

Gödel-type metrics in various dimensions

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Abstract

Gödel-type metrics are introduced and used in producing charged dust solutions in various dimensions. The key ingredient is a $(D-1)$ -dimensional Riemannian geometry which is then employed in constructing solutions to the Einstein–Maxwell field equations with a dust distribution in D dimensions. The only essential field equation in the procedure turns out to be the source-free Maxwell’s equation in the relevant background. Similarly the geodesics of this type of metric are described by the Lorentz force equation for a charged particle in the lower dimensional geometry. It is explicitly shown with several examples that Gödel-type metrics can be used in obtaining exact solutions to various supergravity theories and in constructing spacetimes that contain both closed timelike and closed null curves and that contain neither of these. Among the solutions that can be established using non-flat backgrounds, such as the Tangherlini metrics in $(D-1)$ -dimensions, there exists a class which can be interpreted as describing black-hole-type objects in a Gödel-like universe.

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1. Introduction

Gödel’s metric [1] in general relativity is the solution of Einstein’s field equations with homogeneous perfect fluid distribution having G_5 maximal symmetry [2]. This spacetime admits closed timelike and closed null curves but contains no closed timelike and null geodesics [3]. The Gödel universe is geodesically complete, and does not contain any singularities or horizons. There have been several attempts to generalize the Gödel metric in classical general relativity [4–7]. The main goal of these works has been the elimination of closed timelike and closed null curves.

We call a metric in D dimensions a *Gödel-type metric* if it can be written in the form $g_{\mu\nu} = h_{\mu\nu} - u_\mu u_\nu$ where u^μ is a timelike unit vector and $h_{\mu\nu}$ is a degenerate matrix of

rank $D - 1$ with the additional condition that $h_{\mu\nu}$ be the metric of an Einstein space of a $(D - 1)$ -dimensional Riemannian geometry.

In fact, taken at face value, such a decomposition of spacetime metrics has of course been adopted by several researchers with various aims. These are generally called $3 + 1$ decompositions in general relativity. One well-known work is due to Geroch [8] in $D = 4$ where our u^μ is taken as $u^\mu = \xi^\mu / \sqrt{|\lambda|}$ in which ξ^μ is a Killing vector field to start with and $\lambda = \xi_\mu \xi^\mu$. However, Geroch does not put any restrictions on the three-dimensional metric $h_{\mu\nu}$ unlike our case. Geroch reduces the vacuum Einstein's field equations to a scalar, complex, Ernst-type nonlinear differential equation and develops a technique for generating new solutions of the vacuum Einstein field equations from vacuum spacetimes. Although the Gödel-type metrics we define and use here are of the same type, it must be kept in mind that our $h_{\mu\nu}$ is the metric of an Einstein space of a $(D - 1)$ -dimensional Riemannian geometry. We also do not assume u^μ to be a Killing vector field to start with, but with the other restrictions we impose it turns out to be one. Another major difference is that we look for all possible D -dimensional Gödel-type metrics, and hence u^μ vectors, that produce physically acceptable matter content for Einstein's field equations.

Metrics of this form also look like the well-known Kerr–Schild metrics of classical general relativity [9] which have $g_{\mu\nu} = \eta_{\mu\nu} - \ell_\mu \ell_\nu$ for a null vector ℓ^μ and which we have recently used in constructing accelerated Kerr–Schild geometries for the Einstein–Maxwell null dust [10], Einstein–Born–Infeld null dust field equations [11], and their extensions with a cosmological constant and respective zero acceleration limits [12].

Remarkably the very form of the Gödel-type metrics is also reminiscent of the metrics used in Kaluza–Klein reductions in string theories [13]. However, as will be apparent in the subsequent sections, Gödel-type metrics have a number of characteristics that distinguish them from the Kaluza–Klein metrics. Here the background metric $h_{\mu\nu}$ is taken as positive definite whereas in the Kaluza–Klein case it must be locally Lorentzian. Moreover, contrary to what is done in the Kaluza–Klein mechanism, the Gödel-type metrics are used in obtaining a D -dimensional theory starting from a $(D - 1)$ -dimensional one. The D -dimensional timelike vector u^μ is used in the construction of a Maxwell theory in D dimensions unlike the Kaluza–Klein vector potential which lives and defines a Maxwell theory in $D - 1$ dimensions. Even though Gödel-type metrics are akin to metrics employed in the Kaluza–Klein mechanism at face value, the applications we present here should make their real worth clear and should help in contrasting them with Kaluza–Klein metrics.

Gödel-type metrics also show up in supergravity theories in some dimensions. A special class of Gödel-type metrics is known to be the T-dual of the pp-wave metrics in string theory [14–16]. These metrics are all supersymmetric but contain closed timelike and closed null curves and thus violate causality [17–21]. Recently there has been an attempt to remedy this problem by introducing observer-dependent holographic screens [15, 22]. In [23], a new class of supergravity solutions has been constructed which locally look like the Gödel universe inside a domain wall made out of supertubes and which do not contain any closed timelike curves. There have also been studies that describe black holes embedded in Gödel spacetimes [17, 19, 24] and brane-world generalizations of the Gödel universe [25].

In this work, we consider Gödel-type metrics in a D -dimensional spacetime manifold M . We show that in all dimensions the Einstein equations are classically equivalent to the field equations of general relativity with a charged dust source provided that a simple $(D - 1)$ -dimensional Euclidean source-free Maxwell's equation is satisfied. The energy density of the dust fluid is proportional to the Maxwell invariant F^2 . We next show that the geodesics of the Gödel-type metrics are described by solutions of the $(D - 1)$ -dimensional Euclidean Lorentz force equation for a charged particle. We then discuss the possible existence of examples

of spacetimes containing closed timelike and closed null curves which violate causality and examples of spacetimes without any closed timelike or closed null curves where causality is preserved. We show that the Gödel-type metrics we introduce provide exact solutions to various kinds of supergravity theories in five, six, eight, ten and eleven dimensions. All these exact solutions are based on the vector field u_μ which satisfies the $(D - 1)$ -dimensional Maxwell's equation in the background of some $(D - 1)$ -dimensional Riemannian geometry with metric $h_{\mu\nu}$. In this respect, we do not give only a specific solution but in fact provide a whole class of exact solutions to each of the aforementioned theories. We construct some explicit examples when $h_{\mu\nu}$ is trivially flat, i.e. the identity matrix of $D - 1$ dimensions.

We next consider an interesting class of the Gödel-type metrics by taking a $(D - 1)$ -dimensional non-flat background $h_{\mu\nu}$. We specifically consider the cases when the background $h_{\mu\nu}$ is conformally flat, an Einstein space and, as a subclass, a Riemannian Tangherlini solution. We explicitly construct such examples for $D = 4$ even though these can be generalized to dimensions $D > 4$ as well. When the background is an Einstein space, the corresponding source for the Einstein equations in D dimensions turns out to be a charged perfect fluid with pressure density $p = \frac{1}{2}(3 - D)\Lambda$ (so that $p > 0$ when $\Lambda < 0$) and energy density $\rho = \frac{1}{4}f^2 + \frac{1}{2}(D - 1)\Lambda$, where Λ is the cosmological constant and f^2 denotes the Maxwell invariant. We also discuss the existence of closed timelike and closed null curves in this class of spacetimes and explicitly construct geometries with and without such curves in $D = 4$. We show that when the background is a Riemannian Tangherlini space, the D -dimensional solution turns out to describe a black-hole-type object depending on the parameters. We then finish off with our conclusions and a discussion of possible future work.

