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# Integrability of a Generalized Ito System: The Painlevé Test

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It is shown that a generalized Ito system of four coupled nonlinear evolution equations passes the Painlevé test for integrability in five distinct cases, two of which were introduced recently by Tam *et al.* [H. W. Tam, X. B. Hu, D. L. Wang: J. Phys. Soc. Jpn. **68** (1999) 369]. A conjecture is formulated on integrability of a vector generalization of the Ito system.

KEYWORDS: integrable systems, Painlevé analysis

Recently, Tam *et al.*[1)] introduced the following two systems of coupled nonlinear evolution equations:

$$\begin{aligned} u_t &= v_x, \\ v_t &= -2v_{xxx} - 6\left(uv\right)_x - 12ww_x + 6p_x, \\ w_t &= w_{xxx} + 3uw_x, \\ p_t &= p_{xxx} + 3up_x, \end{aligned} \tag{1}$$

and

$$\begin{aligned} u_t &= v_x, \\ v_t &= -2v_{xxx} - 6\left(uv\right)_x - 6\left(wp\right)_x, \\ w_t &= w_{xxx} + 3uw_x, \\ p_t &= p_{xxx} + 3up_x, \end{aligned} \tag{2}$$

which are generalizations of the well-known integrable Ito system[2)]

$$\begin{aligned} u_t &= v_x, \\ v_t &= -2v_{xxx} - 6\left(uv\right)_x. \end{aligned} \tag{3}$$

Hirota bilinear representations of systems (1) and (2), 4-soliton solutions of system (1) with $p = 0$, and 3-soliton solutions of system (2) were found in ref. 1. More recently, a new type of 3-soliton solutions with constant boundary conditions at infinity was found for system (1) with $p = 0$ in ref. 3. The question of the integrability of systems (1) and (2), in the sense of the existence of Lax pairs, infinitely many conservation laws, and N-soliton solutions, was posed in ref. 1.

In this Letter, we show that the Tam-Hu-Wang systems (1) and (2) must be integrable according to positive results of the Painlevé test. We apply the Painlevé test for integrability to the following generalized Ito system:

$$\begin{aligned} u_t &= v_x, \\ v_t &= -2v_{xxx} - 6\left(uv\right)_x + aww_x + bpw_x + cwp_x \\ &\quad + dpp_x + fw_x + gp_x, \\ w_t &= w_{xxx} + 3uw_x, \\ p_t &= p_{xxx} + 3up_x, \end{aligned} \tag{4}$$

where $a, b, c, d, f, g$ are arbitrary constants. System (4) possesses the Painlevé property under the only constraint imposed on its coefficients: $c = b$. Then, using the affine transformations of $w$ and $p$, we reduce system (4) with $c = b$ to five distinct cases,two of which are eqs. (1) and (2). Later, we conject on the integrability of a multicomponent generalization of the Tam-Hu-Wang systems (1) and (2).

Let us apply the Painlevé test for integrability to system (4), following the Weiss-Kruskal algorithm of singularity analysis.[4,5)] System (4) is a normal system of four partial differential equations of total order ten, and its general solution must contain ten arbitrary functions of one variable. A hypersurface $\phi\left(x, t\right) = 0$ is noncharacteristic of this system if $\phi_x \phi_t \neq 0$, and we set $\phi_x = 1$ without loss of generality. The substitution of the expansions

$$\begin{aligned} u &= u_0\left(t\right)\phi^{\alpha} + \cdots + u_n\left(t\right)\phi^{n+\alpha} + \cdots, \\ v &= v_0\left(t\right)\phi^{\beta} + \cdots + v_n\left(t\right)\phi^{n+\beta} + \cdots, \\ w &= w_0\left(t\right)\phi^{\gamma} + \cdots + w_n\left(t\right)\phi^{n+\gamma} + \cdots, \\ p &= p_0\left(t\right)\phi^{\delta} + \cdots + p_n\left(t\right)\phi^{n+\delta} + \cdots, \end{aligned} \tag{5}$$

into system (4) determines the branches, i.e. the admissible dominant behavior of solutions (values of $\alpha, \beta, \gamma, \delta, u_0, v_0, w_0, p_0$) and the corresponding positions $n$ of the resonances (where arbitrary functions can appear in the expansions (5)).

System (4) allows many branches, but the presence or absence of most of them depends on the values of the parameters $a, b, c, d, f, g$. For this reason, it is useful to start the analysis from the following singular branch:

$$\begin{aligned} &\alpha = \beta = -2, \quad \gamma = \delta = -1, \\ &u_0 = -2, \quad v_0 = -2\phi_t, \quad \forall\, w_0\left(t\right), \quad \forall\, p_0\left(t\right), \end{aligned}$$

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$$n = -1, 0, 0, 1, 1, 2, 4, 5, 5, 6, \tag{6}$$

which is allowed by system (4) at any of its parameters. According to the positions of resonances, branch (6) is generic, i.e., it represents the general solution of system (4). Now, constructing the recursion relations for the coefficients of the expansions (5), and checking the consistency of these recursion relations at the resonances of branch (6), we obtain the compatibility condition $(b-c)(w_{0,t}p_0 - w_0 p_{0,t}) = 0$ at $n = 5$. If $c \neq b$, some logarithmic terms must be introduced into the expansions (5). Therefore system (4) can possess the Painlevé property only if $c = b$. Setting $c = b$ hereafter, we find that the recursion relations are consistent at all resonances of the branch (6).

