



## A Strange Recursion Operator for a New Integrable System of Coupled Korteweg–de Vries Equations

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**Abstract.** A recursion operator is constructed for a new integrable system of coupled Korteweg–de Vries equations by the method of gauge-invariant description of zero-curvature representations. This second-order recursion operator is characterized by unusual structure of its nonlocal part.

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### 1. Introduction

The Painlevé classification of coupled Korteweg–de Vries (KdV) equations was made recently in [2], and the following new system possessing the Painlevé property was found there:

$$\begin{aligned} u_t &= -\frac{\delta c_1}{12} u_{xxx} + \frac{3\delta c_1^2}{4d_1} v_{xxx} + c_1 u_x u - \frac{3c_1^2}{d_1} u_x v - \frac{6c_1^2}{d_1} v_x u, \\ v_t &= -\frac{\delta d_1}{12} u_{xxx} - \frac{7\delta c_1}{12} v_{xxx} + d_1 u_x u - c_1 u_x v - 2c_1 v_x u - \frac{6c_1^2}{d_1} v_x v, \end{aligned} \quad (1)$$

where  $\delta$ ,  $c_1$  and  $d_1$  are arbitrary nonzero constants.

More recently, the coupled KdV equations (1) appeared in [8], also as a system possessing the Painlevé property, in the following form:

$$\begin{aligned} u_t &= u_{xxx} - 9v_{xxx} - 12uu_x + 18vv_x + 18uv_x, \\ v_t &= u_{xxx} - 5v_{xxx} - 12uu_x + 12vv_x + 6uv_x + 18vv_x, \end{aligned} \quad (2)$$

which is related to (1) by the transformation (1)  $\mapsto$  (2):

$$u \mapsto \delta u - \frac{3\delta}{2} v, \quad v \mapsto -\frac{\delta d_1}{2c_1} v, \quad t \mapsto \frac{6}{\delta c_1} t. \quad (3)$$

It was found in [8] that the system (2) admits the zero-curvature representation (ZCR)

$$D_t X = D_x T - [X, T], \quad (4)$$

which is the compatibility condition for the overdetermined linear system

$$\Psi_x = X\Psi, \quad \Psi_t = T\Psi, \quad (5)$$

with

$$X = \begin{pmatrix} \sigma & u - v & \sigma(\frac{8}{3}u - 4v) \\ \frac{3}{2} & -2\sigma & u - v \\ 0 & \frac{3}{2} & \sigma \end{pmatrix}, \quad (6)$$

where  $\sigma$  is an essential ('spectral') parameter,  $D_t$  and  $D_x$  stand for the total derivatives, the square brackets denote the commutator,  $\Psi(x, t)$  is a three-component column, and an explicit (but cumbersome) expression for the  $3 \times 3$  matrix  $T$  can be seen in [8].

In the present paper, we study this new integrable system of coupled KdV equations in its equivalent form, which is characterized by more transparent structure of  $X$  in its ZCR. Namely, making the transformation

$$\begin{aligned} u &\mapsto 2u - \frac{1}{3}v, & v &\mapsto \frac{4}{3}u - \frac{1}{3}v, & t &\mapsto -\frac{1}{2}t, \\ \Psi &\mapsto S\Psi, & X &\mapsto SX S^{-1}, & T &\mapsto ST S^{-1}, & S &= \text{diag}\left(\frac{2}{3}, 1, \frac{3}{2}\right) \end{aligned} \quad (7)$$

in (2), (5) and (6), we obtain the system

$$\begin{aligned} u_t &= 4u_{xxx} - v_{xxx} - 12uu_x + vu_x + 2uv_x, \\ v_t &= 9u_{xxx} - 2v_{xxx} - 12vu_x - 6uv_x + 4vv_x \end{aligned} \quad (8)$$

which admits the ZCR (4) with

$$X = \begin{pmatrix} \sigma & u & \sigma v \\ 1 & -2\sigma & u \\ 0 & 1 & \sigma \end{pmatrix} \quad (9)$$

(the explicit expression for  $T$  is omitted, since not used in what follows).

Our aim is to obtain a recursion operator for the system (8). We derive the recursion operator from the matrix  $X$  (9) of the system's ZCR. We do this, following the method of gauge-invariant description of ZCRs of evolution equations, introduced in [7] (note also a more general gauge-invariant formulation of ZCRs, developed in [4]).

