_1 - k_2)\,\omega_{12}(E_2)$ and

$E_2[k_1] = (k_1 - k_2)\,\omega_{12}(E_1)$,

we deduce that

$\omega_{12}(E_1) = \omega_{12}(E_2) = 0$ at m,

because $k_1 - k_2 \neq 0$.

By Exercise 2 of §6.2. we get

(*) $K = E_2[\omega_{12}(E_1)] - E_1[\omega_{12}(E_2)]$ at m.

Apply E_1 to the first Codazzi equation (both sides are functions) to get

$$E_1E_1[k_2] = (E_1[k_1] - E_1[k_2])\,\omega_{12}(E_2)$$
$$+ (k_1 - k_2)\,E_1[\omega_{12}(E_2)].$$

However, at the point m, $\omega_{12} = 0$ and $k_1 - k_2 > 0$. Thus $E_1[\omega_{12}(E_2)] \geq 0$ at m, because $E_1E_1[k_2] \geq 0$.

Similarly, taking E_2 derivative of the second Codazzi equation

$E_2[\omega_{12}(E_1)] \leq 0$ at m.

Hence by (*) we get $K(m) \leq 0$.

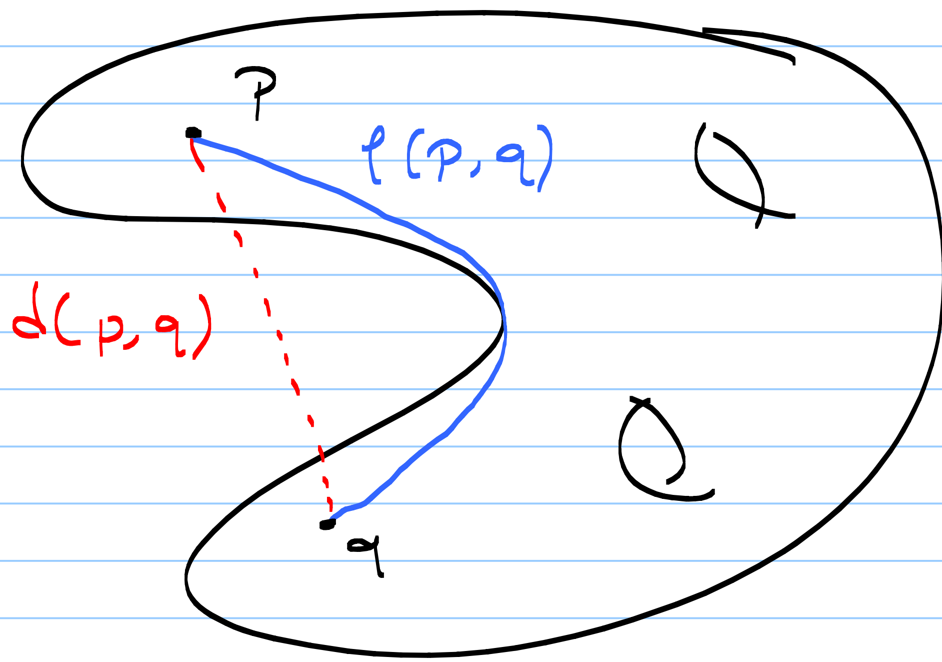
<u>Theorem</u>: (Liebmann)
If M is a compact surface in $\mathbb{R}^3$ with constant Gaussian curvature K, then M is a sphere of radius $1/\sqrt{K}$ (by the compactness of M, $K>0$).

<u>Proof</u>: Since $M \subseteq \mathbb{R}^3$ compact M is orientable by some topological fact, which cannot be proved by the tools of the course. So we have smooth unit normal vector field defined on all of M. Therefore, principal curvature functions are globally

defined on M and $k_1 \geq k_2 \geq 0$ at all points. In particular, k_1 has a maximum at some point say $p \in M$. Since, $K = k_1 k_2$ is constant k_2 has minimum at p. If $k_1(p) > k_2(p)$ then by the previous lemma $K(p) \leq 0$, a contradiction since we have $K > 0$ is a constant function. Thus we must have $k_1(p) = k_2(p)$. This implies $k_1(q) = k_2(q)$ for all $q \in M$, because $k_1 \geq k_2$ and the maximum value of k_1 is equal to the minimum value of k_2. So M is all-umbilic and thus it is a sphere of radius $1/\sqrt{K}$. ∎

§6.4. Isometries and Local Isometries:

§4.1. Definition: If p and q are points of $M \subseteq \mathbb{R}^3$, the intrinsic distance from p to q is defined to be the infimum of lenghts $L(\alpha)$ of all curves α from p to q. It is denoted as $\rho(p,q)$.