Before proceeding to other branches, let us note that, in the case of $c = b$, we can fix all the free parameters of system (4) by means of the affine transformation

$$\begin{aligned} w &\to \xi_1 w + \xi_2 p + \xi_3, \\ p &\to \xi_4 w + \xi_5 p + \xi_6, \end{aligned} \tag{7}$$

with appropriately chosen constants $\xi_1, \dots, \xi_6$, $\xi_1\xi_5 \neq \xi_2\xi_4$. Certainly, this transformation has no effect on the presence or absence of the Painlevé property. If $ad \neq b^2$, then, using transformation (7), we can make

$$b = -6, \quad a = d = f = g = 0. \tag{8}$$

If $ad = b^2, a \neq 0, ag \neq bf$, or if $a = b = 0, d \neq 0, f \neq 0$, we can make

$$a = -12, \quad g = 6, \quad b = d = f = 0. \tag{9}$$

If $ad = b^2, a \neq 0, ag = bf$, or if $a = b = 0, d \neq 0, f = 0$, we can make

$$a = -12, \quad b = d = f = g = 0. \tag{10}$$

If $a = b = d = 0, f \neq 0$, or if $a = b = d = f = 0, g \neq 0$, we can make

$$a = b = d = f = 0, \quad g = 6. \tag{11}$$

And the case of

$$a = b = d = f = g = 0, \tag{12}$$

needs no transformation. These five cases, (8)–(12), of system (4) are not related to each other by transformation (7).

Having reduced system (4) with $c = b$ to five distinct cases, (8)–(12), we find that the following singular nongeneric branches must be studied as well:

$$\begin{aligned} &\alpha = \beta = \gamma = \delta = -2, \\ &u_0 = -4, \quad v_0 = -4\phi_t, \quad w_0 p_0 = -8\phi_t, \\ &\forall\, w_0(t) \text{ or } \forall\, p_0(t), \\ &n = -2, -1, 0, 2, 2, 3, 4, 6, 7, 8, \end{aligned} \tag{13}$$

and

$$\begin{aligned} &\alpha = \beta = -2, \quad \gamma = 0, \quad \delta = -4, \\ &u_0 = -10, \quad v_0 = -10\phi_t, \quad w_0 p_0 = -80\phi_t, \\ &\forall\, w_0(t) \text{ or } \forall\, p_0(t), \\ &n = -5, -4, -1, 0, 2, 4, 6, 7, 8, 12, \end{aligned} \tag{14}$$

in case (8);

$$\begin{aligned} &\alpha = \beta = -2, \quad \gamma = 0, \quad \delta = -4, \\ &u_0 = -10, \quad v_0 = -10\phi_t, \quad p_0 = 80\phi_t, \quad \forall\, w_0(t), \\ &n = -5, -4, -1, 0, 2, 4, 6, 7, 8, 12, \end{aligned} \tag{15}$$

in case (9);

$$\begin{aligned} &\alpha = \beta = \gamma = \delta = -2, \\ &u_0 = -4, \quad v_0 = -4\phi_t, \quad w_0^2 = -8\phi_t, \quad \forall\, p_0(t), \\ &n = -2, -1, 0, 2, 2, 3, 4, 6, 7, 8, \end{aligned} \tag{16}$$

in cases (9) and (10); and

$$\begin{aligned} &\alpha = \beta = -2, \quad \gamma = \delta = -4, \\ &u_0 = -10, \quad v_0 = -10\phi_t, \quad p_0 = 80\phi_t, \quad \forall\, w_0(t), \\ &n = -5, -1, 0, 2, 4, 4, 6, 8, 11, 12, \end{aligned} \tag{17}$$

in case (11). Then, using the *Mathematica* system[6] we prove that the recursion relations are consistent at the resonances of branches (13)–(17), and therefore, no logarithmic terms should be introduced into the expansions (5).

Now, we can conclude that the generalized Ito system (4) passes the Painlevé test for integrability if, and only if, $c = b$, or, up to the equivalence (7), in the five distinct cases of (8)–(12). Cases (8) and (9) correspond to the Tam-Hu-Wang systems (2) and (1), respectively.

The obtained results of the singularity analysis strongly suggest that system (4) with $c = b$ must be integrable in the Lax sense. Moreover, we *conjecture* that, for any constant $k_{ij}$ and $l_i$, $i, j = 1, \dots, m$, and any integer $m$, the following system of $m+2$ coupled nonlinear evolution equations for $u, v, q_1, \dots, q_m$, a vector generalization of the Ito system (3),

$$\begin{aligned} &u_t = v_x, \\ &v_t = -2v_{xxx} - 6(uv)_x + \left( \sum_{i,j} k_{ij} q_i q_j + \sum_i l_i q_i \right)_x, \\ &q_{i,t} = q_{i,xxx} + 3u q_{i,x}, \quad i = 1, \dots, m, \end{aligned} \tag{18}$$

passes the Painlevé test for integrability and possesses a parametric zero-curvature representation.

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