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The Burgers-Hlavaty equation

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Summary. — We study the prolongation stucture of a Painlevé admissible extension of Burgers equation containing two arbitrary analytic functions obtained by Hlavaty. We also obtain the linear scattering problem through the soliton connection. Furthermore, we give an exact solution for the Burgers-Hlavaty equation.

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1. – Introduction

Nonlinear ordinary and partial differential equations play a central role in almost all physical theories. One of the most fundamental and important problems in the investigation of nonlinear dynamical systems is to give a general criterion which decides the integrability. The Painlevé algorithm (or singular-point analysis) [1, 2] is one of the most powerful techniques for identifying integrable systems. The basic idea is whether a general solution of the considered partial differential equation has the expansion

$$u = \phi^n \sum_{j=0}^{\infty} v_j \phi^j \,, \tag{1.1}$$

where ϕ is the analytic function whose vanishing condition defines a noncharacteristic hypersurface. Substituting this expansion into the tested equation gives the conditions on n and recursion relations for the functions v_j. The Painlevé property states that n should be an integer, recursion relations should be consistent and the series expansion (1.1) should contain the correct number of arbitrary functions. The equations passing this test is called Painlevé admissible.

The possibility of using the singular-point analysis for the classification of Painlevé-type ODEs and the extension of this method to the PDEs was proposed by Hlavaty [3, 4]. The classification scheme follows the basic steps of the singular-point analysis. In the first step the types of equations that admit the expansion (1.1) with negative integer n are determined. Afterwards, the resonance analysis and the compatibility conditions of the recursion formula for v_j are used to specify the coefficients of these equations. Applying this method to polynomial second-order partial differential equations in two independent variables, Hlavaty obtained a general extension of the Burgers equation,

$$u_{xx} = -2u_x u + L u_t + (L_x/L)u^2 + (2L_x^2/L^2 - L_{xx}/L)u + S\,, \tag{1.2}$$

where L and S are analytic functions of x and t. This equation is Painlevé admissible and, therefore, should be completely integrable.

In this work eq. (1.2) is studied within the prolongation scheme of Estabrook and Wahlquist [5]. The linear eigenvalue equations and associated time evolution equations are derived. An exact solution depending on x, t and the prolongation variable y is given.

2. – Prolongation structure

For our purposes, it is convenient to write eq. (1.2) as

$$L u_t = u_{xx} + 2u_x u + \lambda_x u - \lambda u(u+\lambda) - S\,, \tag{2.1}$$

where $\lambda = L_x/L$.

Introducing the notation $p = u_x$, eq. (2.1) reduces to a first-order partial differential equation which can be associated with the set of two 2-forms

$$\begin{cases} \alpha_1 \;=\; \mathrm{d}u \wedge \mathrm{d}t - p\,\mathrm{d}x \wedge \mathrm{d}t\,, \\ \alpha_2 \;=\; \mathrm{d}p \wedge \mathrm{d}t + [2pu + \lambda_x u - \lambda u(\lambda+u) - S]\mathrm{d}x \wedge \mathrm{d}t + L\,\mathrm{d}u \wedge \mathrm{d}x\,. \end{cases} \tag{2.2}$$

The set $\{\alpha_1, \alpha_2\}$ constitutes a closed ideal. We seek a set of 1-forms,

$$w_k = \mathrm{d}y_k + F_k\,\mathrm{d}x + G_k\,\mathrm{d}t, \qquad k = 1, \ldots, N \tag{2.3}$$

with $F_k(x,t,u,p,y_i)$ and $G_k(x,t,u,p,y_i)$ having the property that the prolonged ideal $\{\alpha_1, \alpha_2, w_1\,, \ldots, w_N\}$ is closed. This requirement gives the set of partial differential equations for F_k and G_k. Dropping the indices for simplicity we have

$$\begin{cases} F_p = 0\,, \\ G_p - \dfrac{F_u}{L} = 0\,, \\ G_x - F_t - \dfrac{F_u}{L}[2pu + \lambda_x u - \lambda u(\lambda+u) - S] + pG_u + [F,G] = 0\,, \end{cases} \tag{2.4}$$

where

$$[F,G] = \frac{\partial F}{\partial y_i} G_i - \frac{\partial G}{\partial y_i} F_i. \tag{2.5}$$