2. Gödel-type metrics

Let M be a D -dimensional manifold with a metric of the form

$$g_{\mu\nu} = h_{\mu\nu} - u_\mu u_\nu. \quad (1)$$

Here $h_{\mu\nu}$ is a degenerate $D \times D$ matrix with rank equal to $D - 1$. We assume that the degeneracy of $h_{\mu\nu}$ is caused by taking $h_{k\mu} = 0$, where x^k is a fixed coordinate with $0 \leq k \leq D - 1$ (note that x^k does not necessarily have to be spatial), and by keeping the rest of $h_{\mu\nu}$, i.e. $\mu \neq k$ or $\nu \neq k$, dependent on all the coordinates x^α except x^k so that $\partial_k h_{\mu\nu} = 0$. Hence, in the most general case, 'the background' $h_{\mu\nu}$ can effectively be thought of as the metric of a $(D - 1)$ -dimensional non-flat spacetime. As for u^μ , we assume that it is a timelike unit vector, $u_\mu u^\mu = -1$, and that u_μ is independent of the fixed special coordinate x^k , i.e. $\partial_k u_\mu = 0$. These imply that one can take $u^\mu = -\frac{1}{u_k} \delta_k^\mu$.

Now the question we ask is as follows: let us start with a metric of form (1) and calculate its Einstein tensor. Can the Einstein tensor be interpreted as describing the energy momentum tensor of a physically acceptable source? Does one need further assumptions on $h_{\mu\nu}$ and/or u_μ so that 'the left-hand side' of $G_{\mu\nu} \sim T_{\mu\nu}$ can be thought of as giving an acceptable 'right-hand side', i.e. corresponding to a physically reasonable matter source? As you will see in the subsequent sections, the answer is 'yes' provided that one further demands $h_{\mu\nu}$ to be the metric of an Einstein space of a $(D - 1)$ -dimensional Riemannian geometry. We call such a metric $g_{\mu\nu}$ a *Gödel-type of metric*. The sole reason we use this name is because of the fact that some of the spacetimes we find also have closed timelike curves and some of the supergravity solutions we present have already appeared in the literature with a title referring to Gödel.

In the most general case, $u_k \neq \text{constant}$ and the assumptions we have made so far show that u^μ is not a Killing vector. However if one further takes $u_k = \text{constant}$, then it turns out to be one. Throughout this work we will assume that $u_k = \text{constant}$. We will first consider

the simple case of $h_{\mu\nu}$ being flat. For this case, we will examine what can be said and done in classical general relativity in the remaining parts of this section and investigate how one can use flat backgrounds to find solutions to various supergravity theories in section 3. We will consider the case of non-flat backgrounds later in section 4.

2.1. Solutions of Einstein's equations in flat backgrounds

Throughout the rest of this section and in section 3, we will further assume that $h_{0\mu} = 0$, $h_{ij} = \bar{\delta}_{ij}$, the $(D - 1)$ -dimensional Kronecker delta symbol and $\partial_\alpha h_{\mu\nu} = 0$. We take Greek indices to run from $0, 1, \dots$ to $D - 1$ whereas Latin indices range from 1 to $D - 1$. (Our conventions are similar to the conventions of Hawking and Ellis [3].) The determinant of $g_{\mu\nu}$ is then $g = -u_0^2$ and moreover $u^\mu = -\frac{1}{u_0}\delta_0^\mu$. In what follows, we will also assume that $u_0 = 1$ and that $\partial_0 u_\alpha = 0$.

With these assumptions, it is not hard to show that $u^\mu h_{\mu\nu} = 0$ and the inverse of the metric can be calculated to be

$$g^{\mu\nu} = \bar{h}^{\mu\nu} + (-1 + \bar{h}^{\alpha\beta} u_\alpha u_\beta) u^\mu u^\nu + u^\mu (\bar{h}^{\nu\alpha} u_\alpha) + u^\nu (\bar{h}^{\mu\alpha} u_\alpha). \quad (2)$$

Here $\bar{h}^{\mu\nu}$ is the $(D - 1)$ -dimensional inverse of $h_{\mu\nu}$; i.e. $\bar{h}^{\mu\nu} h_{\nu\alpha} = \bar{\delta}_\alpha^\mu$ with $\bar{\delta}_\alpha^\mu$ denoting the $(D - 1)$ -dimensional Kronecker delta: $\delta_\alpha^\mu = \bar{\delta}_\alpha^\mu + \delta_0^\mu \delta_\alpha^0$.

The Christoffel symbols can now be calculated to be

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}(u_\alpha f^\mu_\beta + u_\beta f^\mu_\alpha) - \frac{1}{2}u^\mu(u_{\alpha,\beta} + u_{\beta,\alpha}) \quad (3)$$

where we have used $f_{\alpha\beta} \equiv u_{\beta,\alpha} - u_{\alpha,\beta}$; $u_{\alpha,\beta} \equiv \partial_\beta u_\alpha$ and $f^\mu_\nu = g^{\mu\alpha} f_{\alpha\nu}$. We will also use a semicolon to denote a covariant derivative with respect to the Christoffel symbols given above; $u_{\alpha;\beta} \equiv \nabla_\beta u_\alpha$. One can easily show that $u^\alpha u_{\beta;\alpha} = 0$ and $u_{\beta;\alpha} = \frac{1}{2}f_{\alpha\beta}$, hence u^μ is tangent to a timelike geodesic curve and is a timelike Killing vector.

The Ricci tensor can be calculated to be

$$R_{\mu\nu} = \frac{1}{2}f_\mu^\alpha f_{\nu\alpha} - \frac{1}{2}(u_\mu j_\nu + u_\nu j_\mu) + \frac{1}{4}f^2 u_\mu u_\nu \quad (4)$$

where we have used $f^2 \equiv f^{\alpha\beta} f_{\alpha\beta}$ and $j_\mu \equiv \partial_\alpha f_\mu^\alpha$. (Note that it is not possible to have $j_\mu = k u_\mu$ for a nontrivial constant k . Suppose the contrary is true, i.e. $j_\mu = k u_\mu$

A few remarks regarding the positivity of energy and the character of the geodesics are in order at this point. For a timelike vector ξ^μ , one has $T_{\mu\nu}^f \xi^\mu \xi^\nu \geq 0$ by the very nature of $T_{\mu\nu}^f$ and since $f_{0\mu} = 0$, one has $f^2 = (f_{ij})^2 \geq 0$ as well. Hence it is readily seen that

$$G_{\mu\nu} \xi^\mu \xi^\nu = T_{\mu\nu}^f \xi^\mu \xi^\nu + \frac{1}{4} f^2 (u_\mu \xi^\mu)^2 \geq 0,$$

for all timelike ξ^μ and the weak energy condition is satisfied for spacetimes described by Gödel-type metrics.

As for the behaviour of the geodesics, let us start by taking a geodesic curve Γ on M which is parametrized as $x^\mu(\tau)$. Using (3) and denoting the derivative with respect to the affine parameter τ by a dot, the geodesic equation yields

$$\ddot{x}^\mu + f^\mu{}_\beta \dot{x}^\beta (u_\alpha \dot{x}^\alpha) - u^\mu \dot{x}^\alpha (u_{\alpha,\beta} \dot{x}^\beta) = 0.$$

Noting that $u_{\alpha,\beta} \dot{x}^\beta = \dot{u}_\alpha$, writing $f^\mu{}_\beta$ explicitly via the inverse of metric (2) and using $u^\alpha f_{\mu\alpha} = 0$, this becomes

$$\ddot{x}^\mu + u_\alpha \dot{x}^\alpha (\bar{h}^{\mu\sigma} + u^\mu \bar{h}^{\sigma\nu} u_\nu) f_{\sigma\beta} \dot{x}^\beta - u^\mu (\dot{u}_\alpha \dot{x}^\alpha) = 0, \tag{9}$$

and contracting this with u_μ , one obtains a constant of motion for the geodesic equation as

$$u_\mu \dot{x}^\mu = \dot{x}^0 + u_i \dot{x}^i = -e = \text{constant}. \tag{10}$$

Meanwhile setting the free index $\mu = i$ in (9), one also finds $\ddot{x}^i - e(\bar{h}^{i\sigma} f_{\sigma\beta} \dot{x}^\beta) = 0$, or simply

$$\ddot{x}^i = e f_{ij} \dot{x}^j \quad (i = 1, 2, \dots, D - 1), \tag{11}$$

i.e. the $(D - 1)$ -dimensional Euclidean Lorentz force equation for a charged point particle of charge/mass ratio equal to e . Moreover, contracting (11) further by \dot{x}^i , one obtains a second constant of motion $\dot{x}^i \dot{x}^i = \ell^2 = \text{constant}$. Since $g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = h_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - (u_\mu \dot{x}^\mu)^2 = \ell^2 - e^2$, one concludes that the nature of the geodesics necessarily depends on the sign of $\ell^2 - e^2$.

