Integrability of Kersten–Krasil'shchik coupled KdV–mKdV equations: singularity analysis and Lax pair

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The integrability of a coupled KdV–mKdV system is tested by means of singularity analysis. The true Lax pair associated with this system is obtained by the use of prolongation technique. © *2003 American Institute of Physics.*
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I. INTRODUCTION

Very recently, Kersten and Krasil'shchik[1] constructed the recursion operator for symmetries of a coupled KdV–mKdV system

$$\begin{aligned} u_t &= -u_{xxx} + 6uu_x - 3ww_{xxx} - 3w_x w_{xx} + 3u_x w^2 + 6uww_x, \\ w_t &= -w_{xxx} + 3w^2 w_x + 3uw_x + 3u_x w, \end{aligned} \tag{1}$$

which arises as the classical part of one of the superextensions of the KdV equation. In this work, we study the integrability of this system using the Painlevé test. Then, we use Dodd–Fordy[2] algorithm of the Wahlquist–Estabrook[3] prolongation technique in order to obtain the Lax pair. We find a 3×3 matrix spectral problem for the Kersten–Krasil'shchik system.

II. SINGULARITY ANALYSIS

Let us study the integrability of (1) following the Weiss–Kruskal algorithm of singularity analysis.[4,5] The algorithm is well known and widely used, therefore we omit unessential computational details.

First, we find that a hypersurface $\phi(x,t)=0$ is noncharacteristic for the system (1) if $\phi_x \neq 0$ and set $\phi_x = 1$ without loss of generality. Then we substitute the expansions

$$\begin{aligned} u &= u_0(t)\phi^{\alpha} + \cdots + u_r(t)\phi^{r+\alpha} + \cdots, \\ w &= w_0(t)\phi^{\beta} + \cdots + w_r(t)\phi^{r+\beta} + \cdots, \end{aligned} \tag{2}$$

into (1), and find the following branches (i.e., admissible choices of α, β, u_0, and w_0), together with the positions r of resonances (where arbitrary functions can enter the expansions):

$$\begin{aligned} &\alpha=-2, \quad \beta=-1, \quad u_0=1, \quad w_0=\pm i, \\ &r=-1,1,2,3,4,6, \end{aligned} \tag{3}$$

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$$\alpha=-2,\quad \beta=-1,\quad u_0=2,\quad w_0=\pm 2i,$$
$$r=-2,-1,3,3,4,8, \tag{4}$$

$$\alpha=-2,\quad \beta=2,\quad u_0=2,\quad \forall w_0(t),$$
$$r=-4,-1,0,1,4,6, \tag{5}$$

$$\alpha=-2,\quad \beta=3,\quad u_0=2,\quad \forall w_0(t),$$
$$r=-5,-1,-1,0,4,6, \tag{6}$$

besides those which correspond to the Taylor expansions governed by the Cauchy–Kovalevskaya theorem.

The branch (3) is generic: the expansions (2) with (3) describe the behavior of a generic solution near its singularity. The nongeneric branches (4), (5), and (6) correspond to singularities of special solutions. The branches (4) and (5) admit the following interpretation, in the spirit of Ref. 6: (4) describes the collision of two generic poles (3) with same sign of w_0, whereas (5) describes the collision of two generic poles (3) with opposite signs of w_0. The branch (6) corresponds to (5) with $w_0\to 0$.

Next, we find from (1) the recursion relations for the coefficients $u_n(t)$ and $w_n(t)$ $(n=0,1,2,\ldots)$ of the expansions (2), separately for each of the branches, and check the consistency of those recursion relations at the resonances. The recursion relations turn out to be consistent, therefore the expansions (2) of solutions of (1) are free from logarithmic terms. We conclude that the system (1) passes the Painlevé test for integrability successfully and must be expected to possess a Lax pair.

III. PROLONGATION STRUCTURE

By introducing the variables $p=u_x$, $q=w_x$, $r=p_x$, $s=q_x$, we assume that there exist $N\times N$ matrix functions F and G, depending upon u,w,p,q,r,s, such that

$$y_x=-yF,$$
$$y_t=-yG, \tag{7}$$

where y is a row matrix with elements y^A, $A=1,\ldots,N$. The system of equations in (1) can be represented as the compatibility conditions of (7) if

$$F_t-G_x+[F,G]=0, \tag{8}$$

where $[F,G]$ is the matrix commutator. This requirement gives the set of partial differential equations for F and G:

$$F_p=F_q=F_r=F_s=0,\quad F_u=-G_r,\quad 3wF_u+F_w=-G_s,$$
$$pG_u+qG_w+rG_p+sG_q-3(2up-qs+pw^2+2uwq)F_u-3(w^2q+uq+pw)F_w-[F,G]=0. \tag{9}$$