Integrating eqs. (2.4) we find

$$\text{(2.6)} \qquad \begin{cases} F = Au + B\,, \\ G = p\dfrac{A}{L} + \dfrac{1}{L}\left\{Au^2 + (\lambda A - A_x - [B,A])u + C\right\} \end{cases}$$

and

$$\text{(2.7)} \qquad \begin{cases} A_x - [A, A_x] - [A,[B,A]] + [B,A] = 0\,, \\ LB_t = C_x - \lambda C + [B,C] + AS\,, \\ LA_t = -A_{xx} + 2\lambda A_x - [B,A]_x - [B,A_x] + 2\lambda[B,A] - [B,[B,A]] + [A,C]\,, \end{cases}$$

where A, B, C are functions of x, t and y_i. In particular, if $L = 1$, $S = 0$ (Burgers equation) the results in eq. (2.6) are in agreement with those in ref. [6].

In order to close the algebra defined by eqs. (2.7), let us assume that the set of variables $\{y_i\}$ has one element only; A, B and $[A, B]$ are linearly independent and $A_x, B_x, [A,B]$ and C are given by

$$\text{(2.8)} \qquad \begin{cases} [B,A] = D\,, \\ A_x = \alpha A - B\,, \\ B_x = \gamma A + \delta B + \eta D\,, \\ C = \mu A + \nu B + \rho D\,, \end{cases}$$

where $\alpha, \gamma, \delta, \eta, \mu, \nu, \rho$ are functions of x and t only.

Inserting eqs. (2.8) into eqs. (2.7) and using the Jacobi identity we obtain the finite-dimensional Lie algebra

$$\text{(2.9)} \qquad \begin{cases} [A,D] = \alpha A - B\,, \\ [B,D] = -\alpha B\,, \\ [B,A] = D\,, \\ [B,C] = \mu D - \alpha\rho B\,, \\ [C,A] = \nu D + \rho B - \alpha\rho A\,, \\ [C,D] = \mu\alpha A - (\mu + \nu\alpha)B\,, \end{cases}$$

which can be related to the Lie algebra $SL(2,R)$ [7], and

$$\text{(2.10)} \qquad \begin{cases} A_t = \gamma_1 A + \gamma_2 B + \gamma_3 D\,, \\ B_t = \delta_1 A + \delta_2 B + \delta_3 D\,, \end{cases}$$

where

$$(2.11)\qquad \begin{cases} \gamma_1 \;=\; \dfrac{1}{L}(-\alpha_x - \alpha^2 + 2\lambda\alpha + \alpha\rho + \alpha\eta + \gamma)\,, \\ \gamma_2 \;=\; \dfrac{1}{L}(2\alpha - 2\lambda - \rho - \eta + \delta)\,, \\ \gamma_3 \;=\; \dfrac{1}{L}(-2\alpha + 2\lambda - \nu + \eta - \delta)\,, \\ \delta_1 \;=\; \dfrac{1}{L}(\mu_x - \lambda\mu + \alpha\mu + \nu\gamma - \rho\eta\alpha + S)\,, \\ \delta_2 \;=\; \dfrac{1}{L}(\nu_x - \lambda\nu - \mu + \delta\nu + \rho\eta - \alpha\rho)\,, \\ \delta_3 \;=\; \dfrac{1}{L}(\rho_x - \lambda\rho + \nu\eta + \delta\rho + \alpha\rho + \mu)\,. \end{cases}$$

Using the integrability conditions $A_{xt} = A_{tx},\ B_{xt} = B_{tx},\ D_{xt} = D_{tx}$ and $\alpha_{xt} = \alpha_{tx}$ we obtain

$$(2.12)\qquad \begin{cases} \alpha_t \;=\; \dfrac{1}{L}\{\alpha[\nu_x + \lambda\nu + \rho\eta - \nu^2 + \nu\eta - \nu\alpha - \mu] + (\rho+\nu)\gamma\}\,, \\ \alpha_x \;=\; (2\lambda - \nu + \eta - \alpha)\alpha + \gamma\,, \\ \rho_x \;=\; (\nu + \alpha - \lambda)\rho - \mu\,, \\ \mu_x \;=\; (\lambda - \alpha)\mu + \rho\eta\alpha + \rho\gamma - S\,, \\ \nu_t \;=\; 2\lambda_t - \dfrac{1}{L}[\nu_{xx} - 2\nu_x\nu + \lambda_x\nu - \lambda^2\nu + \lambda\nu^2 + S]\,, \\ \gamma_t \;=\; \dfrac{1}{L}[2\nu_x\gamma + \gamma_x(\rho+\nu) - 2\gamma(2\lambda\rho + \lambda\nu - \rho\nu - \rho\alpha + \mu)]\,, \\ \eta_t \;=\; \dfrac{1}{L}[\nu_x\eta + \eta_x(\rho+\nu) - \eta(2\lambda\rho + \lambda\nu - \rho\nu - \rho\alpha + \mu)]\,, \\ \delta \;=\; 2\lambda - 2\alpha + \eta - \nu\,. \end{cases}$$

