HOMOLOGY OF NON ORIENTABLE REAL ALGEBRAIC VARIETIES

YILDIRAY OZAN

ABSTRACT. Let R be any commutative ring with unity and X a nonsingular compact real algebraic variety with a nonsingular projective complexification $i : X \to X_{\mathbb{C}}$. For a topological component X_0 of X we define $KH_*(X_0, R)$ as the kernel of the induced homomorphism $i_* : H_*(X_0, R) \to H_*(X_{\mathbb{C}}, R)$ and $ImH^*(X_0, R)$ as the image of the homomorphism $i^* : H^*(X_{\mathbb{C}}, R) \to H^*(X_0, R)$. In [6] the author showed that both $KH_*(X_0, R)$ and $ImH^*(X_0, R)$ are independent of the complexification $X \subseteq X_{\mathbb{C}}$ and thus (entire rational) isomorphism invariants of X provided that X_0 is R-orientable. In this note the same result is proved for non R-orientable X_0 under the assumption that $2 \in R$ is a unit. We have also some partial results for $R = \mathbb{Z}$.

1. INTRODUCTION AND THE RESULTS

Let R be any commutative ring with unity. Let X be a nonsingular compact real algebraic variety and $i : X \to X_{\mathbb{C}}$ be the inclusion map into some nonsingular projective complexification. Define $KH_*(X, R)$ as the kernel of the induced map

$$i_* : H_*(X, R) \to H_*(X_{\mathbb{C}}, R)$$

on homology and $ImH^*(X, R)$ as the image of the induced map

$$i^* : H^*(X_{\mathbb{C}}, R) \to H^*(X, R).$$

In [6] it is shown that if X is R-orientable then both $KH_*(X, R)$ and $ImH^*(X, R)$ are independent of the complexification $i : X \to X_{\mathbb{C}}$ and thus are (entire rational) isomorphism invariants of X (see also [3]). Indeed the proof of this result enables us to define $KH_*(X_0, R)$ and $ImH^*(X_0, R)$ for any R-orientable (metric) topological component X_0 of the underlying smooth manifold X. In other words, $KH_*(X_0, R)$ and $ImH^*(X_0, R)$ are independent of the complexification as long as X_0 is R-orientable.

Below is the main result of this note, which extends this result to non R-orientable varieties.

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Theorem 1.1. *Let X_0 be a topological component of any compact nonsingular real algebraic variety X and R is a commutative ring with unity. Then, both $KH_*(X_0, R)$ and $ImH^*(X_0, R)$ are independent of the choice of the smooth projective complexification $i : X \to X_{\mathbb{C}}$ provided that either X_0 is R-orientable or R contains 2 as a unit.*

Even though this theorem excludes the case $R = \mathbb{Z}$ for nonorientable topological component X_0, we have some results in this case also. For any positive integer n consider the exact sequence of Abelian groups

$$0 \to \mathbb{Z} \stackrel{\times n}{\to} \mathbb{Z} \to \mathbb{Z}_n \to 0$$

and the corresponding Bockstein exact sequences for cohomology and homology (cf. see [4])

$$\cdots \to H^{i-1}(X_0, \mathbb{Z}_n) \stackrel{\beta}{\to} H^i(X_0, \mathbb{Z}) \stackrel{\times n}{\to} H^i(X_0, \mathbb{Z}) \to H^i(X_0, \mathbb{Z}_n) \stackrel{\beta}{\to} \cdots$$

and

$$\cdots \to H_{i+1}(X_0, \mathbb{Z}_n) \stackrel{\beta}{\to} H_i(X_0, \mathbb{Z}) \stackrel{\times n}{\to} H_i(X_0, \mathbb{Z}) \to H_i(X_0, \mathbb{Z}_n) \stackrel{\beta}{\to} \cdots .$$