The paper is organized as follows. In Section 2, we find the complete class of coupled evolution equations admitting ZCRs (4) with the matrix  $X$  given by (9). This class is determined implicitly, through a three-term recurrence, which is explicitly solved in Section 3, giving rise to the recursion operator sought. In Section 4, we prove that this operator satisfies the sufficient condition of being a recursion operator in the sense of symmetries. Section 5 contains discussion of the results.

## 2. Class of Systems

Let us find the complete class of coupled evolution equations

$$\begin{aligned} u_t &= f(x, t, u, \dots, u_{x\dots x}, v, \dots, v_{x\dots x}), \\ v_t &= g(x, t, u, \dots, u_{x\dots x}, v, \dots, v_{x\dots x}) \end{aligned} \quad (10)$$

which admit ZCRs (4) with the matrix  $X$  given by (9) and without any restrictions imposed on the  $3 \times 3$  matrix  $T(x, t, u, \dots, u_{x\dots x}, v, \dots, v_{x\dots x}, \sigma)$ . We solve this problem algorithmically, using the technique of cyclic bases of ZCRs, developed in [7] (see also [9] for more examples).

Since the matrix  $X$  (9) does not contain derivatives of  $u$  and  $v$ , the characteristic matrices of the ZCR are simply  $C_u = \partial X / \partial u$  and  $C_v = \partial X / \partial v$ :

$$C_u = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_v = \begin{pmatrix} 0 & 0 & \sigma \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (11)$$

Using the operator  $\nabla_x$ , defined as  $\nabla_x Y = D_x Y - [X, Y]$  for any (here,  $3 \times 3$ ) matrix  $Y$ , we compute  $\nabla_x C_u$ ,  $\nabla_x C_v$ ,  $\nabla_x^2 C_u$ ,  $\nabla_x^2 C_v$ , etc., and find that the cyclic basis (i.e., the maximal sequence of linearly independent matrices  $\nabla_x^i C_u$  and  $\nabla_x^j C_v$ ,  $i, j = 0, 1, 2, \dots$ ) is

$$\{C_u, C_v, \nabla_x C_u, \nabla_x C_v, \nabla_x^2 C_u, \nabla_x^2 C_v, \nabla_x^3 C_u, \nabla_x^4 C_u\}, \quad (12)$$

and that the closure equations for this cyclic basis are

$$\begin{aligned} \nabla_x^5 C_u &= a_1 C_u + a_2 C_v + a_3 \nabla_x C_u + a_4 \nabla_x C_v + a_5 \nabla_x^2 C_u + \\ &\quad + a_6 \nabla_x^2 C_v + a_7 \nabla_x^3 C_u + a_8 \nabla_x^4 C_u, \\ \nabla_x^3 C_v &= b_1 C_u + b_2 C_v + b_3 \nabla_x C_u + b_4 \nabla_x C_v + b_5 \nabla_x^2 C_u + \\ &\quad + b_6 \nabla_x^2 C_v + b_7 \nabla_x^3 C_u + b_8 \nabla_x^4 C_u, \end{aligned} \quad (13)$$

where

$$\begin{aligned} a_1 &= u_{xxx} - 11uu_x + \frac{3}{2}vu_x - 27\sigma^2u_x + 6\sigma^2v_x, \\ a_2 &= -6u_{xxx} + 2v_{xxx} + 54uu_x - 3vu_x - 22uv_x + 3vv_x + \\ &\quad + 54\sigma^2u_x - 18\sigma^2v_x, \\ a_3 &= 4u_{xx} - 22u^2 + 3uv - 99\sigma^2u + \frac{45}{2}\sigma^2v - 81\sigma^4, \\ a_4 &= -24u_{xx} + 7v_{xx} + 54u^2 - 36uv + \frac{9}{2}v^2 + 54\sigma^2u - 9\sigma^2v, \\ a_5 &= 5u_x, \quad a_6 = -36u_x + 8v_x, \\ a_7 &= 13u - \frac{3}{2}v + 18\sigma^2, \quad a_8 = 0, \end{aligned}$$

$$\begin{aligned}
b_1 &= \frac{1}{2}u_x, & b_2 &= -u_x + v_x, & b_3 &= u + \frac{3}{2}\sigma^2, \\
b_4 &= -u + \frac{3}{2}v, & b_7 &= -\frac{1}{2}, & b_5 &= b_6 = b_8 = 0.
\end{aligned} \tag{14}$$

Thus, in the case of the matrix  $X$  determined by (9), the cyclic basis is eight-dimensional. Therefore the singular basis (see [7] for the definition) can be one-dimensional at most; in fact, it consists of the unit matrix  $I = \text{diag}(1, 1, 1)$  and, due to  $\nabla_x I = 0$ , has no effect on the form of the represented Equations (10).

