

Gardner's deformations of the Boussinesq equations

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Abstract

Using the algebraic method of Gardner's deformations for completely integrable systems, we construct recurrence relations for densities of the Hamiltonians for the Boussinesq and the Kaup–Boussinesq equations. By extending the Magri schemes for these equations, we obtain new integrable systems adjoint with respect to the initial ones and describe their Hamiltonian structures and symmetry properties.

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1. Introduction

In this paper we consider the most efficient way to prove the complete integrability of evolutionary PDE systems. Namely, we apply the method of Gardner's deformations [1–6] to the Boussinesq and Kaup–Boussinesq equations. By construction, the deformations consist of the (multi-)Hamiltonian scaling non-invariant parametric extensions \mathcal{E}_ε of the original systems \mathcal{E}_0 and the parameter-dependent Miura transformations $m_\varepsilon: \mathcal{E}_\varepsilon \rightarrow \mathcal{E}_0$. Inverting the Miura transformations from the new systems, one obtains the recurrence relations for infinite sequences of densities of the Hamiltonian functionals.

The deformations determine the Hamiltonians in the hierarchies and thus they contribute to our knowledge of the geometric integrability picture. Below we improve a result [3] by Kupershmidt showing that the sequences of conserved densities for the Boussinesq equation satisfy two different recurrence relations simultaneously. Second, we emphasize that the method of Gardner's deformations is the most powerful instrument for proving the locality [9] of hierarchies. Indeed, the deformations yield local Hamiltonians and do not require calculation of the Nijenhuis tensors for recursion operators or description of the Poisson cohomology groups whose nontriviality precludes infinite motion along the Magri schemes. Third, solutions of the deformation problem determine Gardner's deformations for the (n th)

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modified Boussinesq [10] and Kaup–Boussinesq [11] systems. Next, the deformations constructed in this paper show that the extensions \mathcal{E}_ε must not necessarily interpolate (as it is assumed in [7]) between the equations \mathcal{E}_0 and the modified systems that provide the canonical factorizations for the higher Poisson structures [8]. These counter-examples contribute to the classification of nonlocalities over the Boussinesq systems.

Finally, the search for Gardner’s deformations is motivated by the fact that they result in the new adjoint hierarchies obtained by isolating the flows at higher powers of the deformation parameters. We show that the adjoint Boussinesq equation (5) is C-integrable by solving the related Cauchy problem and we prove that the adjoint Kaup–Boussinesq system (10) is completely integrable by describing the Poisson pencil for it.

The paper is organized as follows. In section 2 we obtain two Gardner’s deformations for the Boussinesq equation using the dispersionless case [17] as a starting point; also, we integrate the adjoint Boussinesq system. In section 3 we construct the deformation of the Kaup–Boussinesq equation and describe the bi-Hamiltonian structure of the adjoint system.

2. The Boussinesq equation

First we consider the Boussinesq equation with dispersion and dissipation, see [3],

$$u_t = \mathcal{V}_x + \alpha \mathcal{U}_{xx}, \quad \mathcal{V}_t = \mathcal{U}_{xxx} + \mathcal{U}\mathcal{U}_x - \alpha \mathcal{V}_{xx}, \quad \alpha \in \mathbb{R}. \quad (1)$$

We note that for any α system (1) is transformed to the equation $\mathcal{U}_{tt} = (\mathcal{U}_{xxx} \cdot (1 + \alpha) + \mathcal{U}\mathcal{U}_x)_x$, which is scaling equivalent to the Boussinesq equation $u_{tt} = (u_{xxx} + uu_x)_x$ whenever $\alpha \neq -1$. Therefore, for $\alpha \neq -1$ we reduce (1) to the Boussinesq equation

$$u_t = v_x, \quad v_t = u_{xxx} + uv_x. \quad (2)$$

In the following, we consider the problem of Gardner’s deformation for system (2).

Next, we observe that a Gardner’s deformation for (2) (and for the Kaup–Boussinesq equation (6) as well) is obtained from the deformation of the dispersionless reduction by adding higher order terms to the equations \mathcal{E}_ε and to the Miura transformations $m_\varepsilon: \mathcal{E}_\varepsilon \rightarrow \mathcal{E}_0$. Indeed, the zero-order terms in the conserved densities, which are obtained recursively by inverting the Miura substitutions, originate from the relations for the dispersionless systems. The deformation for the dispersionless Boussinesq equation is described in [17]; it corresponds to the case $\alpha = -1$ in (1).