3. – Soliton connection and the linear problem

We can write F and G in (2.6) in terms of the generators of $SL(2,R)$ algebra satisfying the commutation relations

$$(3.1)\qquad [X_1, X_2] = 2X_2\,, \qquad [X_1, X_3] = -2X_3\,, \qquad [X_2, X_3] = X_1\,.$$

Choosing

$$(3.2)\qquad \begin{cases} B \;=\; 2\alpha\epsilon X_2\,, \\ A \;=\; \epsilon X_2 + \dfrac{1}{4\epsilon}X_3\,, \\ D \;=\; \dfrac{\alpha}{2}X_1\,, \end{cases}$$

where

$$\epsilon_x = -\alpha\epsilon\,, \quad \epsilon_t = \frac{\epsilon}{L}(\alpha_x + \alpha^2 - 2\alpha\lambda - \alpha\rho)$$

and using the 2×2 matrix representation of the generators X_1, X_2, X_3 we can construct an $SL(2,R)$-valued connection 1-form

$$\Gamma = \begin{pmatrix} \theta_0 & \theta_1 \\ \theta_2 & -\theta_0 \end{pmatrix}, \tag{3.3}$$

where

$$\begin{cases} \theta_0 &= \dfrac{\alpha}{2L}(\rho - u)\mathrm{d}t\,, \\ \theta_1 &= \epsilon\{(u+2\alpha)\mathrm{d}x + \dfrac{1}{L}[p + u^2 + (\lambda+\alpha)u + \mu + 2\alpha\nu]\mathrm{d}t\}\,, \\ \theta_2 &= \dfrac{1}{4\epsilon}\{u\mathrm{d}x + \dfrac{1}{L}[p + u^2 + (\lambda-\alpha)u + \mu]\mathrm{d}t\}\,. \end{cases} \tag{3.4}$$

The connection Γ defines a linear equation $\mathrm{d}\Psi = -\Gamma\Psi$, where Ψ is a column vector with components ψ_1 and ψ_2. The integrability condition yields the Burgers-Hlavaty equation together with eqs. (2.12) where $\gamma = \eta = 0$ for our particular choice of representation.

The associated linear equation does not contain any spectral parameter. To introduce such a parameter, we can perform an $SL(2,R)$ gauge transformation

$$\Gamma' = \Sigma\Gamma\Sigma^{-1} + \Sigma\mathrm{d}\Sigma^{-1}\,, \tag{3.5}$$

where

$$\Sigma = \begin{pmatrix} e^{i\xi x} & 0 \\ 0 & e^{-i\xi x} \end{pmatrix}. \tag{3.6}$$

In this case the linear equations for ψ_1 and ψ_2 are

$$\begin{cases} \psi_{1x} &= i\xi\psi_1 - \epsilon e^{2i\xi x}(u+2\alpha)\psi_2\,, \\ \psi_{1t} &= -\dfrac{\alpha}{2L}(\rho-u)\psi_1 - \dfrac{\epsilon}{L}e^{2i\xi x}[p + u^2 + (\lambda-\alpha)u + \mu + 2\alpha(u+\nu)]\psi_2\,, \end{cases} \tag{3.7}$$

$$\begin{cases} \psi_{2x} &= -\dfrac{1}{4\epsilon}e^{-2i\xi x}u\psi_1 - i\xi\psi_2\,, \\ \psi_{2t} &= -\dfrac{1}{4\epsilon L}e^{-2i\xi x}[\,p + u^2 + (\lambda-\alpha)u + \mu]\psi_1 + \dfrac{\alpha}{2L}(\rho-u)\psi_2\,. \end{cases} \tag{3.8}$$