__Definition:__ An isometry $F: M \to \overline{M}$ of surfaces in $\mathbb{R}^3$ is a 1-1 and onto smooth mapping so that $F_*(v) \cdot F_*(w) = v \cdot w$, for all $v, w \in T_pM$ and $p \in M$.

__Theorem:__ Isometries preserve intrinsic distance: If $F: M \to \overline{M}$ is an isometry then

$$\rho(p,q) = \overline{\rho}(F(p), F(q))$$

for all $p, q \in M$.

__Proof:__ If α is a smooth curve in M with $\alpha(a) = p$ and $\alpha(b) = q$, say, then $L(\alpha) = \int_a^b \|\alpha'(t)\| \, dt$ and

$F(\alpha)$ is a smooth curve in $\overline{M}$ from $F(p)$ to $F(q)$ with length $L(F(\alpha)) = \int_a^b \|(F(\alpha))'(t)\| \, dt$

$$= \int_a^b \|\alpha'(t)\| \, dt$$

$$= L(\alpha).$$

Hence, $L(\alpha) = L(F(\alpha)) \geq \bar{\rho}(F(p), F(q))$.
and thus $\rho(p,q) \geq \bar{\rho}(F(p), F(q))$.
On the other hand, since F is an isometry $F^{-1} : \bar{M} \to M$ exists and is also an isometry. Thus, we get

$$\bar{\rho}(\bar{p}, \bar{q}) \geq \rho(F^{-1}(\bar{p}), F^{-1}(\bar{q})).$$

<u>or</u> letting $\bar{p} = F(p)$, $\bar{q} = F(q)$,

$$\bar{\rho}(F(p), F(q)) \geq \rho(p, q).$$

Therefore, $\rho(p,q) = \bar{\rho}(F(p), F(q))$ for all $p, q \in \bar{M}$. This finishes the proof. ∎

Definition: A local isometry $F: M \to N$ of surfaces is a mapping that preserves dot products of tangent vectors.

Remark If $F: M \to N$ is a local isometry then F_* is an isomorphism ($F_*(v) \cdot F_*(w) = v \cdot w \Rightarrow \|F_*(v)\|^2 = \|v\|^2$ so that if $F_*(v) = 0$ then $v = 0$).

Hence, any local isomorphism is a local diffeomorphism.

Lemma: Let $F: M \to N$ be any smooth mapping. For each patch $x: D \to M$, consider the composite mapping $\bar{x}: F(x): D \to N$. Then F is a local isometry if and only if for each patch x

we have $E=\bar{E}$, $F=\bar{F}$, $G=\bar{G}$.

(Here, $\bar{x}$ need not to be a patch)

Proof: $\bar{x}(u,v)=F(x(u,v))$ and

thus $\bar{x}_u = F_*(x_u)$ and $\bar{x}_v = F_*(x_v)$.

So if F is an isometry then

$$\bar{E} = \bar{x}_u \cdot \bar{x}_u = F_*(x_u)\cdot F_*(x_u)$$
$$= x_u \cdot x_u = E.$$

Similarly, $\bar{F}=F$ and $\bar{G}=G$.

Conversely, let $E=\bar{E}$, $F=\bar{F}$ and

$G=\bar{G}$. If $w_1, w_2 \in T_pM$ then

$w_1 = a_1 x_u + b_1 x_v$ and

$w_2 = a_2 x_u + b_2 x_v$, for some

$a_i, b_i \in \mathbb{R}$ because $\{x_u, x_v\}$ is a

basis for T_pM. Now

$$F_*(w_1)\cdot F_*(w_2) = F_*(a_1 x_u + b_1 x_v)\cdot F_*(a_2 x_u + b_2 x_v)$$

$$= a_1 a_2 F_*(x_u)\cdot F_*(x_u) +$$

$(a_1 b_2 + a_2 b_1)\, F_*(x_u) \cdot F_*(x_v)$

$+\, b_1 b_2 F_*(x_v) \cdot F_*(x_v)$

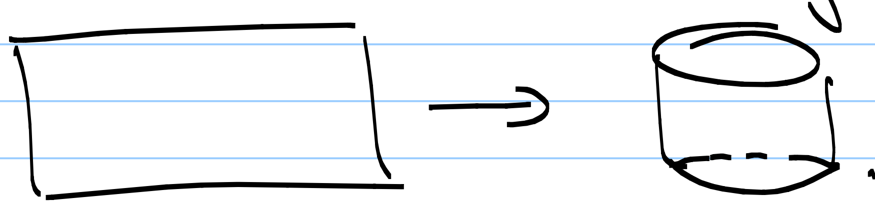
$$= a_1 a_2 x_u \cdot x_u + (a_1 b_2 + a_2 b_1)\, x_u \cdot x_v + b_1 b_2\, x_v \cdot x_v$$