Theorem 1. *There are two deformations $(\mathcal{E}_\varepsilon^\pm, m_\varepsilon^\pm)$ for the Boussinesq equation (2). Both extensions $\mathcal{E}_\varepsilon^\pm$ are Hamiltonian with respect to the structure*

$$\begin{pmatrix} 0 & D_x \\ D_x & 0 \end{pmatrix} \quad (3)$$

and the functionals with the densities

$$H_\pm(\varepsilon) = \frac{1}{6}\tilde{u}^3 - \frac{1}{2}\tilde{u}_x^2 + \frac{1}{2}\tilde{v}^2 + \varepsilon^3\left(\frac{1}{6}\tilde{v}^3 + \frac{1}{2}\tilde{u}_x^2\tilde{v} \pm \frac{1}{2}\tilde{u}_x\tilde{v}^2 \pm \frac{1}{6}\tilde{u}_x^3\right),$$

respectively. The extended Boussinesq equations $\mathcal{E}_\varepsilon^\pm$ are

$$\begin{aligned} \tilde{u}_t &= \tilde{v}_x + \varepsilon^3(\tilde{u}_x\tilde{u}_{xx} \pm \tilde{u}_{xx}\tilde{v} \pm \tilde{u}_x\tilde{v}_x + \tilde{v}\tilde{v}_x), \\ \tilde{v}_t &= \tilde{u}_{xxx} + \tilde{u}\tilde{u}_x - \varepsilon^3(\tilde{u}_{xx}\tilde{v} + 2\tilde{u}_{xx}\tilde{v}_x + \tilde{u}_x\tilde{v}_{xx} \pm \tilde{u}_{xx}^2 \pm \tilde{u}_x\tilde{u}_{xxx} \pm \tilde{v}_x^2 \pm \tilde{v}\tilde{v}_{xx}). \end{aligned} \quad (4)$$

The respective Miura transformations m_ε^\pm from $\mathcal{E}_\varepsilon^\pm$ to (2) are given through

$$\begin{aligned} u &= \tilde{u} \mp 2\varepsilon\tilde{u}_x + 2\varepsilon^2(\tilde{u}_{xx} \pm \tilde{v}_x) + \varepsilon^3(\tilde{u}\tilde{v} \pm \tilde{u}\tilde{u}_x), \\ v &= \tilde{v} \mp 2\varepsilon\tilde{v}_x + 2\varepsilon^2(\tilde{v}_{xx} \pm \tilde{u}_{xxx} \pm \tilde{u}\tilde{u}_x) + \varepsilon^3\left(\frac{1}{3}\tilde{u}^3 + \tilde{v}^2 + \tilde{u}\tilde{u}_{xx} \pm \tilde{v}\tilde{v}_x \pm \tilde{u}_x\tilde{v}\right) \\ &\quad \mp 2\varepsilon^4(\tilde{u}_x\tilde{u}_{xx} \pm \tilde{u}_x\tilde{v}_x \pm \tilde{u}_{xx}\tilde{v} + \tilde{v}\tilde{v}_x) + \varepsilon^6\left(\frac{1}{3}\tilde{v}^3 \pm \frac{1}{3}\tilde{u}_x^3 + \tilde{u}_x^2\tilde{v} \pm \tilde{u}_x\tilde{v}^2\right). \end{aligned}$$

The proposition of theorem 1 is obtained by a symbolic computation using the package [18]. We solve the equation $m_\varepsilon: \mathcal{E}_\varepsilon \rightarrow \mathcal{E}_0$ with respect to the extension \mathcal{E}_ε and the Miura transformation m_ε , here \mathcal{E}_0 stands for (2). We suppose that \mathcal{E}_ε preserves the Poisson structure (3) and the Hamiltonian $H(\varepsilon)$ retracts to $\frac{1}{6}u^3 - \frac{1}{2}u_x^2 + \frac{1}{2}v^2$ at $\varepsilon = 0$. Also, we assume that m_ε and $H(\varepsilon)$ are polynomial in ε . Further, note that the Boussinesq equation (2) is homogeneous with respect to the weights $|u| = 2, |v| = 3, |d/dx| = 1, |d/dt| = 2$. We set $|\varepsilon| = -1$ and generate the ansatz for m_ε and $H(\varepsilon)$. Thus the equation $m_\varepsilon: \mathcal{E}_\varepsilon \rightarrow \mathcal{E}_0$ leads to the algebraic system for the undetermined coefficients. This overdetermined system is processed iteratively by resolving linear relations and substituting them back into the equation for m_ε and \mathcal{E}_ε . The fourth iteration gives the assertion. The same computational scheme is applied to the Kaup–Boussinesq equation (6), see theorem 2.