These equations can be reduced to the following scattering problems for ψ_1 and ψ_2:

$$\begin{cases} \psi_{1xx} &= \{2i\xi - \alpha + [\ln(u+2\alpha)]_x\}\psi_{1x} + \\ & \quad + \left\{\dfrac{u}{4}(u+2\alpha) - i\xi(i\xi - \alpha + [\ln(u+2\alpha)]_x)\right\}\psi_1\,, \\ \psi_{1t} &= -\left\{\dfrac{i\xi}{L(u+2\alpha)}[u_x + u^2 + (\lambda+\alpha)u + \mu + 2\alpha\nu] + \dfrac{\alpha}{2L}(\rho-u)\right\}\psi_1 + \\ & \quad + \dfrac{1}{L(u+2\alpha)}[u_x + u^2 + (\lambda+\alpha)u + \mu + 2\alpha\nu]\psi_{1x}\,, \end{cases} \tag{3.9}$$

$$
(3.10)\qquad \left\{\begin{aligned}
\psi_{2xx} &= \left(\alpha - 2i\xi + \frac{u_x}{u}\right)\psi_{2x} + \left(\xi^2 + i\xi\alpha + i\xi\frac{u_x}{u} + \frac{u^2}{4} + \frac{u\alpha}{2}\right)\psi_2\,,\\
\psi_{2t} &= \frac{1}{Lu}[u_x + u^2 + (\lambda-\alpha)u + \mu]\psi_{2x} +\\
&\quad + \left\{\frac{i\xi}{Lu}[u_x + u^2 + (\lambda-\alpha)u + \mu] + \frac{\alpha}{2L}(\rho - u)\right\}\psi_2\,.
\end{aligned}\right.
$$

4. – Backlund transformations

Assuming that one particular solution of the prolonged ideal α_i, w_k is known, another solution of eq. (1.2) can be written as

$$
(4.1)\qquad \left\{\begin{aligned}
\tilde{u} &= \tilde{u}(u,p,x,t,y)\,,\\
\tilde{p} &= \tilde{p}(u,p,x,t,y)\,.
\end{aligned}\right.
$$

Substituting these into the set of forms

$$
(4.2)\qquad \left\{\begin{aligned}
\tilde{\alpha_1} &= \tilde{\mathrm{d}u}\wedge \mathrm{d}t - \tilde{p}\,\mathrm{d}x\wedge\mathrm{d}t\,,\\
\tilde{\alpha_2} &= \tilde{\mathrm{d}p}\wedge\mathrm{d}t + [2\tilde{p}\tilde{u} + \lambda_x\tilde{u} - \lambda\tilde{u}(\lambda+\tilde{u}) - S]\mathrm{d}x\wedge\mathrm{d}t + L\,\mathrm{d}\tilde{u}\wedge\mathrm{d}x\,,
\end{aligned}\right.
$$

and requiring these be in the ring of prolonged ideal we have the following differential equations:

$$
(4.3)\qquad \left\{\begin{aligned}
&\tilde{u}_p = 0\,,\\
&\tilde{p}_p - \tilde{u}_u = 0\,,\\
&\tilde{u}_u p - \tilde{u}_p[2pu + \lambda_x u - \lambda u(\lambda+u) - S] + \tilde{u}_x - \tilde{u}_y F - \tilde{p} = 0\,,
\end{aligned}\right.
$$

$$
(4.4)\qquad \begin{aligned}
&\tilde{p}_u p - \tilde{p}_p[2pu + \lambda_x u - \lambda u(\lambda+u) - S] + \tilde{p}_x - \tilde{p}_y F - 2\tilde{p}\tilde{u} +\\
&\quad +\lambda_x\tilde{u} - \lambda\tilde{u}(\lambda+\tilde{u}) - S - L\tilde{u}_t + L\tilde{u}_y G = 0\,.
\end{aligned}
$$

Equations in (4.3) imply that

$$
(4.5)\qquad \left\{\begin{aligned}
\tilde{u} &= \tilde{u}(u,x,t,y)\,,\\
\tilde{p} &= \tilde{u}_u p + \tilde{u}_x - \tilde{u}_y F\,.
\end{aligned}\right.
$$