In retrospect, we have shown that the Gödel-type metric (1) solves the Einstein–Maxwell dust field equations in D dimensions provided the flat $(D - 1)$ -dimensional Euclidean source-free Maxwell’s equation (7) holds. Moreover the geodesics of the Gödel-type metrics are described by the $(D - 1)$ -dimensional Euclidean Lorentz force equation (11).

2.2. A special solution to (7)

A solution to (7) is given by the simple choice $u_i = \frac{b}{2} J_{ij} x^j$, where b is a real constant (we keep the $1/2$ factor for later convenience), and J_{ij} is fully antisymmetric with constant components that satisfy

$$J^k{}_j J^i{}_k = -\delta_j^i.$$

(Of course this is only possible when D is odd.) In this case, $f_{ij} = b J_{ij}$ and $f_{0\mu} = 0$ as before. Then $f_\mu{}^\alpha f_{\nu\alpha} = b^2 \delta_{\mu\nu} = b^2 h_{\mu\nu}$ and $f^2 = b^2(D - 1)$. Using (4) and (5) with $j_\mu = 0$, the Einstein tensor can be written as

$$G_{\mu\nu} = \frac{1}{8} b^2 (5 - D) g_{\mu\nu} + \frac{1}{4} b^2 (D + 1) u_\mu u_\nu.$$

This in turn can be interpreted as coming from a perfect fluid source

$$G_{\mu\nu} = T_{\mu\nu} = p g_{\mu\nu} + (p + \rho) u_\mu u_\nu$$

by identifying the pressure p of the fluid as $p = \frac{1}{8} b^2 (5 - D)$ and the mass–energy density ρ with $\rho = \frac{3}{8} b^2 (D - 1)$. Note that in this picture $p = 0$ when $D = 5$ and $p < 0$ when $D > 5$.

Alternatively, one can repeat this analysis by writing

$$h_{\mu\nu} = \frac{1}{b^2} f_\mu{}^\alpha f_{\nu\alpha} \quad \text{and} \quad u_\mu u_\nu = \frac{1}{b^2} f_\mu{}^\alpha f_{\nu\alpha} - g_{\mu\nu}$$

in (4). In this case the Einstein tensor can be written in the form

$$G_{\mu\nu} = \frac{D+1}{4} \left[f_{\mu}{}^{\alpha} f_{\nu\alpha} - \frac{3}{2(D+1)} f^2 g_{\mu\nu} \right].$$

If one is to consider this as an Einstein–Maxwell theory so that $G_{\mu\nu} \sim T_{\mu\nu}^f$, then

$$\frac{3}{2(D+1)} = \frac{1}{4}$$

which yields $D = 5$.

As a result, when $D = 5$ the special solution given above can either be thought of as describing a spacetime filled with dust or as a solution to the Einstein–Maxwell theory. However, in general odd dimensions it can be considered as a solution of Einstein theory coupled with a perfect fluid source where the pressure $p < 0$ when $D > 5$.

2.3. Spacetimes containing closed timelike curves

In this subsection we give a simple solution which corresponds to a spacetime (1) that contains closed timelike or null curves. Here we take $D = 4$ for simplicity but what follows can easily be generalized to higher dimensions.

Obviously the simple choice $u_i = Q_{ij}x^j$, where Q_{ij} is fully antisymmetric with constant components ($i, j = 1, 2, 3$), solves (7). Now let $Q_{13} = Q_{23} = 0$ but $Q_{12} \neq 0$ for simplicity. Then

$$u_{\mu} dx^{\mu} = dt + Q_{12}(x^2 dx^1 - x^1 dx^2),$$

and employing the ordinary cylindrical coordinates (ρ, ϕ, z) this can be written as

$$u_{\mu} dx^{\mu} = dt - Q_{12}\rho^2 d\phi.$$

Using (1), this in turn implies that the line element is

$$ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2 - (dt - Q_{12}\rho^2 d\phi)^2.$$

Consider the curve $C = \{(t, \rho, \phi, z) \mid t = t_0, \rho = \rho_0, z = z_0\}$, where t_0, ρ_0 and z_0 are constants, in the manifold M . The norm of the tangent vector $v^{\mu} = (\partial/\partial\phi)^{\mu}$ to this curve is then

$$v^2 \equiv v_{\mu}v^{\mu} = g_{\phi\phi} = \rho_0^2(1 - (Q_{12})^2\rho_0^2).$$

For a spacelike tangent vector, one has $v^2 > 0$, of course. The spacetime we are studying is obviously homogeneous and there passes a curve such as C from each point of such a spacetime. Since ϕ is a periodic variable with $\phi = 0$ and $\phi = 2\pi$ identified, one then clearly finds that there exist closed timelike and null curves for $\rho_0 \geq 1/|Q_{12}|$ in this spacetime since then $v^2 \leq 0$. One can also show that there exist no closed timelike or null geodesics in this geometry.

2.4. Spacetimes without any closed timelike curves

In this subsection we present a solution which describes a spacetime (1) that does not contain any closed timelike or null curves.

Now let $u_i = s(x^j)\omega_j$ where $\omega^i = \delta^{ij}\omega_j$ is a constant vector and s is a smooth function of the spatial coordinates x^j ($i, j = 1, 2, \dots, D-1$). Hence $f_{ij} = (\partial_i s)\omega_j - (\partial_j s)\omega_i$ and (7) gives

$$\partial_i f_{ij} = (\nabla^2 s)\omega_j - (\partial_i \partial_j s)\omega_i = 0.$$

Then (7) is satisfied if one chooses $\nabla^2 s = 0$ and $\omega_i \partial_i s = \text{constant}$, which can further be set equal to zero.

Now let us take $D = 4$ specifically, but the following discussion can of course be generalized to $D > 4$ as well. As a simple example, choose $\omega_i = \delta_i^3$ above. Then any function s which is harmonic in the (x^1, x^2) variables will do. Now

$$u_\mu dx^\mu = dt + s(x^1, x^2) dx^3,$$

and using the cylindrical coordinates again, the line element becomes

$$ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2 - (dt + s(\rho, \phi) dz)^2.$$

Consider the curve C and its tangent vector v^μ we used in subsection 2.3 again. Now the norm of v^μ is

$$v^2 = g_{\phi\phi} = \rho_0^2,$$

and this is obviously always positive definite, $v^2 > 0$, i.e. v^μ is always spacelike. Hence we see that the closed curve we used in the previous subsection is no longer timelike.