Now, we rewrite the ZCR (4) in its equivalent (characteristic) form

$$fC_u + gC_v = \nabla_x T \tag{15}$$

and decompose the matrix  $T$  over the cyclic basis as

$$\begin{aligned}
T &= p_1 C_u + p_2 C_v + p_3 \nabla_x C_u + p_4 \nabla_x C_v + p_5 \nabla_x^2 C_u + \\
&\quad + p_6 \nabla_x^2 C_v + p_7 \nabla_x^3 C_u + p_8 \nabla_x^4 C_u,
\end{aligned} \tag{16}$$

where the coefficients  $p_1, \dots, p_8$  are functions of  $x, t, u, \dots, u_{x\dots x}, v, \dots, v_{x\dots x}$  and  $\sigma$ . Taking into account the closure equations (13) and the linear independence of the matrices (12), we obtain from (15) and (16) the following relations:

$$\begin{aligned}
f &= D_x p_1 + a_1 p_8 + b_1 p_6, & g &= D_x p_2 + a_2 p_8 + b_2 p_6, \\
p_1 &= -D_x p_3 - a_3 p_8 - b_3 p_6, & p_2 &= -D_x p_4 - a_4 p_8 - b_4 p_6, \\
p_3 &= -D_x p_5 - a_5 p_8 - b_5 p_6, & p_4 &= -D_x p_6 - a_6 p_8 - b_6 p_6, \\
p_5 &= -D_x p_7 - a_7 p_8 - b_7 p_6, & p_7 &= -D_x p_8 - a_8 p_8 - b_8 p_6.
\end{aligned} \tag{17}$$

These relations (17), together with (14), determine  $p_1, p_2, p_3, p_4, p_5, p_7, f$  and  $g$  in terms of some linear differential operators applied to the two (still undetermined) functions  $p_6$  and  $p_8$ . For what follows, we need no expression for  $T$ : it is sufficient to know that  $T$  exists. The expressions for the right-hand sides  $f$  and  $g$  of the represented Equations (10) are

$$h = (M + \lambda L + \lambda^2 K)r, \tag{18}$$

where

$$h = \begin{pmatrix} f \\ g \end{pmatrix}, \quad \lambda = 9\sigma^2, \quad r = \begin{pmatrix} p_8 \\ p_6 \end{pmatrix}, \tag{19}$$

$$K = \begin{pmatrix} D_x & 0 \\ 0 & 0 \end{pmatrix}, \tag{20}$$

$$L = \begin{pmatrix} -2D_x^3 + (11u - \frac{5}{2}v)D_x + (8u_x - \frac{11}{6}v_x) & -\frac{1}{6}D_x \\ (-6u + v)D_x - v_x & 0 \end{pmatrix}, \tag{21}$$

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \tag{22}$$

with

$$\begin{aligned} M_{11} = & D_x^5 + \left(-13u + \frac{3}{2}v\right)D_x^3 + \left(-34u_x + \frac{9}{2}v_x\right)D_x^2 + \\ & + \left(-33u_{xx} + \frac{9}{2}v_{xx} + 22u^2 - 3uv\right)D_x + \\ & + \left(-11u_{xxx} + \frac{3}{2}v_{xxx} + 33uu_x - \frac{3}{2}vu_x - 3uv_x\right), \end{aligned} \quad (23)$$

$$M_{12} = \frac{1}{2}D_x^3 - uD_x - \frac{1}{2}u_x, \quad (24)$$

$$\begin{aligned} M_{21} = & (-36u_x + 8v_x)D_x^2 + \\ & + \left(-48u_{xx} + 9v_{xx} - 54u^2 + 36uv - \frac{9}{2}v^2\right)D_x + \\ & + (-18u_{xxx} + 3v_{xxx} - 54uu_x + 33vu_x + 14uv_x - 6vv_x), \end{aligned} \quad (25)$$

$$M_{22} = D_x^3 + \left(u - \frac{3}{2}v\right)D_x - \frac{1}{2}v_x. \quad (26)$$