$$= (a_1 x_u + b_1 x_v) \cdot (a_2 x_u + b_2 x_v)$$

$$= w_1 \cdot w_2 .$$

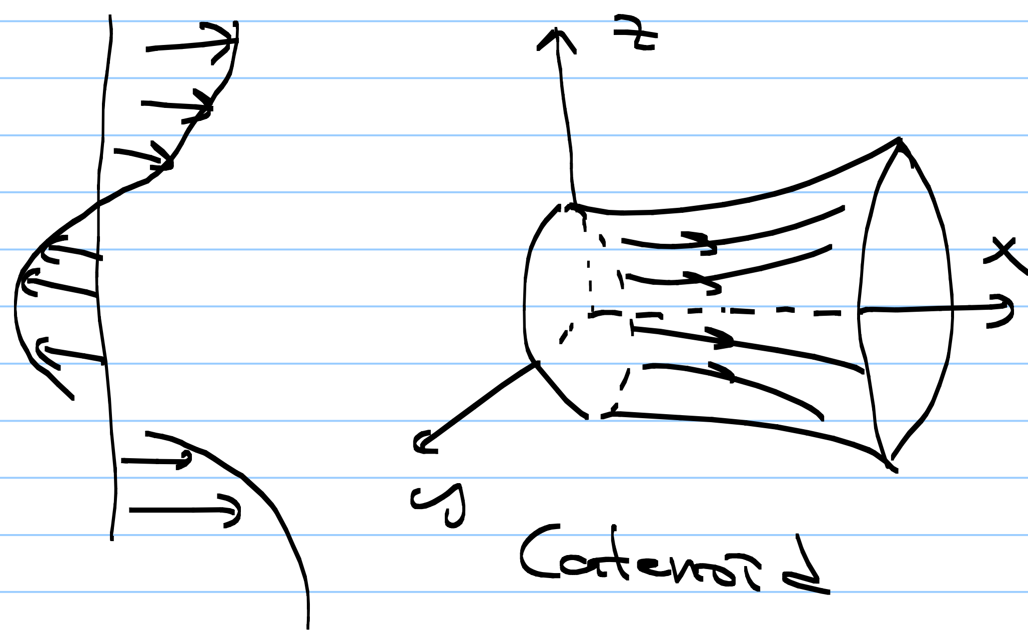
Hence, F is a local isometry. $\blacksquare$

<u>Ex</u> 1) Let $M : x^2 + y^2 = r^2$ be a cyclinder and $x : \mathbb{R}^2 \to M$ be the parametrization given by $x(u,v) = (r\cos \frac{u}{r}, r\sin\frac{u}{r}, v)$. We've computed before that $E = 1$, $F = 0$ and $G = 1$. Thus x is a local isometry



2) Local Isometry of a helicoid onto a catenoid.

Let $x(u,v) = (u\cos v, u\sin v, v)$ be a patch for the helicoid and $y(u,v) = (g(u), h(u)\cos v, h(u)\sin v)$ where $g(u) = \sinh^{-1} u$ and $h(u) = \sqrt{1+u^2}$ a parametrization for the catenoid:



Helicoid

One can see easily that $F(x(u,v)) = y(u,v)$ is an isometry with $E=\bar{E}=1$, $F=\bar{F}=0$ and $G=\bar{G}=1+u^2=h^2(u,v)$.

F just wraps the helicoid around the catenoid.

<u>Definition:</u> A mapping $F: M \to N$ of surfaces is conformal if there is a smooth real valued function $\lambda: M \to (0,\infty)$ so that $\|F_*(v_p)\| = \lambda(p)\,\|v_p\|$ for all $v_p \in T_pM$ and $p \in M$.

§6.5. Intrinsic Geometry of Surfaces in $\mathbb{R}^3$.

<u>Lemma</u>: The connection form $\omega_{12} = -\omega_{21}$ is the only 1-form that satisfies the first structural equations

$$d\theta_1 = \omega_{12} \wedge \theta_2, \quad d\theta_2 = \omega_{21} \wedge \theta_1.$$

<u>Proof</u>: $d\theta_1(E_1, E_2) = \omega_{12}(E_1)\, \theta_2(E_2) - \omega_{12}(E_2)\, \theta_2(E_1)$

$\Rightarrow d\theta_1(E_1, E_2) = \omega_{12}(E_1)$ and

$d\theta_2(E_1, E_2) = \omega_{21}(E_1)\, \theta_1(E_2) - \omega_{21}(E_2)\, \theta_1(E_1)$

$\Rightarrow d\theta_2(E_1, E_2) = \omega_{21}(E_2)$.

