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ON SOLITON SOLUTIONS OF NONLINEAR SIGMA MODELS OF SYMMETRIC SPACES

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The inverse scattering transform technique of Belinskii–Zakharov for the integration of nonlinear *sigma* model equations is reviewed. N-soliton solutions of the principal chiral field equations are given. The explicit two-complex pole soliton solutions of *vacuum* and *electro-vacuum* Ernst equations are constructed.

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1. Introduction

Among the techniques developed for the generation of new solutions of Einstein equations from known ones, the inverse scattering method of Belinskii and Zakharov[1,2] has turned out to be one of the most fruitful.[3–7] The Belinskii–Zakharov integration technique was first applied by the authors, to the Einstein vacuum field equations where the metric tensor depends only on two of the space–time coordinates. They obtained all multi-soliton solutions of Einstein equations for stationary axially symmetric vacuum and colliding plane gravitational wave space–times. Starting from some "seed" solution, the technique was based on the construction of a scattering matrix as a function of a complex spectral parameter. The method was later extended by Alekseev[8] to construct soliton solutions of the stationary axially symmetric Einstein–Maxwell equations. For complex poles, the vacuum N-pole solitons in Alekseev formalism were equivalent to $2N$-pole solitons in the Belinskii–Zakharov formalism. In both of the formulations the parametrization of the problem was such that the associated linear eigenvalue equation contained the space–time metric functions directly. It was demonstrated in Refs. 9–13 that the method was also applicable to the field equations of nonlinear *sigma* models of *symmetric* spaces provided that one can solve the *reduction* problem; thus untangling the constraint which makes the sigma model nonlinear. In this case the poles of scattering matrix must be complex and even in number. In this work we review the integration technique of Belinskii–Zakharov in a coordinate free way and discuss the two reduction problems. In Sec. 2, vacuum[14] and electro-vacuum[15] Ernst equations as nonlinear sigma models on symmetric spaces are briefly reviewed. In Sec. 3, N-soliton solutions of principal chiral field equations are given and two different reduction problems are discussed. In Sec. 4, the complex two-pole soliton solutions of vacuum and electro-vacuum Ernst equations on arbitrary background are constructed and explicit solutions with flat backgrounds are given.

2. Stationary Gravitational Fields with Axial Symmetry

Stationary axially symmetric Einstein fields are invariantly described by space–time geometries admitting two commuting Killing vectors, one of which is timelike while the other one is spacelike. Physically, provided that they are asymptotically flat, these fields may describe the exterior gravitational fields of uniformly rotating isolated mass distributions. The existence of the two commuting Killing vectors imply that one can find a particular coordinate system in which the metric functions are independent of two of the coordinates. As first shown by Weyl[16] the line element describing such a geometry can be written in the form

$$dS^2 = F(d\rho^2 + dz^2) + g_{ab}dx^a dx^b \,, \tag{2.1}$$

where the metric coefficient F and g_{ab} $(a, b = 0, 1)$ are functions of ρ, z only and x^0, x^1 correspond to the two Killing directions t, ϕ respectively. It is well known that the vacuum Einstein equations for the metric (2.1) separate into two groups. The first group is of the form

$$\boldsymbol{\nabla} \cdot [\rho(\boldsymbol{\nabla} G)G^{-1}] = 0 \,; \tag{2.2}$$

determining the 2×2 symmetric matrix G with components (g_{ab}) satisfying the condition

$$\det G = -\rho^2 \,, \tag{2.3}$$

where $\boldsymbol{\nabla}$ stands for two-dimensional flat space gradient operator in Cartesian coordinates (ρ, z). The second group of equations can be written in the form

$$\begin{aligned} (\ln F)_{,\rho} &= -\frac{1}{\rho} + \frac{\rho}{4}\,\mathrm{Tr}[(G_{,\rho}\, G^{-1})^2 - (G_{,z}\, G^{-1})^2] \,, \\ (\ln F)_{,z} &= \frac{\rho}{2}\,\mathrm{Tr}[(G_{,\rho}\, G^{-1})(G_{,z}\, G^{-1})] \,. \end{aligned} \tag{2.4}$$

Since the integrability of these equations for the metric coefficient F is guaranteed if G is a solution of Eq. (2.2), these equations determine F for a given G. Choosing a particular parametrization, the matrix G can be written as

$$G = \begin{pmatrix} -f & fw \\ fw & \rho^2 f^{-1} - fw^2 \end{pmatrix} , \tag{2.5}$$

which puts the line element in (2.1) into Weyl–Papapetrou canonical form

$$dS^2 = -f(dt - w\, d\phi)^2 + f^{-1}[e^{2\gamma}(dz^2 + d\rho^2) + \rho^2\, d\phi^2] \,. \tag{2.6}$$