If  $\sigma$  in (9) is a fixed constant, then (18) and (19)–(26) give the solution of our problem: a continual class of coupled evolution equations, containing two arbitrary functions  $p_6$  and  $p_8$ , is represented by (4) with (9). However,  $\sigma$  is a free parameter in (9), and we need to take into account that the condition  $\partial h/\partial \lambda = 0$  must be satisfied for (18). Using the series expansion of  $r$

$$r = r_0 + \lambda r_1 + \lambda^2 r_2 + \lambda^3 r_3 + \lambda^4 r_4 + \dots \quad (27)$$

(if a singularity is suspected in  $r$  at  $\lambda = 0$ , one may use (27) after an infinitesimal shift of  $\lambda$ , which does not affect final results), we obtain from (18) that

$$h = Mr_0 \quad (28)$$

and

$$0 = Mr_1 + Lr_0, \quad (29a)$$

$$0 = Mr_2 + Lr_1 + Kr_0, \quad (29b)$$

$$0 = Mr_3 + Lr_2 + Kr_1, \quad (29c)$$

$$0 = Mr_4 + Lr_3 + Kr_2, \quad (29d)$$

$\dots$ .

This gives us the following implicit solution of our problem: the system of coupled evolution equations (10) admits the ZCR (4) with the matrix  $X$  given by (9) if and only if its right-hand side  $h = (f, g)^T$  is determined through (28) by the set of two-component functions  $r_0, r_1, r_2, \dots$  satisfying the three-term recurrence (29), where the matrix differential operators  $M, L$  and  $K$  are defined by (20)–(26).

### 3. Recursion of Systems

Let us solve the recurrence (29), using formal inversion of differential operators. We say that the operator  $P^{-1}$  is formally inverse to a linear differential operator  $P$  if  $P^{-1}P = PP^{-1} = I$ , where  $I$  is the unit operator. In other words,  $P^{-1}a$  denotes any function  $b$  of local variables, such that  $a = Pb$ , if  $b$  exists. We have to remind that, in the present case, differential operators contain total (not partial) derivatives  $D_x$  and act on differential functions (in Olver's sense [6], i.e., functions of independent variables, dependent variables and finite-order derivatives of dependent variables). For this reason,  $D_x^{-1}0 = \phi(t)$  with any function  $\phi$ , whereas  $(D_x^2 + u)^{-1}0 = 0$  and  $(3D_x^3 - 4vD_x - 2v_x)^{-1}v_x = -1/2$ , to give some examples.

For the operators  $K$ ,  $L$  and  $M$ , defined by (20)–(26), we notice that the inverse operators  $M^{-1}$  and  $L^{-1}$  exist, but  $K^{-1}$  does not exist, and therefore we cannot use inversion of  $K$  in what follows. We obtain

$$r_0 = -L^{-1}Mr_1 \quad (30)$$

from (29a), and

$$r_1 = (KL^{-1}M - L)^{-1}Mr_2 \quad (31)$$

from (29b), and notice that the form of (31) is different from the form of (30), because  $KL^{-1}M \neq 0$  for the operators defined by (20)–(26). Then we obtain

$$r_2 = (-K(KL^{-1}M - L)^{-1}M - L)^{-1}Mr_3 \quad (32)$$

from (29c), and find that, fortunately, it is possible to rewrite (32) as

$$r_2 = (KL^{-1}M - L)^{-1}Mr_3 \quad (33)$$

owing to the condition

$$KL^{-1}K = 0 \quad (34)$$

satisfied by  $K$  (20) and  $L$  (21). We see that the form of (33) coincides with the form of (31). The same is true, also owing to (34), for the relation between  $r_3$  and  $r_4$ , which follows from (29d); and, in general,

$$r_i = (KL^{-1}M - L)^{-1}Mr_{i+1}, \quad i = 1, 2, 3, \dots \quad (35)$$

We have to emphasize that the recursion (35) is caused by the specific property (34) of the operators  $K$  (20) and  $L$  (21), and that it would be impossible to solve the three-term recurrence (29) for generic operators.