The extended equations (4) consist of the original Boussinesq flows and the adjoint flows at ε^3 , which will be further discussed in more detail. The Poisson structure (3) for the extensions $\mathcal{E}_\varepsilon^\pm$ together with the Miura transformations $\mathcal{E}_\varepsilon^\pm \rightarrow \mathcal{E}_0$ induce [8] the Poisson structures $\hat{A}_1 \mp \varepsilon^3 \hat{A}_2$ for Boussinesq’s equation (2), here \hat{A}_1 given through (3) and

$$\hat{A}_2 = \begin{pmatrix} 8D_x^3 + uD_x + D_x \circ u & 3vD_x + v_x \\ 3vD_x + 2v_x & D_x^5 + 5(uD_x^3 + D_x^3 \circ u) - 3(u_{xx}D_x + D_x \circ u_{xx}) + uD_x \circ u \end{pmatrix}$$

are its first and second structures, respectively (see [10] and references therein). The Hamiltonians for the extensions $\mathcal{E}_\varepsilon^\pm$ are inherited from the original functionals, which are described in proposition 1 below, by using the Miura substitutions m_ε^\pm .

Proposition 1. *Densities of the Hamiltonian functionals for the Boussinesq equation can be obtained using two different recurrence relations, which are*

$$\begin{aligned} \tilde{u}_0 &= u, & \tilde{v}_0 &= v, & \tilde{u}_1 &= \pm 2u_x, \\ \tilde{v}_1 &= \pm 2v_x, & \tilde{u}_2 &= 2u_{xx} \mp 2v_x, & \tilde{v}_2 &= 2v_{xx} \mp 2u_{xxx} \mp 2uu_x, \\ \tilde{u}_n &= \pm 2D_x(\tilde{u}_{n-1}) - 2D_x^2(\tilde{u}_{n-2}) \mp 2D_x(\tilde{v}_{n-2}) + \sum_{k+\ell=n-3} [-\tilde{u}_k \tilde{v}_\ell \mp \tilde{u}_k D_x(\tilde{u}_\ell)], & n &\geq 3, \\ \tilde{v}_3 &= \pm 2D_x(\tilde{v}_2) \mp 2D_x^3(\tilde{u}_1) - 2D_x^2(\tilde{v}_1) \mp [uD_x(\tilde{u}_1) + \tilde{u}_1 u_x] \\ &\quad - \frac{1}{3}u^3 - v^2 - uu_{xx} \mp uv_x \mp u_x v, \\ \tilde{v}_n &= \pm 2D_x(\tilde{v}_{n-1}) \mp 2D_x^3(\tilde{u}_{n-2}) - 2D_x^2(\tilde{v}_{n-2}) \\ &\quad \mp \sum_{k+\ell=n-2} 2\tilde{u}_k D_x(\tilde{u}_\ell) - \sum_{k+\ell+m=n-3} \frac{1}{3}\tilde{u}_k \tilde{u}_\ell \tilde{u}_m \\ &\quad + \sum_{k+\ell=n-3} [-\tilde{v}_k \tilde{v}_\ell - \tilde{u}_k D_x^2(\tilde{u}_\ell) \mp \tilde{u}_k D_x(\tilde{v}_\ell) \mp D_x(\tilde{u}_k) \tilde{v}_\ell] \\ &\quad + \sum_{k+\ell=n-4} 2 \cdot [\pm D_x(\tilde{u}_k) D_x^2(\tilde{u}_\ell) + D_x(\tilde{u}_k) D_x(\tilde{v}_\ell) + D_x^2(\tilde{u}_k) \tilde{v}_\ell \pm \tilde{v}_k D_x(\tilde{v}_\ell)], \\ &\quad n = 4, 5, \\ \tilde{v}_n &= \pm 2D_x(\tilde{v}_{n-1}) \mp 2D_x^3(\tilde{u}_{n-2}) - 2D_x^2(\tilde{v}_{n-2}) \\ &\quad \mp \sum_{k+\ell=n-2} 2\tilde{u}_k D_x(\tilde{u}_\ell) - \sum_{k+\ell+m=n-3} \frac{1}{3}\tilde{u}_k \tilde{u}_\ell \tilde{u}_m \\ &\quad + \sum_{k+\ell=n-3} [-\tilde{v}_k \tilde{v}_\ell - \tilde{u}_k D_x^2(\tilde{u}_\ell) \mp \tilde{u}_k D_x(\tilde{v}_\ell) \mp D_x(\tilde{u}_k) \tilde{v}_\ell] \end{aligned}$$