It is worth pointing out that the consideration of a curve of the form $\bar{C} = \{(t, \rho, \phi, z) \mid t = t_0, \rho = \rho_0, \phi = \phi_0, z \in [0, 2\pi)\}$, where t_0, ρ_0 and ϕ_0 are constants, and its tangent vector $\bar{v}^\mu = (\partial/\partial z)^\mu$ gives

$$\bar{v}^2 = g_{zz} = 1 - (s(\rho_0, \phi_0))^2,$$

which at first sight indicates the existence of closed timelike or null curves in this geometry. On the other hand, we do not confine ourselves to the small patch of spacetime where the z coordinate is on S^1 , we are interested in the universal covering of this patch and thus take z to be on the real line \mathbb{R} .

We thus conclude that the solutions we present correspond to spacetimes that contain both closed timelike and null curves and that contain neither of these depending on how one solves (7).

3. Solutions of various supergravity theories with flat backgrounds

In this section, we use the results we have obtained so far in constructing solutions to some supergravity theories in dimensions $D \geq 5$ with flat backgrounds.

3.1. Five dimensions

The bosonic part of the minimal supergravity in $D = 5$ has the following field equations [18, 19]:

$$R_{\mu\nu} = 2(F_{\mu\alpha}F_\nu^\alpha - \frac{1}{6}g_{\mu\nu}F^2) \Leftrightarrow G_{\mu\nu} = 2T_{\mu\nu}^F, \tag{12}$$

$$\nabla_\mu F^{\mu\nu} = \frac{1}{2\sqrt{3}}\eta^{\alpha\beta\gamma\mu\nu}F_{\alpha\beta}F_{\gamma\mu}, \tag{13}$$

where the Levi-Civita tensor η is given in terms of the Levi-Civita tensor density ϵ by $\eta^{\alpha\beta\gamma\mu\nu} = \epsilon^{\alpha\beta\gamma\mu\nu}/\sqrt{-g}$.

Let $A_\mu = bu_\mu$, where b is a real constant. One then has $F^{\mu\nu} = bf^{\mu\nu}$. Now let us concentrate on (13) first. By (8), $\nabla_\mu F^{\mu\nu} = \frac{b}{2}f^2u^\nu$ and since $u^\nu = -\delta_0^\nu$, one finds that

$$\nabla_\mu F^{\mu\nu} = -\frac{b}{2}f^2\delta_0^\nu.$$

Now when $\nu = i$, one of the first four indices of η on the right-hand side of (13) has to be 0 and since $F_{0k} = 0$, (13) is satisfied identically in this case. When $\nu = 0$, $\nabla_\mu F^{\mu 0} = -\frac{b}{2}f^2$. On the other hand, the right-hand side of (13) gives

$$\frac{1}{2\sqrt{3}}\epsilon^{\alpha\beta\gamma\mu 0}F_{\alpha\beta}F_{\gamma\mu} = \frac{b^2}{2\sqrt{3}}\epsilon^{ijkl}f_{ij}f_{kl}$$

since $\sqrt{-g} = 1$. So if one chooses f_{ij} to be further (anti) self-dual in Euclidean \mathbb{R}^4 , i.e. $f_{ij} = \pm\frac{1}{2}\epsilon_{ijkl}f_{kl}$ (in addition to condition (7)), then the right-hand side of (13) gives $\pm\frac{b^2}{\sqrt{3}}f^2$ and comparing with the left-hand side, one finds $b = \mp\frac{\sqrt{3}}{2}$.

Paraphrasing, (13) is satisfied provided $b = \mp\frac{\sqrt{3}}{2}$, $f_{\mu\nu}$ satisfies (8) and f_{ij} is (anti) self-dual in the Euclidean \mathbb{R}^4 space. Such an $f_{\mu\nu}$ can easily be constructed by choosing $u_i = J_{ij}x^j$ ($i, j = 1, 2, 3, 4$), where all components of the fully antisymmetric J_{ij} are constants with $J^k{}_j J^i{}_k = -\delta^i_j$, i.e. J_{ij} defines an almost complex structure in \mathbb{R}^4 . Moreover, for f_{ij} to be (anti) self-dual, J_{ij} must itself be (anti) self-dual in \mathbb{R}^4 , i.e. $J_{ij} = \pm\frac{1}{2}\epsilon_{ijkl}J_{kl}$.

With the special choice of u_i as above,

$$T_{\mu\nu}^f = \frac{1}{4}f^2u_\mu u_\nu \quad \text{and} \quad T_{\mu\nu}^F = b^2T_{\mu\nu}^f.$$

Looking back at (12) and using (6) of section 2.1, one finds that $T_{\mu\nu}^f = \frac{1}{2(4b^2-1)}f^2u_\mu u_\nu$. Comparing with the $T_{\mu\nu}^f$ above, this again yields $b = \mp\frac{\sqrt{3}}{2}$ as before. In fact this solution is supersymmetric [15, 16, 18, 19].

3.2. Six dimensions

In our conventions, the bosonic part of the $D = 6, N = 2$ supergravity theory [26] reduces to the following field equations when all the scalars of the hypermatter ϕ^a and the 2-form field $B_{\mu\nu}$ in the theory are set to zero and when one assumes the dilaton φ to be constant with $\mu \equiv e^{\sqrt{2}\varphi}$:

$$R_{\mu\nu} = 2\mu F_{\mu\rho}F_{\nu}{}^\rho + \mu^2 G_{\mu\rho\sigma}G_{\nu}{}^{\rho\sigma}, \tag{14}$$

$$\nabla_\mu F^{\mu\nu} - \mu G^{\nu\rho\sigma}F_{\rho\sigma} = 0, \tag{15}$$

$$\nabla_\mu G^{\mu\nu\rho} = 0, \tag{16}$$

$$\frac{1}{3}\mu G_{\mu\nu\rho}G^{\mu\nu\rho} + \frac{1}{2}F_{\mu\nu}F^{\mu\nu} = 0. \tag{17}$$

Here all Greek indices run from 0 to 5 and $G_{\mu\nu\rho}$ is given by

$$G_{\mu\nu\rho} = F_{\mu\nu}A_\rho + F_{\nu\rho}A_\mu + F_{\rho\mu}A_\nu \tag{18}$$

and instead of a Yang–Mills field, we have taken an ordinary vector field A^μ to be present.

Let $A_\mu = \lambda u_\mu$, where λ is a real constant. Then (16) is satisfied identically since $u^\mu = -\delta_0^\mu$ for our choice. One also finds that with this A_μ , $G_{\mu\nu\rho}G^{\mu\nu\rho} = -3\lambda^4 f^2$ and F_F

3.3. Eight dimensions

The bosonic part of the gauged $D = 8, N = 1$ supergravity theory coupled to n vector multiplets [27] has field equations which are very similar to the field equations of $D = 6, N = 2$ supergravity that we have examined in subsection 3.2. Taking an ordinary vector field instead of a Yang–Mills field and setting the 2-form field B_{MN} equal to zero (as was done in subsection 3.2), one again has

$$G_{MNP} = F_{MN}A_P + F_{NP}A_M + F_{PM}A_N, \tag{19}$$

similarly to (18), where now capital Latin indices run from 0 to 7. We also set all the scalars in the theory to zero but assume that the dilaton σ is constant with $\mu \equiv e^\sigma$. These assumptions lead to the following field equations (see (26) of [27])

$$R_{MN} = 2\mu F_{MP}F_N{}^P + \mu^2 G_{MPS}G_N{}^{PS}, \tag{20}$$

$$\nabla_M F^{MN} = \mu G^{NPS}F_{PS}, \tag{21}$$

$$\nabla_M G^{MNP} = 0, \tag{22}$$

$$\frac{2}{9}\mu^2 G_{MNP}G^{MNP} + \frac{1}{3}\mu F_{MN}F^{MN} = 0, \tag{23}$$

which have the same form as (14), (15), (16) and (17), respectively.