Theorem 1.2. *Let X_0 be a nonorientable topological component of a compact nonsingular real algebraic variety X. Then, $ImH^i(X_0, \mathbb{Z}) \otimes \mathbb{Q}$ and the image of the Bockstein homomorphism restricted to $ImH^{i-1}(X_0, \mathbb{Z}_n)$, $\beta(ImH^{i-1}(X_0, \mathbb{Z}_n))$, which is a subgroup of n-torsion elements in $H^i(X_0, \mathbb{Z})$, are independent of the complexification $i : X \to X_{\mathbb{C}}$, provided that $n = 2$ or is a positive odd integer.*

Similarly for homology, $KH_i(X_0, \mathbb{Z}) \otimes \mathbb{Q}$ and the image of the Bockstein homomorphism restricted to $KH_{i+1}(X_0, \mathbb{Z}_n)$, $\beta(KH_{i+1}(X_0, \mathbb{Z}_n))$, which is a subgroup of n-torsion elements in $H_i(X_0, \mathbb{Z})$, are independent of the complexification $i : X \to X_{\mathbb{C}}$, provided that $n = 2$ or is a positive odd integer.

2. Proofs

All real algebraic varieties under consideration in this report are nonsingular. It is well known that real projective varieties are affine (Proposition 2.4.1 of [1] or Theorem 3.4.4 of [2]). Moreover, compact affine real algebraic varieties are projective (Corollary 2.5.14 of [1]) and therefore, we will not distinguish between real compact affine varieties and real projective varieties.

For real algebraic varieties $X \subseteq \mathbb{R}^r$ and $Y \subseteq \mathbb{R}^s$ a map $F : X \to Y$ is said to be entire rational if there exist $f_i, g_i \in \mathbb{R}[x_1, \ldots, x_r]$, $i = 1, \ldots, s$, such that each g_i vanishes nowhere on X and $F = (f_1/g_1, \ldots, f_s/g_s)$. We say X and Y are isomorphic if there are entire rational maps $F : X \to Y$ and $G : Y \to X$ such that $F \circ G = id_Y$ and $G \circ F = id_X$. Isomorphic algebraic varieties will be regarded the same. We refer the reader for the basic definitions and facts about real algebraic geometry to [1, 2].

We will only prove the statements of the above theorems involving cohomology, because proof of the statements about homology are completely analogous.

Let X_0 be a nonorientable topological component of a nonsingular real algebraic variety X. Since X_0 is nonorientable so is X. The smooth orientation double cover of X is diffeomorphic to a nonsingular real algebraic variety $\tilde{X}$, possibly not unique, on which the corresponding $\mathbb{Z}_2$ deck transformation group acts algebraically and the quotient map $p : \tilde{X} \to X$ is entire rational. This can be seen as follows: The determinant line bundle of X is (strongly) algebraic and is non trivial on X_0. Hence, the $f : X \to \mathbb{R}P^N$ be an entire rational map classifying this line bundle. Now, $\tilde{X} \to X$ can be taken to be the pull back of the algebraic double covering $S^N \to \mathbb{R}P^N$ (cf. see [7, 8]). Now, we have the following result:

Lemma 2.1. *Assume that X is a nonsingular compact real algebraic variety and X_0 a nonorientable topological component of X. Let $i : X \to X_{\mathbb{C}}$ be any smooth projective complexification and $p : \widetilde{X} \to X$ any real algebraic orientation double cover as above. Then if R contains 2 as a unit then $(p^*)^{-1}(ImH^*(\widetilde{X}, R)) = ImH^*(X_0, R)$. Moreover, $ImH^*(X_0, R)$ is independent of the complexification.*

One can easily state and prove the above lemma for homology.