$$\begin{aligned}
 &+ \sum_{k+\ell=n-4} 2[\pm D_x(\tilde{u}_k)D_x^2(\tilde{u}_\ell) + D_x(\tilde{u}_k)D_x(\tilde{v}_\ell) + D_x^2(\tilde{u}_k)\tilde{v}_\ell \pm \tilde{v}_k D_x(\tilde{v}_\ell)] \\
 &+ \sum_{k+\ell+m=n-6} \left[-\frac{1}{3}\tilde{v}_k\tilde{v}_\ell\tilde{v}_m \mp \frac{1}{3}D_x(\tilde{u}_k)D_x(\tilde{u}_\ell)D_x(\tilde{u}_m) \right. \\
 &\quad \left. - D_x(\tilde{u}_k)D_x(\tilde{u}_\ell)\tilde{v}_m + D_x(\tilde{u}_k)\tilde{v}_\ell\tilde{v}_m \right], \quad n \geq 6.
 \end{aligned}$$

The ambiguity of signs of the differential terms does not affect the nontrivial conserved densities \tilde{u}_{3k} and \tilde{v}_{3k} . The densities with subscripts $3k + 1, 3k + 2$ are trivial for all $k \geq 0$.

We have described two distinct recurrence relations for both sequences of the conserved densities for (2). Thus we improve a result of [3], where one recurrence relation for only one sequence was obtained. The relations listed in proposition 1 are also valid for those Hamiltonians of the modified Boussinesq equation [10] which are obtained using the Miura substitutions to (2). The modified fields themselves do not start these sequences of conserved densities and thus cannot be regarded as their negative terms $\tilde{u}_{-3}, \tilde{v}_{-3}$.

Further, consider the flow at ε^3 in the rhs of equation (4) and fix the minus sign; the second case in (4) is analogous. Thus we obtain the adjoint Boussinesq system

$$u_\tau = \frac{1}{2}D_x((v - u_x)^2), \quad v_\tau = \frac{1}{2}D_x^2((v - u_x)^2). \tag{5}$$

It is Hamiltonian w.r.t. the structure (3) and the functional with density

$$H' = \frac{1}{6}v^3 + \frac{1}{2}u_x^2v - \frac{1}{2}u_xv^2 - \frac{1}{6}u_x^3.$$

The symbol of evolutionary system (5) is nonconstant contrarily to (2).

The adjoint Boussinesq system (5) is symmetry integrable (see [19] for methods, some classifications, and references), admitting two infinite sequences of Hamiltonian symmetries $(u_s, v_s) = \varphi_{[s]}$ for all weights $|s| = -(6k + 3 \pm 2), k \in \mathbb{N}_{\geq 0}$ (we note that system (5) is homogeneous w.r.t. the weights $|u| = 2, |v| = 3, |d/d\tau| = 5, |d/dx| = 1$). These flows reproduce (with proper modifications) the scheme for the symmetries of the original Boussinesq hierarchy. Namely, they are arranged according to the diagram

$$\varphi_{[1]} \rightarrow \varphi_{[7]} \rightarrow \varphi_{[13]} \rightarrow \dots, \quad \varphi_{[5]} \rightarrow \varphi_{[11]} \rightarrow \dots.$$

The flow $\varphi_{[5]}$ is the right-hand side of the adjoint Boussinesq equation (5) itself. The two sequences of Hamiltonians for the higher symmetries of (5) are obtained from the functionals for system (4) by isolating the coefficients of the highest powers of ε . For example, the components $\varphi_{[7]}^{1,2}$ of the Hamiltonian symmetry $\varphi_{[7]}$ are

$$\varphi_{[7]}^1 = D_x((v - u_x)(v_{xx} - u_{xxx}) + \frac{1}{2}(v_x - u_{xx})^2), \quad \varphi_{[7]}^2 = D_x(\varphi_{[7]}^1).$$