Letting

$$g_{MN} = h_{MN} - u_M u_N$$

(as in section 2.1) and $A_M = \lambda u_M$ (with λ real), and following similar steps as in subsection 3.2, it immediately follows that one obtains exact solutions to gauged $D = 8, N = 1$ supergravity with matter couplings provided $\mu\lambda^2 = 1/2$. Once again the conditions on u_M under which these solutions are supersymmetric should be studied further.

3.4. Ten dimensions

The following field equations can be obtained from a five-dimensional action which is itself obtained by a Kaluza–Klein reduction of the type IIB supergravity theory with only a dilaton, a Ramond–Ramond 2-form gauge potential and a graviton. (The details of the reduction process, the corresponding splitting of the ten-dimensional coordinates and the metric ansatz employed are explained in detail in [19] and we directly make use of the results of that article here.)

$$\nabla_\mu F^{\mu\nu} = -\frac{1}{2}H^{\nu\rho\sigma}F_{\rho\sigma}, \tag{24}$$

$$\nabla_\mu H^{\mu\nu\rho} = 0, \tag{25}$$

$$H_{\mu\nu\rho}H^{\mu\nu\rho} = -3F_{\mu\nu}F^{\mu\nu}, \tag{26}$$

$$G_{\mu\nu} = F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{4}g_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} + \frac{1}{4}(H_{\mu\alpha\beta}H_\nu{}^{\alpha\beta} - \frac{1}{6}g_{\mu\nu}H_{\rho\alpha\beta}H^{\rho\alpha\beta}). \tag{27}$$

Here all Greek indices run from 0 to 4.

Note the striking resemblance of these equations to the equations of the $D = 6, N = 2$ supergravity theory of subsection 3.2. We want to see whether our Gödel-type metric ansatz (1) and choice $A_\mu = \lambda u_\mu$, with λ a constant, solves equations (24)–(27). We take the 2-form field B to be zero to that effect and following [19] find that $H_{\mu\nu\rho}$ is given by ($H = -A \wedge dA$)

$$H_{\mu\nu\rho} = -(F_{\mu\nu}A_\rho + F_{\nu\rho}A_\mu + F_{\rho\mu}A_\nu)$$

which already resembles the $G_{\mu\nu\rho}$ of subsection 3.2. Following similar steps to what was done in subsection 3.2, one can easily show that our Gödel-type metric ansatz (1) and choice of A_μ solve equations (24)–(27) provided that $\lambda^2 = 1$.

Following the discussion of [19], if one further assumes that u_μ is chosen in such a way that the 3-form field $H = -\star dA$, where \star denotes Hodge duality with respect to the Gödel-type metric (1), and that the gauge field A is rescaled as $A \rightarrow 2A/\sqrt{3}$, this five-dimensional solution can be further uplifted as the solution

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu + \left(dy + \frac{2}{\sqrt{3}} A_\mu dx^\mu\right)^2 + ds^2(\mathbb{T}^4), \tag{28}$$

$$\hat{H} = \frac{2}{\sqrt{3}} dA \wedge \left(dy + \frac{2}{\sqrt{3}} A\right) - \frac{2}{\sqrt{3}} \star dA, \tag{29}$$

of the type IIB supergravity theory. Here $ds^2(\mathbb{T}^4)$ is the metric on a flat four-torus and y denotes one of the singled out extra dimensions. (See [19] for details.)

3.5. Eleven dimensions

The solution we gave in subsection 3.1 can also be uplifted to eleven dimensions as well [18]. The field equations for the bosonic part of $D = 11$ supergravity are as follows [28]:

$$R_{AB} = \frac{1}{12} \left(H_{ACDE} H_B{}^{CDE} - \frac{1}{12} H^2 g_{AB}\right), \tag{30}$$

$$\partial_A(\sqrt{-g} H^{ABCD}) = \frac{1}{2(4!)^2} \epsilon^{BCDMNKLPRST} H_{MNKL} H_{PRST}. \tag{31}$$

Here capital Latin indices run from 0 to 10. Now split the spacetime into $x^A = (x^\mu, x^m)$ where $\mu = 0, 1, 2, 3, 4$ of subsection 3.1, $m = 5, 6, \dots, 10$ and let $u_A = (u_\mu, 0)$. With this choice of u_A , take the metric to be of Gödel-type (1) with

$$g_{AB} = h_{AB} - u_A u_B. \tag{32}$$

Next define a 1-form field A as $A_A = k u_A$ where k is a real constant. Then $F = dA = kf$ where f has components $f_{\mu\nu}$ as in subsection 3.1. Moreover one can also define a second 2-form \mathcal{F} as

$$\mathcal{F} = \frac{2}{\sqrt{3}}(dx^5 \wedge dx^6 + dx^7 \wedge dx^8 + dx^9 \wedge dx^{10})$$

and the 3-form potential \mathcal{G} as $\mathcal{G} = \mathcal{F} \wedge A$. Then the 4-form $H = d\mathcal{G} = k\mathcal{F} \wedge f$. Using the property that $\mathcal{F}_{AB} f^{BC} = 0$, one then obtains

$$H_{ACDE} H_B{}^{CDE} = 3k^2 [f^2 \mathcal{F}_{AC} \mathcal{F}_B{}^C + \mathcal{F}^2 f_{AC} f_B{}^C].$$

Note that the way \mathcal{F} is constructed implies that $\mathcal{F}_{AC} \mathcal{F}_B{}^C = \frac{4}{3} \delta_{AB}$ and $\mathcal{F}^2 = 8$. Substituting these into (30), one gets

$$R^\mu{}_\nu = 2k^2 [f^{\mu\sigma} f_{\nu\sigma} - \frac{1}{6} f^2 \delta_5^\mu{}_\nu]$$

where $\delta_5^\mu{}_\nu$ denotes the five-dimensional Kronecker delta. This is of exactly the same form as (12) in $D = 5$. For the remaining field equation (31), first note that $\sqrt{-g} = 1$ and

$$\partial_A H^{ABCD} = k(\mathcal{F}^{BC} \partial_A f^{AD} + \mathcal{F}^{DB} \partial_A f^{AC} + \mathcal{F}^{CD} \partial_A f^{AB}) \tag{33}$$

and the way \mathcal{F} and f are constructed implies that only one of the terms on the right-hand side of (33) survives, say for $B = \nu, C = 2a + 1, D = 2a + 2$ ($0 \leq \nu \leq 4; 2 \leq a \leq 4$). Then by (8) and $u^\mu = -\delta_0^\mu$, (31) is equivalent to

$$-\frac{k}{\sqrt{3}} f^2 \delta_0^\nu = \frac{1}{2(4!)^2} \epsilon^{\nu 2a+1 2a+2 MNKLPRST} H_{MNKL} H_{PRST}. \tag{34}$$

When $v = i$ ($1 \leq i \leq 4$), one of the last eight indices of ϵ on the right-hand side of (34) must be a 0 and since $H_{0MNS} = 0$, (31) is satisfied identically in this case. When $v = 0$, the right-hand side of (34) has nonzero contributions from terms of the form (henceforth $1 \leq i, j, k, l \leq 4$ and $5 \leq m, n, p, q \leq 10$)

$$\frac{1}{2(4!)^2} \epsilon^{02a+12a+2ijmnpq} H_{ijmn} H_{klpq}. \tag{35}$$

However the way H and \mathcal{F} are constructed also implies that (35) is equal to

$$\frac{1}{2(4!)^2} \left(\frac{4!}{2!2!}\right)^2 8 \left(\frac{2k}{\sqrt{3}}\right)^2 \epsilon^{ijkl} f_{ij} f_{kl}$$

which now must equal to $-\frac{k}{\sqrt{3}} f^2$ from the left-hand side of (34). Remember that in $D = 5$, we chose f_{ij} to be (anti) self-dual in Euclidean \mathbb{R}^4 , which implies that

$$-\frac{k}{\sqrt{3}} = \pm \frac{2k^2}{3}$$

or $k = \mp\sqrt{3}/2$ for a solution. This is exactly the value of b found in $D = 5$.