So, $\omega_{12}(E_1) = d\theta_1(E_1, E_2)$ and

$$\omega_{12}(E_2) = d\theta_2(E_1, E_2).$$

Since any 1-form is determined by its values on a basis the proof finishes. ∎

Lemma: Let $F: M \to \overline{M}$ be an isometry, and let E_1, E_2 be a tangent frame field on M. ~~If~~ $\overline{E}_1, \overline{E}_2$ is the transferred frame field on $\overline{M}$

1) $\theta_1 = F^*(\overline{\theta}_1)$, $\theta_2 = F^*(\overline{\theta}_2)$;

2) $\omega_{12} = F^*(\overline{\omega}_{12})$.

Here $\overline{E}_1 = F_*(E_1)$ and $\overline{E}_2 = F_*(E_2)$.

Proof $F^*(\overline{\theta}_i)(E_j) = \overline{\theta}_i(F_*(E_j))$

$$= \overline{\theta}_i(\overline{E}_j)$$

$$= \delta_{ij}.$$

Hence, $\theta_i = F^*(\overline{\theta}_i)$.

This proves (1).

By the previous lemma it is enough to show that

Since $\boxed{d\bar{\theta}_1 = \bar{\omega}_{12} \wedge \bar{\theta}_2}$ and $\boxed{d\bar{\theta}_2 = \bar{\omega}_{21} \wedge \bar{\theta}_1}$ we have

$$F^*(d\bar{\theta}_1) = F^*(\bar{\omega}_{12} \wedge \bar{\theta}_2)$$

$$\Rightarrow d(F^*\bar{\theta}_1) = F^*(\bar{\omega}_{12}) \wedge F^*(\bar{\theta}_2)$$

$$\Rightarrow \boxed{d\theta_1 = F^*(\bar{\omega}_{12}) \wedge \theta_2}$$ and similarly

$$\boxed{d\theta_2 = F^*(\bar{\omega}_{21}) \wedge \theta_1.}$$

Now by the previous lemma $F^*(\bar{\omega}_{12}) = \omega_{12}$.

Theorem (Gauss's Theorema Egregium)

Gaussian curvature is an isometric invariant. Explicitly if $F: M \to \bar{M}$ is an isometry then $K(p) = \bar{K}(F(p))$, for all $p \in M$.

Proof: Let $p \in M$ be any point and E_1, E_2 a tangent frame at p. Also let $\bar{E}_i = F_*(E_i), i = 1, 2$. By the previous lemma $F_*(\bar{\omega}_{12}) = \omega_{12}$. On the other hand, we know by §6.2

$$d\bar{\omega}_{12} = -\bar{K}\,\bar{\theta}_1 \wedge \bar{\theta}_2.$$

Thus

$$F^*(d\bar{\omega}_{12}) = F^*(-\bar{K}\,\bar{\theta}_1 \wedge \bar{\theta}_2)$$

$$d(F^*\bar{\omega}_{12}) = -\bar{K}(F)\, F^*(\bar{\theta}_1 \wedge \bar{\theta}_2)$$

$$d\omega_{12} = -\bar{K}(F)\, F^*(\bar{\theta}_1) \wedge F^*(\bar{\theta}_2)$$

$$d\omega_{12} = -\bar{K}(F)\, \theta_1 \wedge \theta_2$$

However, since $d\omega_{12} = -K\,\theta_1 \wedge \theta_2$ we see that $K = \bar{K}(F)$.

Corollary Any part of a sphere is <u>not</u> isometric to any part of a plane.

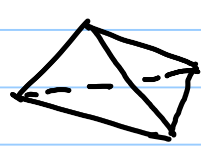
Gauss–Bonnet Theorem:

If Σ_g is compact genus g surface in $\mathbb{R}^3$ then

$$\int_{\Sigma_g} K(p)\, dS = 4\pi(g-1),$$

where $K(p)$ is the Gaussian curvature of Σ_g and dS is the area form on Σ_g.

Proof Proof uses a topological fact: Let Σ_g be triangulated so that the triangulation has v vertices, e edges and f faces. Then $v-e+f = 2-2g$. This number is called the Euler characteristic of the surface Σ_g and denoted as $\chi(\Sigma_g)$.