The symmetry $\varphi_{[11]}$ that succeeds the equation in the hierarchy has the components

$$\varphi_{[11]}^1 = D_x(w w_{6x} + 3w_x w_{5x} + 7w_{xx} w_{4x} + \frac{9}{2}w_{xxx}^2), \quad \varphi_{[11]}^2 = D_x(\varphi_{[11]}^1),$$

where we put $w = v - u_x$. Moreover, the adjoint Boussinesq hierarchy is C-integrable. It is not hard to prove that $u_{\tau_i} = \varphi_i[v - u_x]$ and $v_{\tau_i} = D_x(\varphi_i)$ for all i . Therefore, $u_{\tau_i} = \varphi_i[w]$ and $w_{\tau_i} = 0$. Hence we have $w(\tau_i, x) = w(0, x), u(\tau_i, x) = u(0, x) + \tau_i \cdot \varphi_i[w(0, x)]$ and $v(\tau_i, x) = w(0, x) + u_x(\tau_i, x)$.

3. The Kaup–Boussinesq equation

Now we construct the Gardner deformation for the Kaup–Boussinesq equation

$$u_t = uu_x + v_x, \quad v_t = (uv)_x + u_{xxx}. \tag{6}$$

Theorem 2. *The integrable extension \mathcal{E}_ε of the Kaup–Boussinesq equation (6) is the system*

$$\begin{aligned} \tilde{u}_t &= \tilde{u}\tilde{u}_x + \tilde{v}_x + \varepsilon(\tilde{u}\tilde{u}_{xx} + \tilde{u}_x^2 + (\tilde{u}\tilde{v})_x), \\ \tilde{v}_t &= (\tilde{u}\tilde{v})_x + \tilde{u}_{xxx} - \varepsilon(2\tilde{u}_x\tilde{u}_{xx} + \tilde{u}\tilde{u}_{xxx} + \tilde{u}_x\tilde{v}_x + \tilde{u}\tilde{v}_{xx} - \tilde{v}\tilde{v}_x). \end{aligned} \tag{7}$$

System (7) is Hamiltonian with respect to the structure (3) and the functional

$$\mathcal{H}(\varepsilon) = \int \left(\frac{1}{2}\tilde{u}^2\tilde{v} + \frac{1}{2}\tilde{v}^2 - \frac{1}{2}\tilde{u}_x^2 + \frac{1}{2}\varepsilon[\tilde{u}\tilde{u}_x^2 + 2\tilde{u}\tilde{u}_x\tilde{v} + \tilde{u}\tilde{v}^2] \right) dx.$$

The Miura transformation $m_\varepsilon: \mathcal{E}_\varepsilon \rightarrow \mathcal{E}_0$ is given through

$$u = \tilde{u} + \varepsilon(\tilde{u}_x + \tilde{v}), \quad v = \tilde{v} + \varepsilon(\tilde{u}\tilde{u}_x + \tilde{u}_{xx} + \tilde{u}\tilde{v} + \tilde{v}_x). \tag{8}$$

The recurrence relations upon densities of the Hamiltonian functionals for (6) are

$$\begin{aligned} \tilde{u}_0 &= u, & \tilde{v}_0 &= v, & \tilde{u}_k &= -D_x(\tilde{u}_{k-1}) - \tilde{v}_{k-1}, \\ \tilde{v}_k &= -D_x^2(\tilde{u}_{k-1}) - D_x(\tilde{v}_{k-1}) - \sum_{\ell+m=k-1} [\tilde{u}_\ell D_x(\tilde{u}_m) + \tilde{u}_\ell \tilde{v}_m], & k > 0. \end{aligned} \tag{9}$$

Relations (9) do not produce any auxiliary trivial conserved densities.

Relations (9) provide the formulae for the Hamiltonians of the (twice- and thrice-) modified Kaup–Boussinesq equations [11]. The modified fields are conserved; nevertheless, they are not obtained from the Hamiltonians for (6) by the Miura substitutions and the modified fields cannot be used as the negative terms in the general scheme of (9).