Substituting these into eq. (4.4) where F and G are given in (2.6), we have

$$
(4.6)\qquad \tilde{u} = W(x,t,y)u + \sigma(x,t,y)\,,
$$

where

$$
(4.7)\qquad W_y = \frac{1}{A}(W^2 - W)\,,
$$

$$
(4.8)\qquad W_x = W_y B - W\sigma\,,
$$

$$(4.9) \qquad \begin{aligned} &-2AW_{xy} + 2ABW_{yy} + \sigma_{yy}A^2 - \sigma_y(2W - 1 - A_y)A - \\ &\quad -W_y(2A_x - A_yB - B_yA + 2WB + 2\sigma A - A\lambda + D) + \\ &\quad +2W_xW - \lambda W(W-1) = 0\,, \end{aligned}$$

$$(4.10) \qquad \begin{aligned} LW_t = &-2BW_{xy} - 2AB\sigma_{yy} + W_{yy}B^2 - 2A\sigma_{xy} + W_{xx} - \\ &-W_y(B_x - B_yB + 2\sigma B - C) - \\ &-\sigma_y(2A_x - A_yB - B_yA + 2WB + 2\sigma A - A\lambda + D) + \\ &+2W_x\sigma + 2\sigma_xW - 2\lambda W\sigma\,, \end{aligned}$$

$$(4.11) \qquad \begin{aligned} L\sigma_t = &-2\sigma_{xy}B + \sigma_{yy}B^2 - \sigma_yB_x + \sigma_yB_yB + \sigma_yC - 2\sigma_y\sigma B + \\ &+2\sigma_x\sigma + \sigma_{xx} + \lambda_x\sigma - \lambda\sigma(\sigma+\lambda) + WS - S\,. \end{aligned}$$

The integrability condition $W_{xy} = W_{yx}$ gives

$$(4.12) \qquad \sigma_y = \frac{1}{A}\left[W\sigma + \frac{1}{A}(W-1)(\alpha A - B + D)\right]\,.$$

With eqs. (4.7), (4.8) and (4.12), eq. (4.9) holds identically. On the other hand, the integrability condition $W_{xt} = W_{tx}$ gives

$$(4.13) \qquad \alpha = \frac{B^2 - D^2}{2AB}\,, \quad \gamma = 0\,,$$

where $\alpha_y = 0$. The integrability conditions $W_{xt} = W_{tx}$, $\sigma_{yt} = \sigma_{ty}$ with eq. (4.13) gives us

$$(4.14) \qquad W = 1,\ \ \sigma = 0\,,$$

which implies that we do not have Backlund transformation other than identity map.

Now let us consider eqs. (4.7)-(4.11) and assume that $W = 0$. Then from eq. (4.9) we have

$$(4.15) \qquad \sigma = f(x,t) + \frac{1}{A}(\alpha A - B + D)\,,$$

where $\alpha = (B^2 - D^2)/2AB$. Inserting (4.15) into (4.10) and (4.11), we obtain

$$(4.16) \qquad f(x,t) = \frac{1}{2}(2\nu - 3\lambda)\,,$$

$$(4.17) \qquad \nu_{xx} = \frac{1}{8}[6\lambda_{xx} - 12\lambda_x\lambda + 4\lambda_x\nu + 2\lambda_tL + 12\nu_x\lambda + 3\lambda^3 - 4\lambda^2\nu]\,.$$

Thus the expression

$$(4.18) \qquad \sigma(x,t,y) = \frac{1}{2}(2\nu - 3\lambda) - \frac{1}{2AB}(B-D)^2$$

is an exact solution for the Burgers-Hlavaty equation provided that the relations in (2.12) and (4.17) are satisfied.

Notice that for the time-independent case where the functions L and S depend on x only, we have

$$\nu = -u(x, -t)\,, \tag{4.19}$$

where u is a solution of eq. (1.2). If, in addition, the functions ν and L satisfy eq. (4.17), we have a Backlund transformation,

$$\tilde{u} = -u(x, -t) - \frac{1}{2}\left[3\lambda + \frac{(B-D)^2}{AB}\right]. \tag{4.20}$$

5. – Summary

We have shown that the Burgers-Hlavaty equation, an equation with variable coefficients, possesses non-Abelian prolongations. Using the representation of the quotient Lie algebra we have derived the linear eigenvalue and associated time evolution equations. We have found an exact solution, given in eq. (4.18), for the Burgers-Hlavaty equation. After completing this work we noticed that the equation under consideration is linearisable by Forsyth-Hopf-Cole transformation [8].

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