Next, we integrate equations (9) and find

$$F=\left(uw-\frac{w^3}{2}\right)X_1+\frac{w^2}{2}X_2+uX_3+wX_4+X_5, \tag{10}$$

where X_1,X_2,X_3,X_4,X_5 are constant matrices of integration. It is immediately seen that X_1 is in the center of prolongation algebra.[3] Hence, we can take it to be zero and find G as

$$G=(-r-ws-q^2+2u^2-w^4-w^2u)X_3-(s-w^3-3uw)X_4-(p+wq)X_6-uwX_7-\left(\frac{w^2}{2}+u\right)X_8$$
$$-qX_9-\frac{w^2}{2}X_{10}-wX_{11}+X_0, \tag{11}$$

where X_0 is a constant matrix of integration. The remaining elements are

$$X_6=[X_5,X_3],\quad X_7=[X_4,X_6],\quad X_8=[X_5,X_6],$$
$$X_9=[X_5,X_4],\quad X_{10}=[X_4,X_9],\quad X_{11}=[X_5,X_9]. \tag{12}$$

The integrability conditions impose the following restrictions on X_i $(i=0,\ldots,11)$:

$$[X_2,X_3]=0,\quad [X_5,X_0]=0,\quad [X_3,[X_3,X_6]]=0,\quad [X_2,[X_4,X_3]]=0,$$
$$[X_3,[X_4,X_3]]=0,\quad [X_3,[X_4,[X_4,X_3]]]=0,\quad [[X_4,[X_4,X_3]],[X_3,X_6]]=0,$$
$$2X_6+[X_5,X_2]=0,\quad [X_3,X_0]-[X_5,X_8]=0,\quad [X_4,X_2]+4[X_4,X_3]=0,$$
$$[X_4,X_0]-[X_5,X_{11}]=0,\quad 3X_6-\tfrac{1}{2}[X_5,[X_3,X_6]]-[X_3,X_8]=0,$$
$$3X_2-3[X_4,[X_4,X_3]]-[X_2,X_6]+[X_3,X_6]=0,$$
$$X_7+2[X_5,[X_4,X_3]]-[X_3,X_9]=0,$$
$$[X_2,X_0]-2[X_4,X_{11}]-[X_5,X_8]-[X_5,X_{10}]=0,$$
$$[X_2,[X_5,[X_4,X_3]]]+[X_2,X_7]+\tfrac{1}{2}[X_2,[X_2,X_9]]=0, \tag{13}$$
$$3X_9-[X_3,X_{11}]-[X_4,X_8]-[X_5,X_7]-2[X_5,[X_5,[X_4,X_3]]]=0,$$
$$[X_3,X_7]+\tfrac{1}{2}[X_4,[X_3,X_6]]+[X_3,[X_5,[X_4,X_3]]]=0,$$
$$X_9-\tfrac{1}{2}([X_2,X_{11}]+[X_4,X_8]+[X_4,X_{10}])-\tfrac{1}{3}([X_5,[X_5,[X_4,X_3]]]+[X_5,X_7])-\tfrac{1}{6}[X_5,[X_2,X_9]]$$
$$=0,$$
$$\tfrac{1}{2}[X_2,X_5]+\tfrac{1}{4}([X_2,X_8]+[X_2,X_{10}])+\tfrac{1}{3}([X_4,X_7]+[X_4,[X_5,[X_4,X_3]]])+\tfrac{1}{6}[X_4,[X_2,X_9]]=0,$$
$$3X_6-\tfrac{1}{2}([X_2,X_8]+[X_3,X_8]+[X_3,X_{10}])-[X_4,X_7]-2[X_5,[X_3,X_6]]-[X_4,[X_5,[X_4,X_3]]]$$
$$-2[X_5,[X_4,[,X_4,X_3]]]=0,$$
$$8[X_4,X_3]+\tfrac{1}{4}[X_2,[X_2,X_9]]-2[X_4,[X_4,[X_4,X_3]]]-\tfrac{1}{6}([X_3,[X_2,X_9]]+11[X_4,[X_3,X_6]])=0..$$