Remark 1. The Gardner deformation for (6) generates a parametric family of the Bäcklund autotransformations for the Burgers equation. First, recall that the Kaup–Boussinesq equation (6) is related to the Kaup–Broer system $u_t = u_{xx} + uu_x + w_x$, $w_t = (uw)_x - w_{xx}$ by using the invertible substitution $w = v - u_x$ (the field u remains unchanged). The further reduction $w = 0$ results in the Burgers equation $u_t = u_{xx} + uu_x$. Lifting the constraint $w = 0$ onto the extension (7), we get the equation $\tilde{u}_t = \tilde{u}_{xx} + \tilde{u}\tilde{u}_x$, which holds for all ε . Thus from (8) we derive the one-parametric Bäcklund autotransformation

$$u = \tilde{u} + 2D_x \ln(1 + \varepsilon\tilde{u})$$

for the Burgers equation; this expression generalizes the Fokas’ formula [20, 21].

Substitution (8) provides the canonical factorization of the extended Poisson structure $\hat{A}_1 + 2\varepsilon\hat{A}_2$ for the Kaup–Boussinesq equation, where \hat{A}_1 is defined in (3) and

$$\hat{A}_2 = \begin{pmatrix} D_x & \frac{1}{2}D_x \circ u \\ \frac{1}{2}uD_x & D_x^3 + \frac{1}{2}D_x \circ v + \frac{1}{2}vD_x \end{pmatrix}.$$

The two sequences of Hamiltonians, see (9), for the deformed Kaup–Boussinesq equation (7) are inherited from the original functionals for the Kaup–Boussinesq equation (6) by using the Miura substitution (8). The correlation between the higher flows for (7) and the symmetries of (6) is standard, see [3] and references therein.

Taking the flow at ε in the extension (7), we obtain the adjoint Kaup–Boussinesq equation

$$\begin{aligned} u_\tau &= uu_{xx} + u_x^2 + u_xv + uv_x, \\ v_\tau &= -(2u_xu_{xx} + uu_{xxx} + u_xv_x + uv_{xx} - vv_x). \end{aligned} \tag{10}$$

System (10) is bi-Hamiltonian with respect to two compatible local Poisson structures Γ_1 and Γ_2 , where Γ_1 is (3) and

$$\Gamma_2 = \begin{pmatrix} 0 & D_x \circ u_x + D_x \circ v \\ u_xD_x + vD_x & -u_{xx}D_x - D_x \circ u_{xx} - v_xD_x - D_x \circ v_x \end{pmatrix}. \tag{11}$$

We conclude that the adjoint Kaup–Boussinesq equation (10) is completely integrable.

Remark 2. Both Poisson structures (3) and (11) for (10) are of differential order 1. This indicates that the system can be further extended with a higher order symbol such that its complete integrability is preserved. The situation is analogous to the Gardner extension of the Korteweg–de Vries equation [1]; we recall that the extension of KdV resulted in the dispersionless modified KdV equation whose Poisson structures are of order 1.

Remark 3. The ‘minus’ Kaup–Boussinesq equation $u_t = uu_x + v_x$, $v_t = (uv)_x - u_{xxx}$ admits a unique real quadratic extension

$$\begin{aligned}\tilde{u}_t &= \tilde{u}\tilde{u}_x + \tilde{v}_x + \varepsilon(\tilde{u}\tilde{u}_{xx} + \tilde{u}_x^2), \\ \tilde{v}_t &= (\tilde{u}\tilde{v})_x - \tilde{u}_{xxx} - \varepsilon(\tilde{u}_x\tilde{v}_x + \tilde{u}\tilde{v}_{xx}) - \varepsilon^2(\tilde{u}^2\tilde{u}_{xxx} + \tilde{u}\tilde{u}_x\tilde{u}_{xx} + \tilde{u}_x^3).\end{aligned}\quad (12)$$

System (12) is assigned by the Poisson structure (3) to the Hamiltonian

$$\mathcal{H}'(\varepsilon) = \int \left(\frac{1}{2}\tilde{u}^2\tilde{v} + \frac{1}{2}\tilde{v}^2 + \frac{1}{2}\tilde{u}_x^2 + \varepsilon\tilde{u}\tilde{u}_x\tilde{v} + \frac{1}{2}\varepsilon^2\tilde{u}^2\tilde{u}_x^2 \right) dx.$$

The invertible Miura transformation from the extension (12) to the ‘minus’ Kaup–Boussinesq equation is

$$\{u = \tilde{u}, v = \tilde{v} + \varepsilon\tilde{u}\tilde{u}_x\} \iff \{\tilde{u} = u, \tilde{v} = v - \varepsilon uu_x\}. \quad (13)$$

The recurrence relation obtained from (13) provides only the Casimirs $\int u \, dx$ and $\int v \, dx$; all the conserved densities \tilde{v}_k are trivial if $k > 0$. Therefore, the deformation (12), (13) of the ‘minus’ Kaup–Boussinesq equation is trivial.