For convenience in what follows, we rewrite (35) as

$$r_i = M^{-1}LM^{-1}RML^{-1}Mr_{i+1}, \quad i = 1, 2, 3, \dots, \quad (36)$$

where

$$R = MN^{-1} \quad (37)$$

with

$$N = ML^{-1}K - L. \quad (38)$$

The explicit expressions for  $N$  and  $R$  are

$$N = \begin{pmatrix} -D_x^3 + (-5u + \frac{5}{2}v)D_x + (-5u_x + \frac{11}{6}v_x) & \frac{1}{6}D_x \\ -6D_x^3 + 8vD_x + 4v_x & 0 \end{pmatrix} \quad (39)$$

and

$$R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \quad (40)$$

with

$$R_{11} = 3D_x^2 - 6u - 3u_x D_x^{-1}, \quad (41)$$

$$\begin{aligned} R_{12} = & [-2D_x^5 + (2u + 3v)D_x^3 + (-4u_x + 8v_x)D_x^2 + \\ & + (-6u_{xx} + 7v_{xx} + 4u^2 - 6uv)D_x + \\ & + (-2u_{xxx} + 2v_{xxx} + 6uu_x - 3vu_x - 4uv_x) + \\ & + u_x D_x^{-1} v_x](3D_x^3 - 4vD_x - 2v_x)^{-1}, \end{aligned} \quad (42)$$

$$R_{21} = 6D_x^2 + (6u - 9v) - 3v_x D_x^{-1}, \quad (43)$$

$$\begin{aligned} R_{22} = & [-3D_x^5 + (-18u + 12v)D_x^3 + (-27u_x + 18v_x)D_x^2 + \\ & + (-21u_{xx} + 14v_{xx} + 12u^2 + 12uv - 9v^2)D_x + \\ & + (-6u_{xxx} + 4v_{xxx} + 12uu_x + 6vu_x + 6uv_x - 9vv_x) + \\ & + v_x D_x^{-1} v_x](3D_x^3 - 4vD_x - 2v_x)^{-1}. \end{aligned} \quad (44)$$

Now, using (28), (30) and (36), we obtain the following expression for  $h$ :

$$h = -R^n M L^{-1} M r_{n+1}, \quad n = 0, 1, 2, \dots \quad (45)$$

Since the leading-order terms of the operator  $R$  (40)–(44) are given by

$$R = \begin{pmatrix} 3D_x^2 + \dots & -\frac{2}{3}D_x^2 + \dots \\ 6D_x^2 + \dots & -D_x^2 + \dots \end{pmatrix} \quad (46)$$

and the orders of derivatives of  $u$  and  $v$  in  $h$  must be finite, we can choose a sufficiently large  $n$  in (45), so that the orders of derivatives of  $u$  and  $v$  in  $M L^{-1} M r_{n+1}$  are less than 2 for this  $n$ . In this case, we conclude, taking into account the form of  $M$  (22)–(26) and  $L$  (21), that  $L^{-1} M r_{n+1}$  must be a function of  $x$  and  $t$  only, that the orders of derivatives of  $u$  and  $v$  in  $M r_{n+1}$  must be less than 2, and that  $r_{n+1} = (0, 0)^T$ . Then we find that

$$L^{-1} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2\phi(t) \end{pmatrix}, \quad (47)$$

where the function  $\phi(t)$  is arbitrary, and the factor 2 is taken for convenience. Finally, (45) gives us

$$h = \phi(t)R^n \begin{pmatrix} u_x \\ v_x \end{pmatrix}, \quad \forall \phi(t), \quad n = 0, 1, 2, \dots, \quad (48)$$

or, equivalently,

$$h = R^n \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad n = 1, 2, 3, \dots, \quad (49)$$

because  $R(0, 0)^T = (\phi(t)u_x, \phi(t)v_x)^T$  with any  $\phi(t)$ .

Thus, the problem, formulated in the beginning of Section 2, is solved explicitly: the relation (49) determines the right-hand sides  $h = (f, g)^T$  of all systems (10) which admit the linear problem (5) with the matrix  $X$  (9). The constructed operator  $R$  (40)–(44) turns out to be a recursion operator in the Lax sense: it generates a hierarchy of integrable systems which all possess Lax pairs with the predetermined spatial part. The system (8) corresponds to  $n = 2$  in (49), and the next member of the hierarchy ( $n = 3$ ) is the fifth-order system