Hence our Gödel-type metric (32) and choice of A^A and \mathcal{F} yield a class of exact solutions to $D = 11$ supergravity theory. (In fact, it has also been shown that this class preserves 5/8 of the supersymmetry [18].)

4. Gödel-type metrics with $(D - 1)$ -dimensional non-flat backgrounds

So far we have assumed that $h_{0\mu} = 0$ and $h_{ij} = \delta_{ij}$, the $(D - 1)$ -dimensional Kronecker delta symbol, in the metric (1). We have also taken $u_0 = 1$ and $\partial_0 u_\alpha = 0$. These assumptions simplified the calculation of the Ricci tensor (4) and we showed that for the metric (1) to be an exact solution to the Einstein–Maxwell dust field equations in D dimensions, one had the $(D - 1)$ -dimensional Euclidean source-free Maxwell’s equations (7) to solve. Now let us take $h_{\mu\nu}$ to be a general $(D - 1)$ -dimensional non-flat spacetime and for simplicity take $u_k = 1$.

One now finds that $u^\mu h_{\mu\nu} = 0$ and the inverse of the metric is given by (2) again. However the determinant of $g_{\mu\nu}$ is now different, $g = -h$, where h is the determinant of the $(D - 1) \times (D - 1)$ submatrix obtained by deleting the k th row and the k th column of $h_{\mu\nu}$. The new Christoffel symbols of $g_{\mu\nu}$ are given by

$$\tilde{\Gamma}^\mu_{\alpha\beta} = \gamma^\mu_{\alpha\beta} + u^\mu u_\sigma \gamma^\sigma_{\alpha\beta} + \frac{1}{2}(u_\alpha f^\mu_{\beta} + u_\beta f^\mu_{\alpha}) - \frac{1}{2}u^\mu(u_{\alpha,\beta} + u_{\beta,\alpha}), \tag{36}$$

where $\gamma^\mu_{\alpha\beta}$ are the Christoffel symbols of $h_{\mu\nu}$ and we assume that the indices of u_μ and $f_{\alpha\beta}$ are raised and lowered by the metric $g_{\mu\nu}$. By using a vertical stroke to denote a covariant derivative with respect to $h_{\mu\nu}$ so that $u_{\alpha|\beta} = u_{\alpha,\beta} - \gamma^v_{\alpha\beta} u_v$, (36) can simply be written as

$$\tilde{\Gamma}^\mu_{\alpha\beta} = \gamma^\mu_{\alpha\beta} + \frac{1}{2}(u_\alpha f^\mu_{\beta} + u_\beta f^\mu_{\alpha}) - \frac{1}{2}u^\mu(u_{\alpha|\beta} + u_{\beta|\alpha}). \tag{37}$$

Thus the ordinary commas in (3) have been replaced with vertical strokes and the Christoffel symbols of $h_{\mu\nu}$ have been added to obtain the Christoffel symbols (37) of $g_{\mu\nu}$.

To further remove any ambiguity, let us also denote a covariant derivative with respect to $g_{\mu\nu}$ by $\tilde{\nabla}_\mu$, thus

$$\tilde{\nabla}_\beta u_\alpha = u_{\alpha,\beta} - \tilde{\Gamma}^v_{\beta\alpha} u_v.$$

Using these preliminaries one can in fact show that $u^\alpha \tilde{\nabla}_\alpha u_\beta = 0$ and $\tilde{\nabla}_\alpha u_\beta = \frac{1}{2} f_{\alpha\beta}$, hence u^μ is still tangent to a timelike geodesic curve and is still a timelike Killing vector.

The Ricci tensor turns out to be

$$\tilde{R}_{\mu\nu} = \hat{r}_{\mu\nu} + \frac{1}{2}f_{\mu}^{\alpha}f_{\nu\alpha} + \frac{1}{2}(u_{\mu}\tilde{j}_{\nu} + u_{\nu}\tilde{j}_{\mu}) + \frac{1}{4}f^2u_{\mu}u_{\nu}, \quad (38)$$

where $f^2 = f^{\alpha\beta}f_{\alpha\beta}$ as before, $\tilde{j}_{\mu} \equiv f^{\alpha}{}_{\mu|\alpha}$ and $\hat{r}_{\mu\nu}$ is the Ricci tensor of $h_{\mu\nu}$. The Ricci scalar is now readily obtained as

$$\tilde{R} = \hat{r} + \frac{1}{4}f^2 + u^{\mu}\tilde{j}_{\mu},$$

where \hat{r} denotes the Ricci scalar of $h_{\mu\nu}$. (Note that $\hat{r} = g^{\alpha\beta}\hat{r}_{\alpha\beta} = \bar{h}^{\alpha\beta}\hat{r}_{\alpha\beta}$ by using $u^{\mu} = -\delta_k^{\mu}$, (2) and $u^{\mu}\gamma^{\nu}{}_{\mu\alpha} = 0$ in the explicit calculation of \hat{r} .) Setting $\tilde{j}_{\mu} = 0$, the Einstein tensor is found to be

$$\tilde{G}_{\mu\nu} = \hat{r}_{\mu\nu} - \frac{1}{2}h_{\mu\nu}\hat{r} + \frac{1}{2}T_{\mu\nu}^f + \left(\frac{1}{4}f^2 + \frac{1}{2}\hat{r}\right)u_{\mu}u_{\nu}, \quad (39)$$

where $T_{\mu\nu}^f$ denotes the Maxwell energy–momentum tensor for $f_{\mu\nu}$ as before.

Note that in fact

$$\tilde{j}_{\mu} = (g^{\alpha\beta}f_{\beta\mu})_{|\alpha} = (\bar{h}^{\alpha\beta}f_{\beta\mu})_{|\alpha}.$$

This follows by using (2), $u^{\alpha}f_{\mu\alpha} = 0$, $u^{\mu} = -\delta_k^{\mu}$ and the initial assumptions on $h_{\mu\nu}$. Hence $\tilde{j}_{\mu} = 0$ equivalently implies that $\bar{h}^{\mu\nu}\tilde{j}_{\nu} = 0$ or

$$\partial_{\alpha}(\bar{h}^{\alpha\mu}\bar{h}^{\beta\nu}\sqrt{|\bar{h}|}f_{\mu\nu}) = 0. \quad (40)$$

Hence we find that the Einstein tensor of the $(D - 1)$ -dimensional background $h_{\mu\nu}$ acts as a source term for the Einstein equations obtained for the D -dimensional Gödel-type metric and that the curvature scalar of the background contributes to the energy density of the dust fluid provided that the $(D - 1)$ -dimensional source-free Maxwell equation (40) in the background holds. In the following subsection we give a class of such solutions in the background of some spaces of constant curvature.

Note that all the theories we discussed in subsections 3.2 to 3.4 have Gödel-type metrics as exact solutions with the Ricci flat background metric $h_{\mu\nu}$ where the 3-form field $H_{\mu\nu\alpha}$ and the 2-form field $F_{\mu\nu}$ are given in exactly the same way as those defined in these subsections, the dilaton field is taken to be zero and the vector field u_{α} now satisfies the Maxwell equation (40) in the background $h_{\mu\nu}$. Hence the bosonic field equations of all of these supergravity theories have effectively reduced to the Maxwell equation (40)! In subsection 4.3, we will present solutions of this type by taking the $(D - 1)$ -dimensional Tangherlini solution as the background $h_{\mu\nu}$.