4. Conclusion

The Gardner deformations $(\mathcal{E}_\varepsilon, m_\varepsilon)$ were constructed for the (Kaup–)Boussinesq equations \mathcal{E}_0 by using symbolic computations, and explicit recurrence relations for infinite sequences of densities for the Hamiltonian functionals were obtained by expanding the Miura substitutions $m: \mathcal{E}_\varepsilon \rightarrow \mathcal{E}_0$ in the parameters ε . Integrability of the adjoint (Kaup–)Boussinesq systems was established by revealing the bi-Hamiltonian structure for (10) and by solving the Cauchy problem for (5). A one-parametric family of Bäcklund transformations for the Burgers equation was obtained by a reduction of the Gardner deformation for the Kaup–Boussinesq system.

In this paper, Gardner’s deformations were constructed for hierarchies of evolution equations \mathcal{E}_0 (see [3] for a deformation of the hyperbolic sine-Gordon equation). Recall that the ambient system \mathcal{E}_{EL} for the hierarchy of an evolution equation \mathcal{E}_0 is an Euler–Lagrange’s hyperbolic system such that the flows within the hierarchy are Noether’s symmetries of \mathcal{E}_{EL} , see [22, 23]. The ambient systems for KdV-type equations are Liouville-type: the 2D Toda lattice associated with the root system A_2 is [23] the ambient hyperbolic equation for the Boussinesq equation (2) and the ambient system \mathcal{E}_{EL} for the modified Kaup–Boussinesq hierarchy is an integrable extension [16] of the Liouville equation. It is straightforwardly proved that both 2D Toda-type systems \mathcal{E}_{EL} are fragile with respect to the Gardner deformations of \mathcal{E}_0 . Nevertheless, suppose a 2D Liouville-type Euler–Lagrange system \mathcal{E}_{EL} is known for an evolution equation \mathcal{E}_0 . Then the ambient system specifies admissible Miura’s transformations for \mathcal{E}_0 , which can be obstructions to the presence of Gardner’s deformations if they are non-invertible. Hence the search for the ambient Euler–Lagrange systems \mathcal{E}_{EL} helps to solve the problem of constructing Gardner’s deformations.

We conclude that Gardner’s deformations result in two types of the adjoint systems obtained by isolating the higher powers of the deformation parameters. In the first case, the adjoint systems inherit the Poisson structure proportional to D_x , see (3), and the Hamiltonians

from the original equations. Simultaneously, these adjoint systems may lose the higher Poisson structures. Thus we obtain completely integrable systems which are not manifestly bi-Hamiltonian. We conjecture that this is precisely the situation realized for the $N = 2$ SKdV _{$a=1$} equation [7, 24]. Hence a Gardner's deformation and a bi-Hamiltonian formulation for these systems are restored by constructing the 'adjoint to adjoint,' that is, the non-extended system. Secondly, the adjoint systems acquire new Magri's schemes similarly to the Poisson pencil (3), (11) for (10). In this case the problem of constructing Gardner's deformations is closely related to the purely algebraic homotopy procedure for extending the Poisson structures.

The results presented in this paper are essentially used for illustrating the general algebraic approach to the problem of Gardner's deformations and to the construction of two types of integrable extensions for the Magri schemes. The approach is based on the notion of coverings over PDE [12] and their parametric families constructed using the Frölicher–Nijenhuis bracket [13–15]. We emphasize that the algebraic scheme is essentially the same for constructing Gardner's deformations and the Bäcklund transformations; moreover, the former is relatively simpler than the latter. Hence the present paper provides new natural applications of the framework developed in [13]. We discover that Gardner's deformations, being dual to the Bäcklund transformations, are inhomogeneous generalizations of the infinitesimal symmetries. The algebraic formalism yields a cohomological description of Gardner's deformations and specifies the obstructions for existence of the deformations as non-invertible Miura's transformations. This will be the object of a subsequent publication, see [16].

We further intend to address the discretization problem for the spatial variables in Gardner's deformations. This amounts to the task of discretizing the extensions \mathcal{E}_ε and to the question whether the deformation parameters ε always remain continuous and whether the nature of the algebraic obstructions for the existence of deformations is preserved.

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