Together with the Jacobi identities we obtain further relations

$$[X_2,X_6]+2[X_3,X_6]=0,\quad [X_4,X_{11}]-[X_5,X_{10}]=0,$$
$$[X_5,[X_3,X_6]]-[X_3,X_8]=0,\quad [X_2,X_8]-[X_5,[X_2,X_6]]=0,$$
$$[X_5,[X_4,X_3]]+[X_3,X_9]-X_7=0,$$

$$
\begin{aligned}
&-4[X_5,[X_4,X_3]]+[X_2,X_9]+2X_7=0,\\
&[X_2,[X_5,[X_4,X_3]]]+2[[X_4,X_3],X_6]=0,\\
&[X_3,[X_5,[X_4,X_3]]]-[[X_4,X_3],X_6]=0,\\
&[X_3,[X_2,X_9]]-[X_2,[X_3,X_9]]=0,\\
&[X_4,X_3]=0,\quad [X_2,X_7]=0,\quad [X_3,X_7]=0,\\
&[X_3,X_{10}]=0,\quad [X_4,X_7]=0,\quad [X_5,X_7]=X_9,\\
&[X_2,[X_2,X_9]]=0,\quad [X_4,[X_3,X_6]]=0,\\
&[X_5,X_8]+[X_5,X_{10}]=0.
\end{aligned}
\tag{14}
$$

In order to find the Lie algebra generated by F and matrix representations of the generators $\{X_i\}_0^{11}$, we follow the strategy of Dodd–Fordy.[3] First we reduce the number of elements. By using Eqs. (12)–(14), we get $X_2=-2X_3$. Next, we locate nilpotent and neutral elements. The Eqs. (12) and (13) together with $X_2=-2X_3$ give that $[X_5,X_3]=X_6$ and $[X_3,X_6]=2X_3$, hence X_3 is nilpotent and X_6 is the neutral element. Let us note that the system of equations in (1) has the following scale symmetry:

$$
x\to\lambda^{-1}x,\quad t\to\lambda^{-3}t,\quad u\to\lambda^2u,\quad w\to\lambda w,
\tag{15}
$$

which implies that the elements X_i must satisfy

$$
\begin{aligned}
&X_0\to\lambda^3X_0,\quad X_3\to\lambda^{-1}X_3,\quad X_4\to X_4,\quad X_5\to\lambda X_5,\\
&X_6\to X_6,\quad X_7\to X_7,\quad X_8\to\lambda X_8,\quad X_9\to\lambda X_9,\\
&X_{10}\to\lambda X_{10},\quad X_{11}\to\lambda^2X_{11},
\end{aligned}
\tag{16}
$$

where λ is a constant. By using the basis elements, we try to embed the prolongation algebra into $\mathrm{sl}(n+1,c)$. Starting from the case $n=1$, we found that $\mathrm{sl}(2,c)$ cannot be the whole algebra. The simplest nontrivial closure is in terms of $\mathrm{sl}(3,c)$. We take

$$
X_3=e_{-\alpha_1},\quad X_6=h_1,
\tag{17}
$$

where we use the standart Cartan–Weyl basis[7] of A_2. Together with the scale symmetries we find that

$$
\begin{aligned}
&X_0=-4c_2^2\lambda^4e_{-\alpha_1}-36c_1^3\lambda^3(h_1+2h_2)-4c_2\lambda^2e_{\alpha_1},\\
&X_4=d_1(h_1+2h_2)+d_2\lambda^{-1}e_{\alpha_2}+d_3\lambda^2e_{-\alpha_1-\alpha_2},\\
&X_5=e_{\alpha_1}+c_1\lambda(h_1+2h_2)+c_2\lambda^2e_{-\alpha_1},\\
&X_7=d_2\lambda^{-1}e_{\alpha_2}+d_3\lambda^2e_{-\alpha_1-\alpha_2},\\
&X_8=-2e_{\alpha_1}+2c_2\lambda^2e_{-\alpha_1},\\
&X_9=d_2\lambda^{-1}e_{\alpha_1+\alpha_2}-d_3\lambda^2e_{-\alpha_2}+3c_1d_2e_{\alpha_2}-3c_1d_3\lambda^3e_{-\alpha_1-\alpha_2},
\end{aligned}
\tag{18}
$$

$$X_{10}=-d_2d_3\lambda(h_1+2h_2)-6c_1d_2d_3\lambda^2e_{-\alpha_1},$$

$$X_{11}=(9c_1^2+c_2)d_2\lambda e_{\alpha_2}+6c_1d_3\lambda^3e_{-\alpha_2}+6c_1d_2e_{\alpha_1+\alpha_2}+(9c_1^2+c_2)d_3\lambda^4e_{-\alpha_1-\alpha_2},$$