With this parametrization, Eq. (2.2) turns into two equations for the metric coefficients f and w. These are the fundamental equations for the Ernst formulation of the stationary axially symmetric vacuum field equations. They can be combined into a single complex equation by defining a complex function

$$\varepsilon = f + i\psi \,, \tag{2.7}$$

satisfying the so-called Ernst[14] equation

$$\mathrm{Re}\,\varepsilon \boldsymbol{\nabla} \cdot (\rho \boldsymbol{\nabla} \varepsilon) - \rho (\boldsymbol{\nabla} \varepsilon)^2 = 0 \,, \tag{2.8}$$

where ψ is a function of ρ, z satisfying

$$\hat{n}_\phi \times \boldsymbol{\nabla} \psi = \rho^{-1} f^2 \boldsymbol{\nabla} w \,. \tag{2.9}$$

It has been pointed out[12,13] that Eq. (2.8) also appears as the equation of motion of SU$(1,1)$/U(1)-sigma model. That is, if one considers a symmetric space scalar field theory with values in the space SU$(1,1)$/U(1), the dynamics of this theory are governed by the Ernst equation. For this reason it is possible to construct the 2×2 matrix[13]

$$P = \frac{2}{\varepsilon + \bar{\varepsilon}} \begin{pmatrix} 1 & \frac{i}{2}(\bar{\varepsilon} - \varepsilon) \\ \frac{i}{2}(\bar{\varepsilon} - \varepsilon) & \bar{\varepsilon}\varepsilon \end{pmatrix} , \tag{2.10}$$

satisfying

$$P = P^\dagger \,, \qquad (\Gamma P)^2 = I \,, \tag{2.11}$$

where

$$\Gamma = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} , \qquad (\Gamma)^2 = I \,, \tag{2.12}$$

in terms of which the Ernst equation can be written as

$$\boldsymbol{\nabla} \cdot [\rho (\boldsymbol{\nabla} P) P^{-1}] = 0 \,. \tag{2.13}$$

Once the solution for P is found, one can determine the metric coefficient $e^{2\gamma}$ by integrating the equations

$$\begin{aligned} (\gamma)_{,\rho} &= \frac{\rho}{8} \mathrm{Tr}[(P_{,\rho} P^{-1})^2 - (P_{,z} P^{-1})^2] \,, \\ (\gamma)_{,z} &= \frac{\rho}{4} \mathrm{Tr}[(P_{,\rho} P^{-1})(P_{,z} P^{-1})] \,. \end{aligned} \tag{2.14}$$

Thus, we have two alternate formulations of stationary axially symmetric vacuum Einstein field equations. One can either start from the particular parametrization (2.5) of a real symmetric matrix G satisfying $\det G = -\rho^2$ and solve Eq. (2.2) or one can start from parametrization (2.10) of 2×2 Hermitian matrix P satisfying $(\Gamma P)^2 = I$ and solve Eq. (2.13) as the first step in the construction of stationary axially symmetric vacuum Einstein fields. In both formulations the form of the master equation is the same, the only difference being in the parametrization of the matrices. These equations are also valid for stationary axially symmetric space–times with an electromagnetic field as a source for the geometry. For these space–times the line element can be written in Weyl–Papapetrou form (2.6), with the source field depending on ρ, z. By a choice of gauge the only nonvanishing components of electromagnetic potential are A_ϕ and A_t. It is known for a long time, that the field equations for this metric written in terms of Ernst potentials,[15] enables one to formulate the stationary axially symmetric Einstein–Maxwell equations as the equation of motion of the $\mathrm{SU}(2,1)/\mathrm{SU}(2)\times\mathrm{U}(1)$-nonlinear sigma model on symmetric space.[13] So we can write the Einstein–Maxwell field equations in terms of 3×3 Hermitian matrix P satisfying Eqs. (2.13) and (2.14) where

$$P=\frac{2}{\varepsilon+\bar{\varepsilon}+2\Phi\bar{\Phi}}\begin{pmatrix} 1 & \sqrt{2}\Phi & \frac{i}{2}(\bar{\varepsilon}-\varepsilon+\Phi\bar{\Phi}) \\ \sqrt{2}\bar{\Phi} & -\frac{1}{2}(\bar{\varepsilon}+\varepsilon-2\Phi\bar{\Phi}) & -i\sqrt{2}\bar{\Phi}\varepsilon \\ \frac{i}{2}(\bar{\varepsilon}-\varepsilon-2\Phi\bar{\Phi}) & i\sqrt{2}\Phi\bar{\varepsilon} & \bar{\varepsilon}\varepsilon \end{pmatrix}. \tag{2.15}$$