Proof. Let $i : X \to X_{\mathbb{C}}$ be any fixed smooth projective complexification. Then the entire rational covering map $p : \widetilde{X} \to X$ will extend to a rational map, and after blowing up some smooth centers away from the real locus, to some smooth projective complexification $p_{\mathbb{C}} : \widetilde{X_{\mathbb{C}}} \to X_{\mathbb{C}}$ (possibly a branched double covering projection) making the diagram below commutative:

$$\begin{array}{ccc} \widetilde{X_0} & \xrightarrow{j} & \widetilde{X_{\mathbb{C}}} \\ p \downarrow & & \downarrow p_{\mathbb{C}} \\ X_0 & \xrightarrow{i} & X_{\mathbb{C}} \end{array}$$

where $\widetilde{X_0}$ is $p^{-1}(X_0)$, clearly a topological component of $\widetilde{X}$. The $\mathbb{Z}_2$ action on $\widetilde{X}$ extends to an algebraic action on $\widetilde{X_{\mathbb{C}}}$ so that $p_{\mathbb{C}} : \widetilde{X_{\mathbb{C}}} \to X_{\mathbb{C}}$ is a, possibly branched, double covering, and the vertical maps are equivariant. This diagram yields the following commutative diagram

$$\begin{array}{ccc} H^i(\widetilde{X_{\mathbb{C}}}, R) & \xrightarrow{j^*} & H^i(\widetilde{X_0}, R) \\ p_{\mathbb{C}}^* \uparrow & & \uparrow p^* \\ H^i(X_{\mathbb{C}}, R) & \xrightarrow{i^*} & H^i(X_0, R). \end{array}$$

Since the smooth projective complexification $i : X \to X_{\mathbb{C}}$ is arbitrary the commutativity of the above diagram implies that

$$ImH^*(X_0, R) \subseteq (p^*)^{-1}(ImH^*(\widetilde{X_0}, R)).$$

To see the other inclusion, let τ and $\tau_{\mathbb{C}}$ denote the involutions of the $\mathbb{Z}_2$-actions on $\widetilde{X}$ and $\widetilde{X_{\mathbb{C}}}$, respectively. It is well known that the vertical maps in the above diagrams are injective with images $H^i(\widetilde{X_0},R)^{\tau^*}$ and $H^i(\widetilde{X_{\mathbb{C}}},R)^{\tau^*_{\mathbb{C}}}$, the subgroups of invariant classes (cf. see page 193 of [5]).

Note that to finish the proof of the first assertion we need to prove the following : Let $a \in H^i(X_0,R)$ be such that $p^*(a) \in ImH^i(\widetilde{X_0},R)$. Then $a \in ImH^i(X_0,R)$. To prove this let $b \in H^i(\widetilde{X_{\mathbb{C}}},R)$ be such that $j^*(b) = p^*(a)$. By the above paragraph $2p^*(a) = j^*(b+\tau^*_{\mathbb{C}}(b)) = p^*_{\mathbb{C}}(c)$, for some $c \in H^i(X_{\mathbb{C}},R)$. It follows from the commutativity of the above diagram and the injectivity of p^* that $2a = i^*(c)$. Since 2 is a unit we have $a = i^*(\frac{c}{2}) \subseteq Im(i^*)$.

For the last assertion just note that the complexification $i : X \to X_{\mathbb{C}}$ is arbitrary and independent from the choice of the double covering $p : \widetilde{X} \to X$. □

Note that Theorem 1.1 follows from the above Lemma 2.1.

Proof of Theorem 1.2. Consider the exact sequence of Abelian groups

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/Z \to 0$$

and the corresponding Bockstein exact sequences for X_0 and $X_{\mathbb{C}}$

$$\begin{array}{ccccccccc} \cdots \to & H^{i-1}(X_0,\mathbb{Q}/Z) & \to & H^i(X_0,\mathbb{Z}) & \to & H^i(X_0,\mathbb{Q}) & \to & H^i(X_0,\mathbb{Q}/Z) & \to \cdots \\ & \uparrow i^* & & \uparrow i^* & & \uparrow i^* & & \uparrow i^* & \\ \cdots \to & H^{i-1}(X_{\mathbb{C}},\mathbb{Q}/Z) & \to & H^i(X_{\mathbb{C}},\mathbb{Z}) & \to & H^i(X_{\mathbb{C}},\mathbb{Q}) & \to & H^i(X_{\mathbb{C}},\mathbb{Q}/Z) & \to \cdots . \end{array}$$

Tensoring the above sequences with $\mathbb{Q}$ and using Theorem 1.1 we get that $ImH^i(X_0,\mathbb{Z}) \otimes \mathbb{Q}$ is independent of the complexification $i : X \to X_{\mathbb{C}}$.