$$\begin{aligned} u_t &= 6u_{xxxxx} - \frac{5}{3}v_{xxxxx} - 60uu_{xxx} + \frac{40}{3}vu_{xxx} + \frac{50}{3}uv_{xxx} - \\ &\quad - \frac{10}{3}vv_{xxx} - 150u_xu_{xx} + 40v_xu_{xx} + \frac{125}{3}u_xv_{xx} - 10v_xv_{xx} + \\ &\quad + 120u^2u_x - 40uvu_x + \frac{5}{3}v^2u_x - \frac{80}{3}u^2v_x + \frac{20}{3}uvv_x, \\ v_t &= 15u_{xxxxx} - 4v_{xxxxx} - 120uu_{xxx} + 10vu_{xxx} + 30uv_{xxx} - \\ &\quad - \frac{5}{3}vv_{xxx} - 360u_xu_{xx} + 80v_xu_{xx} + 90u_xv_{xx} - \frac{55}{3}v_xv_{xx} + \\ &\quad + 160uvu_x - 40v^2u_x + 40u^2v_x - 60uvv_x + \frac{35}{3}v^2v_x. \end{aligned} \quad (50)$$

#### 4. Recursion of Symmetries

The operator  $R$ , defined by (40)–(44), is also a recursion operator with respect to symmetries of the coupled KdV equations (8). In order to see this, we have to prove that  $R$  satisfies the following condition [6]:

$$D_t R = [H, R], \quad (51)$$

where  $H$  is the Fréchet derivative of the right-hand side of the system (8),

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \quad (52)$$



with

$$\begin{aligned}
H_{11} &= 4D_x^3 + (-12u + v)D_x + (-12u_x + 2v_x), \\
H_{12} &= -D_x^3 + 2uD_x + u_x, \\
H_{21} &= 9D_x^3 - 12vD_x - 6v_x, \\
H_{22} &= -2D_x^3 + (-6u + 4v)D_x + (-12u_x + 4v_x).
\end{aligned} \tag{53}$$

It is convenient to study the condition (51) in its equivalent form:

$$HM - D_t M = R(HN - D_t N), \tag{54}$$

which follows from (51) through the definition of  $R$  (37) and the identity  $D_t N^{-1} = -N^{-1}(D_t N)N^{-1}$ . Direct computations show that the condition (54) is satisfied identically by  $H$  (52)–(53),  $M$  (22)–(26),  $R$  (40)–(44),  $N$  (39) and the expressions (8) for  $u_t$  and  $v_t$ . Since the condition (51) is satisfied, the system of coupled KdV equations (8) possesses infinitely many symmetries: the recursion operator  $R$  generates a ‘new’ symmetry (but not necessarily a local one) from any ‘old’ symmetry.

## 5. Conclusion

The recursion operator  $R$  (40)–(44), constructed in this paper for the new integrable system of coupled KdV equations (8), is characterized by unusual structure of its nonlocal part, which contains not only the conventional operator  $D_x^{-1}$  but also the strange nonlocal operator  $(3D_x^3 - 4vD_x - 2v_x)^{-1}$ . Such a phenomenon has not been observed in the literature as yet. This operator  $(3D_x^3 - 4vD_x - 2v_x)^{-1}$  cannot be expressed in terms of a finite number of operators  $D_x^{-1}$ , if one uses only local variables, as we did throughout the paper. However, if we introduce the nonlocal variable  $w: v = 3w_{xx}/w$ , we can represent the obtained recursion operator in a more conventional form, using the factorization  $(3D_x^3 - 4vD_x - 2v_x)^{-1} = \frac{1}{3}w^2 D_x^{-1} w^{-2} D_x^{-1} w^{-2} D_x^{-1} w^2$ .

We found the recursion operator of the system (8) by the technique of cyclic bases of ZCRs. It is interesting if this recursion operator can be obtained by different methods as well, e.g., by those developed in [1, 3, 5].

We have seen that the relation (49) generates local expressions for  $h$  at  $n = 1, 2, 3$ . It is easy to check that the seventh-order system of this hierarchy ( $n = 4$ ) is also a local expression. Nevertheless, we have not proven as yet that this is the case for any  $n$ .

Though our method produces the recursion operator in the quotient form  $MN^{-1}$  (37), the operators  $M$  (22)–(26) and  $N$  (39) are not Hamiltonian operators. It is still unknown if the obtained recursion operator is hereditary and if the system (8) admits a bi-Hamiltonian structure.

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