4.1. Solutions with $(D - 1)$ -dimensional conformally flat backgrounds

Let us now take the special fixed coordinate x^k as x^0 (i.e. $k = 0$) and let the background h_{ij} be conformally flat so that $h_{ij} = e^{2\psi}\delta_{ij}$. Here Latin indices run from 1 to $D - 1$. If we denote the radial distance of \mathbb{R}^{D-1} by $r = \sqrt{x^i x^i}$, take $\psi = \psi(r)$ and use a prime to denote the derivative with respect to r , then one finds that (see, e.g., [29])

$$\hat{r}_{ij} = \frac{1}{2}$$

If $\psi'' + \frac{\psi'}{r} = 0$, then one finds that $\psi = a \ln r + b$ for some constants a and b . Taking $b = 0$, this gives

$$\hat{r}_{ij} - \frac{1}{2}h_{ij}\hat{r} = \frac{a(a+2)}{r^4}x_i x_j.$$

If one chooses $a = -2$, then $h_{ij} = \frac{1}{r^4}\delta_{ij}$ and now both \hat{r}_{ij} and \hat{r} vanish. One now has to solve (40) in this background to find the Gödel-type metric $g_{\mu\nu}$ which solves (39) with $\hat{r}_{\mu\nu} = \hat{r} = 0$. To construct such a solution, take $u_i = s(r)Q_{ij}x^j$ where Q_{ij} is fully antisymmetric with constant components. Equation (40) implies that $(r^2s'' + 6rs' + 4s)Q_{j\ell}x^\ell = 0$, and in general one obtains

$$u_i = \left(\frac{A}{r^4} + \frac{B}{r}\right) Q_{i\ell}x^\ell$$

for some real constants A and B .

Thus the line element corresponding to the Gödel-type metric in $D = 4$ in this three-dimensional conformally flat background

$$ds^2 = \frac{1}{r^4}(dx^2 + dy^2 + dz^2) - \left(dt + \left(\frac{A}{r^4} + \frac{B}{r}\right) Q_{i\ell}x^\ell dx^i\right)^2$$

solves the $D = 4$ Einstein charged dust field equations.

Let us further set $Q_{13} = Q_{23} = 0$ but $Q_{12} \neq 0$ for simplicity and write the resultant line element using cylindrical coordinates. One finds

$$ds^2 = \frac{1}{r^4}(d\rho^2 + \rho^2 d\phi^2 + dz^2) - \left(dt - Q_{12}\frac{\rho^2}{r^4}(A + Br^3) d\phi\right)^2.$$

Employing the curve C of subsection 2.3 and its tangent vector v^μ , one finds that

$$v^2 = g_{\phi\phi} = \frac{\rho_0^2}{r_0^4} \left(1 - (Q_{12})^2 \frac{\rho_0^2}{r_0^4} (A + Br_0^3)^2\right)$$

with $r_0^2 = \rho_0^2 + z_0^2$. Since v^2 is not positive definite in its full generality, we conclude that there exist closed timelike and closed null curves in this spacetime.

4.2. Solutions with $(D - 1)$ -dimensional Einstein spaces as backgrounds (which are themselves conformally flat)

Let us again take the special fixed coordinate x^k as x^0 and the background h_{ij} to be conformally flat so that $h_{ij} = e^{2\psi}\delta_{ij}$. However let us now assume that $\hat{r}_{ij} = \Lambda h_{ij}$, where Λ denotes the cosmological constant; i.e. the background is a $(D - 1)$ -dimensional Einstein space as well. This yields

$$\hat{r} = (D - 1)\Lambda \quad \text{and} \quad \hat{r}_{ij} - \frac{1}{2}h_{ij}\hat{r} = \left(\frac{3 - D}{2}\right) \Lambda e^{2\psi}\delta_{ij}.$$

Substituting these into (39), one finds that when the background $h_{\mu\nu}$ is a $(D - 1)$ -dimensional Einstein space, the Gödel-type metric $g_{\mu\nu}$ provides a solution to

$$\tilde{G}_{\mu\nu} = \left(\frac{3 - D}{2}\right) \Lambda g_{\mu\nu} + \frac{1}{2}T_{\mu\nu}^f + \left(\frac{1}{4}f^2 + \Lambda\right) u_\mu u_\nu,$$

which describes a charged perfect fluid source with

$$p = \frac{1}{2}(3 - D)\Lambda \quad \text{and} \quad \rho = \frac{1}{4}f^2 + \frac{1}{2}(D - 1)\Lambda.$$

(For $D \geq 4$, Λ must be negative in order to have a positive pressure density p .)

We now further assume that $\psi = \psi(z)$ and take $D = 4$ for simplicity. (One can again generalize the arguments we present here to $D > 4$.) Such a choice of ψ yields

$$\hat{r}_{ij} - \frac{1}{2}h_{ij}\hat{r} = -\partial_i\partial_j\psi + (\partial_i\psi)(\partial_j\psi) + \bar{\delta}_{ij}\nabla^2\psi = (-\psi'' + (\psi')^2)\bar{\delta}_i^3\bar{\delta}_j^3 + \psi''\bar{\delta}_{ij},$$

where a prime denotes the derivative with respect to z . One then has to solve $-\psi'' + (\psi')^2 = 0$ which yields $\psi = b - \ln|z + a|$ for some real constants a and b . Further demanding $\hat{r} = 3\Lambda$ fixes the constant b so that

$$h_{ij} = \frac{-2}{\Lambda(z+a)^2}\bar{\delta}_{ij}.$$

Thus for a physically acceptable background, it must be that $\Lambda < 0$.

Hence one now has to solve (40) in this background so that the Gödel-type metric $g_{\mu\nu}$ solves (39) in the form

$$\tilde{G}_{\mu\nu} = -\frac{1}{2}\Lambda g_{\mu\nu} + \frac{1}{2}T_{\mu\nu}^f + \left(\frac{1}{4}f^2 + \Lambda\right)u_\mu u_\nu,$$

i.e. the charged perfect fluid source has pressure density $p = -\frac{1}{2}\Lambda$, (and $p > 0$ when $\Lambda < 0$) and energy density $\rho = \frac{1}{4}f^2 + \frac{3}{2}\Lambda$, and $\Lambda < 0$ must be chosen properly so that $\rho > 0$.

To find a solution to (40), which simply takes the form $\partial_i((z+a)f_{ij}) = 0$ in this background, let us use the ansatz $u_i = \bar{\delta}_i^3 s(x, y, z)$. Then $f_{ij} = \bar{\delta}_j^3 \partial_i s - \bar{\delta}_i^3 \partial_j s$ and when the free index j is equal to 3, one finds $\partial_\ell((z+a)\partial_\ell s) = 0$, where the index ℓ runs over 1 and 2. When $j = \ell \neq 3$, one similarly obtains $\partial_3((z+a)\partial_\ell s) = 0$, which is easily integrated to give

$$\partial_\ell s = \frac{c_\ell(x, y)}{z+a}$$

for some ‘integration constants’ $c_\ell(x, y)$. Consistency with the $j = 3$ equation above further constrains c_ℓ to satisfy $\partial_\ell c_\ell = 0$. Hence letting $c_\ell = \partial_\ell c$ for a potential $c(x, y)$, one finds that

$$s(x, y, z) = \frac{c(x, y)}{z+a}$$

for a function $c(x, y)$ which is harmonic in the (x, y) variables.