Ex  Tetrahedron may be regarded as a triangulation of the sphere S^2. In this case, $v=4$, $e=6$ and $f=4$. Then

$$\chi(S^2)=v-e+f=4-6+4=2$$

Also we need a geometric fact about integrals of the Gaussian curvature:

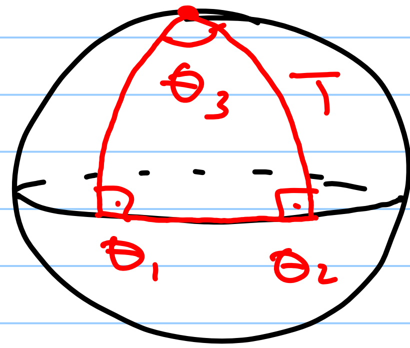
<u>Theorem:</u> Let T be a triangle on a surface $\Sigma\subseteq\mathbb{R}^3$ so that the edges of T are geodesic curves on Σ. Then

$$\int_T K(p)\,dS=\theta_1+\theta_2+\theta_3-\pi,$$

where θ_i's are the interior angles of the triangle T.

We'll use this fact without proof.

Ex:



$K(p) = \frac{1}{r^2}$

Let Σ be a sphere of radius r.

Then $K(p) = \frac{1}{r^2}$ is the constant function.

So by the above fact

$$\theta_1 + \theta_2 + \theta_3 - \pi = \int_T K(p)\, dS$$

$$= \int_T \frac{1}{r^2}\, dS$$

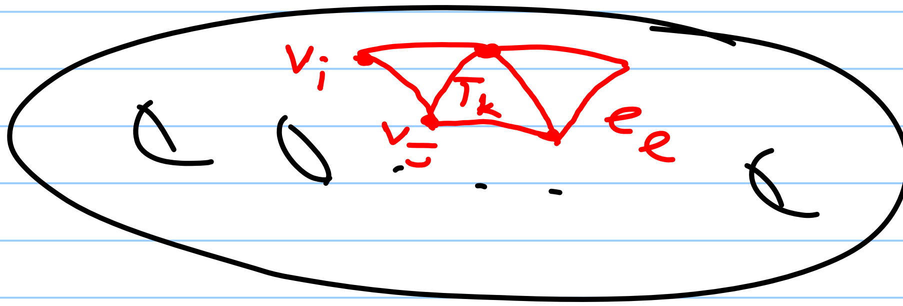
$$= \frac{1}{r^2} \int_T dS$$

$$= \frac{1}{r^2} \text{Area}(T)$$

$$\Rightarrow \text{Area}(T) = \left(\frac{\pi}{2} + \frac{\pi}{2} + \theta_3 - \pi\right) r^2$$

$$= \theta_3 r^2.$$

Consider the surface Σ_g with a geodesic triangulation:



Let the number of vertices, edges and faces of the triangulation are v, e and f, respectively.

By the topological fact we stated above $v-e+f=\chi(\Sigma_g)=2-2g$.
Let $T_1,\dots,T_f$ be the list of all triangles in the triangulation.
Note that Σ_g is the union of the triangles: $\Sigma_g=T_1\cup T_2\cup\dots\cup T_f$.
Also let $\theta_1^i,\theta_2^i,\theta_3^i$ be the interior angles of the triangle T_i.

Now, $\displaystyle\int_{\Sigma_g} \mathcal{K}(p)\, dS = \int_{\bigcup_{i=1}^{f} \overline{T_i}} K(p)\, dS$

$$= \sum_{i=1}^{f} \int_{\overline{T_i}} \mathcal{K}(p)\, dS$$

$$= \sum_{i=1}^{f} \left(\theta_1^i + \theta_2^i + \theta_3^i - \pi\right)$$

$$= -f\pi + \sum_{i=1}^{f} \theta_1^i + \theta_2^i + \theta_3^i$$

The above sum is the sum of interior angles of all the triangles. That sum is clearly $2\pi \cdot V$, where V is the number of all vertices. Hence, we have

$$\int_{\Sigma_g} \mathcal{K}(p)\, dS = -f\pi + 2\pi V.$$

Finally, we make the following observation: Each triangle has 3 edges and every edge is the edge of exactly two triangles. Hence, $3f = 2e$. So

$$\int_{\Sigma_g} \mathcal{K}(p)\,dS = -f\pi + 2\pi v$$

$$= 2\pi\left(v - \frac{f}{2}\right)$$

$$= 2\pi\left(v - e + e - \frac{f}{2}\right)$$

$$= 2\pi\left(v - e + \frac{3f}{2} - \frac{f}{2}\right)$$

$$= 2\pi(v - e + f)$$

$$= 2\pi\,\chi(\Sigma_g).$$

This finishes the proof. ∎