where $\{c_i\}_1^2$ and $\{d_i\}_1^3$ are constants with conditions

$$d_1d_2=0,\quad d_1d_3=0,\quad d_2d_3=6c_1,\quad c_2=9c_1^2. \tag{19}$$

We choose $d_1=0$, $c_1=d_2=1$. So that, $X_7=X_4$ and $X_0=-36\lambda^2X_5$. Then, we obtain the matrix representations of the generators X_i as

$$\begin{gathered}
X_3=\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix},\quad X_4=\begin{pmatrix}0&0&0\\-\lambda^{-1}&0&0\\0&6\lambda^2&0\end{pmatrix},\\
X_5=\begin{pmatrix}-\lambda&0&1\\0&2\lambda&0\\9\lambda^2&0&-\lambda\end{pmatrix},\quad X_6=\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix},\\
X_8=\begin{pmatrix}0&0&-2\\0&0&0\\18\lambda^2&0&0\end{pmatrix},\quad X_9=\begin{pmatrix}0&6\lambda^2&0\\-3&0&\lambda^{-1}\\0&-18\lambda^3&0\end{pmatrix},\\
X_{10}=\begin{pmatrix}6\lambda&0&0\\0&-12\lambda&0\\-36\lambda^2&0&6\lambda\end{pmatrix},\quad X_{11}=\begin{pmatrix}0&-36\lambda^3&0\\-18\lambda&0&6\\0&108\lambda^4&0\end{pmatrix}.
\end{gathered}\tag{20}$$

By substituting the matrix representations of the generators into Eqs. (10) and (11) we can construct the Lax pair, $\Psi_x=X\Psi$, $\Psi_t=T\Psi$, for the system (1), with the following matrices X and T:

$$X=\begin{pmatrix}\lambda&w\lambda^{-1}&w^2-u-9\lambda^2\\0&-2\lambda&-6w\lambda^2\\-1&0&\lambda\end{pmatrix}, \tag{21}$$

$T=\{\{p+wq+3\lambda w^2-36\lambda^3,(w^3+2uw-s)\lambda^{-1}-3q-18\lambda w,r+ws+q^2-2u^2+w^4+w^2u-9\lambda^2 w^2+18\lambda^2u+324\lambda^4\}$, $\{6q\lambda^2-36\lambda^3w,-6\lambda w^2+72\lambda^3,\ 6(s-w^3-2uw)\lambda^2-18q\lambda^3+108\lambda^4w\}$, $\{-w^2-2u+36\lambda^2,q\lambda^{-1}+6w,-p-wq+3\lambda w^2-36\lambda^3\}\}$, where the matrix T is written by rows and $X=-F^\dagger$, $T=-G^\dagger$, $\Psi=y^\dagger$.

The forms of X and T are unusual in the sense of the dependence on λ. It is possible to obtain equivalent matrices by the gauge transformation,

$$X'=SXS^{-1},\quad T'=STS^{-1}, \tag{22}$$

where

$$S=\begin{pmatrix}1&0&0\\0&0&-1\\0&\lambda^{-1}&0\end{pmatrix}. \tag{23}$$

The result is

$$X' = \begin{pmatrix} \lambda & u - w^2 + 9\lambda^2 & w \\ 1 & \lambda & 0 \\ 0 & 6\lambda w & -2\lambda \end{pmatrix}, \tag{24}$$

$T' = \{\{p + wq + 3\lambda w^2 - 36\lambda^3, \quad -r - ws - q^2 + 2u^2 - w^4 - w^2 u + 9\lambda^2 w^2 - 18\lambda^2 u - 324\lambda^4, \quad w^3 + 2uw - s - 3q\lambda - 18\lambda^2 w\}, \quad \{w^2 + 2u - 36\lambda^2, \quad -p - wq + 3\lambda w^2 - 36\lambda^3, -q - 6w\lambda\}, \quad \{6q\lambda - 36\lambda^2 w, -6(s - w^3 - 2uw)\lambda + 18q\lambda^2 - 108\lambda^3 w, -6\lambda w^2 + 72\lambda^3\}\}.$

IV. CONCLUDING REMARKS

The matrix X' gives us exactly the spectral problem for the KdV equation when $w=0$. But X' does not reduce to the one for mKdV equation when $u=0$. This result should be expected because the Kersten–Krasil'shchik system, when $u=0$, gives not only mKdV equation, as stated in Ref. 1, but also an ordinary differential equation in w. Finally, we note that the Lax pair obtained from (7) with (24) is a true Lax pair since the parameter λ cannot be removed from X' by a gauge transformation, as can be proven by a gauge-invariant technique.[8]

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