Now let us concentrate on torsion elements in $H^i(X_0,\mathbb{Z})$. Let n be a positive odd integer. Then 2 is a unit in $\mathbb{Z}_n$. Considering a similar diagram as above corresponding to the short exact sequence of Abelian groups

$$0 \to \mathbb{Z} \stackrel{\times n}{\to} \mathbb{Z} \to \mathbb{Z}_n \to 0$$

and the corresponding Bockstein sequences

$$\begin{array}{ccccccccc} \cdots \to & H^{i-1}(X_0,\mathbb{Z}_n) & \stackrel{\beta}{\to} & H^i(X_0,\mathbb{Z}) & \stackrel{\times n}{\to} & H^i(X_0,\mathbb{Z}) & \to & H^i(X_0,\mathbb{Z}_n) & \to \cdots \\ & \uparrow i^* & & \uparrow i^* & & \uparrow i^* & & \uparrow i^* & \\ \cdots \to & H^{i-1}(X_{\mathbb{C}},\mathbb{Z}_n) & \stackrel{\beta}{\to} & H^i(X_{\mathbb{C}},\mathbb{Z}) & \stackrel{\times n}{\to} & H^i(X_{\mathbb{C}},\mathbb{Z}) & \to & H^i(X_{\mathbb{C}},\mathbb{Z}_n) & \to \cdots \end{array}$$

we deduce that the image of the Bockstein homomorphism restricted to $ImH^{i-1}(X_0,\mathbb{Z}_n)$, $\beta(ImH^{i-1}(X_0,\mathbb{Z}_n))$, which is a subgroup of n-torsion elements in $H^i(X_0,\mathbb{Z})$, is independent of the complexification $i : X \to X_{\mathbb{C}}$ by Theorem 1.1. On the other hand, since $\mathbb{Z}_2$ is a field $ImH^i(X_0,\mathbb{Z}_2)$ is also independent of the complexification ([6]). Now using the exact sequence

$$0 \to \mathbb{Z} \stackrel{\times 2}{\to} \mathbb{Z} \to \mathbb{Z}_2 \to 0,$$

in a similar fashion, we see that the image of the Bockstein homomorphism restricted to $\beta(ImH^{i-1}(X_0,\mathbb{Z}_2))$, which is a subgroup of 2-torsion elements in $H^i(X_0,\mathbb{Z})$, is also independent of the complexification $i: X \to X_{\mathbb{C}}$. $\square$

References

[1] Akbulut, S., King, H.: Topology of real algebraic sets, M.S.R.I. book series, Springer-Verlag, New York, 1992

[2] Bochnak, J., Coste, M., Roy, M. F.: Real Algebraic Geometry, Ergebnisse der Math. vol. 36, Springer-Verlag, Berlin, 1998

[3] Bochnak, J., Kucharz, W.: Complexification of real algebraic varieties and vanishing of homology classes. *Bull. London Math. Soc.* **33**, 32-40 (2001)

[4] Bredon, G. E.: Topology and Geometry, Springer-Verlag, New York, 1993

[5] Dieck T. tom : Transformation groups, Walter de Gruyter, Berlin, 1987

[6] Ozan, Y.: On homology of real algebraic varieties. *Proc. Amer. Math. Soc.* **129**, 3167-3175 (2001)

[7] ———, Quotients of real algebraic sets via finite groups. *Turkish J. Math.* **21**, 493-499 (1997)

[8] ———, Real algebraic principal abelian fibrations, *Contemp. Math.* **182**, 121-133 (1995)

Department of Mathematics, Middle East Technical University, 06531 Ankara, Turkey

E-mail address: ozan@metu.edu.tr