Substituting these into the metric $g_{\mu\nu}$, one finds that the line element corresponding to this $D = 4$ example is given in cylindrical coordinates as

$$ds^2 = \frac{-2}{\Lambda(z+a)^2}(d\rho^2 + \rho^2 d\phi^2 + dz^2) - \left(dt + \frac{c(x, y)}{z+a} dz\right)^2.$$

One then finds that the norm of the tangent vector v^μ to the curve C of subsection 2.3 is

$$v^2 = g_{\phi\phi} = \frac{-2\rho_0^2}{\Lambda(z_0+a)^2}.$$

One again sees that for v^2 to be positive definite Λ must be $\Lambda < 0$. (In that case the pressure density of the perfect fluid is also positive, $p > 0$.) So we see that the closed curve of subsection 2.3 may be timelike and also the discussion we give at the end of subsection 2.4 regarding the universal covering can similarly be repeated here.

4.3. Spacetimes with $(D - 1)$ -dimensional Riemannian Tangherlini solutions as backgrounds

Let the $(D - 1)$ -dimensional background metric $h_{\mu\nu}$ be the metric of an Einstein space,

$$\hat{r}_{\mu\nu} = \frac{2\Lambda}{3-D}h_{\mu\nu} \quad (D \neq 3).$$

Then the full D -dimensional Einstein tensor becomes

$$\tilde{G}_{\mu\nu} = \frac{1}{2}T_{\mu\nu}^f + \Lambda g_{\mu\nu} + \left(\frac{1}{4}f^2 + \frac{2\Lambda}{3-D}\right)u_\mu u_\nu \quad (D \neq 3).$$

Hence our metric (1) solves the Einstein's field equations with a charged dust source and a cosmological constant provided the source-free Maxwell equation (40) holds. The energy density of the dust is

$$\rho = \frac{1}{4}f^2 + \frac{D-1}{3-D}\Lambda,$$

where Λ must be chosen so that $\rho > 0$.

Consider the line element corresponding to the $(D-1)$ -dimensional Riemannian Tangherlini solution with a cosmological constant Λ

$$ds_{D-1}^2 = \zeta dt^2 + \frac{dr^2}{\zeta} + r^2 d\Omega_{D-3}^2,$$

where $\zeta = 1 - 2V$ with

$$V = \begin{cases} mr^{4-D} + \frac{\Lambda}{(D-3)(D-2)}r^2 & (D \geq 5), \\ m + \frac{\Lambda}{2}r^2 & (D = 4), \end{cases}$$

where m is the constant mass parameter, $d\Omega_{D-3}^2$ is the metric on the $(D-3)$ -dimensional unit sphere and we take the static limit so that all acceleration parameters vanish [12].

Let the special fixed coordinate x^k be x^{D-1} this time. Let us also assume that $u_\mu = u(r)\delta_\mu^0 + \delta_\mu^{D-1}$. Then $f_{\mu\nu} = (\delta_\mu^r \delta_\nu^0 - \delta_\mu^0 \delta_\nu^r)u'$, where a prime denotes the derivative with respect to r . The only nontrivial component of (40) is obtained when $\beta = 0$ and in that case $(r^{D-3}u')' = 0$, which yields

$$u(r) = \begin{cases} ar^{4-D} + b & (D \geq 5), \\ a \ln r + b & (D = 4), \end{cases}$$

for some real constants a and b . Here b is irrelevant since it can be gauged away and taken as zero.

Substituting these into metric (1), the D -dimensional line element becomes

$$ds^2 = \zeta dt^2 + \frac{dr^2}{\zeta} + r^2 d\Omega_{D-3}^2 - (u(r) dt + dx^{D-1})^2.$$

For this solution one has

$$f^2 = \begin{cases} 2a^2(D-4)^2r^{6-2D} & (D \geq 5), \\ 2a^2/r^2 & (D = 4), \end{cases}$$

and one finds that the energy density of the dust diverges at $r = 0$. In the simple case $a = 0$, the Maxwell part of the full energy momentum tensor vanishes and one just has a dust source with

$$\rho = \frac{D-1}{3-D}\Lambda,$$

and for $\rho > 0$, Λ must be negative. For $D = 4$, $\zeta = 0$ when $r^2 = (1 - 2m)/\Lambda$.

Note that the Tangherlini solutions we start with are locally Riemannian metrics and the parameter m is no longer the 'mass constant' in the D -dimensional spacetime M . The local coordinates chosen here are $(t, r, \theta_1, \theta_2, \dots, \theta_{D-3}, x^{D-1})$. Here x^{D-1} plays the role of the 'time' coordinate and the rest of the coordinates $(t, r, \theta_1, \theta_2, \dots, \theta_{D-3})$ are the $(D-1)$ -dimensional cylindrical coordinates. Here the $t = \text{constant}$ surfaces are planes perpendicular

to the t -axis and the $r = \text{constant}$ surfaces are the cylinders containing the set of points $r = 0$, i.e. the t -axis. Hence in our solution the set of points $\zeta = 0$ defines a $(D - 1)$ -dimensional cylinder. As an illustration, when $D = 5$ and $\Lambda = 0$, we have $\zeta = 1 - \frac{2m}{r}$ and

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) - \left(\frac{a}{r} dt + dx^4\right)^2,$$

with $f^2 = \frac{2a^2}{r^4}$. Hence at $r = 2m$ (a cylinder in five-dimensions), $\zeta = 0$. This is not a spacetime singularity and does not describe an event horizon either. Inside the cylinder ($r < 2m$), the signature of the spacetime changes from $(+, +, +, +, -)$ to $(-, -, +, +, -)$. The spacetime singularity is located at $r = 0$ (inside the cylinder) which is the t -axis. It is clear that the interior region of this cylinder is not physical. The solutions we give by using the Tangherlini metrics describe physical spacetimes only in those regions where $\zeta > 0$. If the t -coordinate is assumed to be closed ($t \in [0, 2\pi]$), the cylinders mentioned above should be replaced by tori.

As we pointed out earlier, the solutions presented here are also solutions of the supergravity theories listed in section 3 with the 2- and 3-form fields defined in exactly the same way as those given in subsections 3.2–3.4 and with a vanishing dilaton field. The only field equation we had to solve was the Maxwell equation (40) in the Riemannian Tangherlini background.

5. Conclusion

We have introduced and used Gödel-type metrics to find charged dust solutions to the Einstein's field equations in D dimensions. We started with a $(D - 1)$ -dimensional Riemannian background (which could be taken as either flat or non-flat) and showed that solutions to D -dimensional Einstein–Maxwell theory with a dust source could be obtained provided the source-free Maxwell's equation is satisfied in the relevant background. The corresponding geodesics were also found to be described by the Lorentz force equation for a charged particle in the background geometry. We gave examples of spacetimes which contained closed timelike and closed null curves and others that contained neither of these. We used the Gödel-type metrics to find exact solutions to various kinds of supergravity theories. By constructing the 2-form and 3-form fields out of the vector field u_μ and by assuming a vanishing dilaton field, we demonstrated that the bosonic field equations of these supergravity theories could effectively be reduced to a simple source-free Maxwell's equation (40) in the relevant background $h_{\mu\nu}$.

In the case of non-flat backgrounds, we constructed explicit solutions for $D = 4$ when the background was taken to be conformally flat, an Einstein space and a Riemannian Tangherlini solution. We showed that the Gödel-type metrics described a black-hole-like object depending on the parameters in the latter case. We also discussed the existence of closed timelike or closed null curves for conformally flat and Einstein space backgrounds.

It would be worth studying to see how much of the supersymmetry is preserved in the solutions we have given to various supergravity theories here and to further seek whether Gödel-type metrics can be employed in finding new (possibly supersymmetric) solutions to others that we have not considered. Throughout this work, we assumed the component of u_μ along the fixed special coordinate x^k to be constant. Another interesting point to investigate would be to generalize this assumption to non-constant u_k . One would then expect to construct solutions to the Einstein–Maxwell dilaton 3-form field equations. Work along these lines is in progress and we expect to report our results soon.

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