The symmetric space condition (2.11) of the theory is satisfied by P with

$$\Gamma=\begin{pmatrix} 0 & 0 & i \\ 0 & -1 & 0 \\ -i & 0 & 0 \end{pmatrix}. \tag{2.16}$$

Here $\varepsilon=f-\Phi\bar{\Phi}+i\psi$ and $\Phi=A_t+iA'_\phi$ are complex Ernst potentials. The original metric function w and the potential A_ϕ are related to the Ernst potentials by the equations

$$\begin{aligned} \hat{n}_\phi\times\boldsymbol{\nabla}\psi &= \rho^{-1}f^2\boldsymbol{\nabla}w-2\hat{n}_\phi\times\mathrm{Im}(\bar{\Phi}\boldsymbol{\nabla}\Phi)\,, \\ \hat{n}_\phi\times\boldsymbol{\nabla}A'_\phi &= \rho^{-1}f^2(\boldsymbol{\nabla}A_\phi-w\boldsymbol{\nabla}A_t)\,. \end{aligned} \tag{2.17}$$

The final form of the field equations in terms of P is suitable for application of the inverse scattering transform technique of Belinskii and Zakharov. Furthermore, it can be applied to any field theory which can be formulated as a nonlinear sigma model of a symmetric space. In the next section, we will review this method and the problems of reduction in a coordinate free language.

3. Formulation for Principal Chiral Fields

The equation satisfied by the matrix G or equivalently the matrix P, is known as the equation of motion for the principal chiral fields which are the elements of the matrix. It can be derived from the variational principle $\delta I=0$ where

$$I=\int\rho\,\mathrm{Tr}[(dP)P^{-1}\wedge({}^*dP)P^{-1}]\,. \tag{3.1}$$

Here $\wedge$ denotes the exterior (wedge) product, d is the exterior derivative, and the "Hodge" dual operator $(^*)$ is defined by $^*d\rho = dz$, $^*dz = -d\rho$. The general $n \times n$ matrix $P(\rho, z)$ satisfies the field equation

$$d[\rho(^*dP)P^{-1}] = 0\,. \tag{3.2}$$

We shall first consider this equation as it is without any additional condition like the matrix being symmetric or Eq. (2.11). Then, these conditions will be imposed respectively giving rise to the two reduction problems.

Using the general $n \times n$ matrix $P(\rho, z)$ one can define a connection one-form

$$W = (-dP)P^{-1}\,, \tag{3.3}$$

whose curvature vanishes identically, $dW + W \wedge W = 0$, where the field equation (3.2) can be written in terms of W as $d(\rho^*W) = 0$. In order to be able to cast the problem into a form to which inverse scattering techniques are applicable we need a connection one-form whose curvature vanishes modulo the field equation. To this end we define

$$\Omega(\lambda, \rho, z) = aW + b\rho^*W\,, \tag{3.4}$$

where λ is a complex spectral parameter and $a(\lambda, \rho, z)$ and $b(\lambda, \rho, z)$ are complex scalar functions satisfying

$$\lim_{\lambda\to 0} a(\lambda, \rho, z) = 1\,, \qquad \lim_{\lambda\to 0} b(\lambda, \rho, z) = 0\,. \tag{3.5}$$

These imply $\lim_{\lambda\to 0}\Omega = W$. Next we introduce a generalized exterior derivative operator D, which contains differentiations with respect to spectral parameter λ, as

$$D = d - (\partial\Theta/\partial\lambda)^{-1}d\Theta(\partial/\partial\lambda)\,, \tag{3.6}$$

with $D^2 = 0$ and $\Theta(\lambda, \rho, z)$ is any complex scalar function with the property $\lim_{\lambda\to 0} D = d$. Now the question is what are the conditions on functions a and b so that the curvature of Ω vanishes by virtue of the field equation, and hence

$$D\Omega + \Omega \wedge \Omega = 0\,. \tag{3.7}$$

Substituting (3.4) into (3.7) we obtain

$$a^2 + b^2\rho^2 - a = 0\,, \tag{3.8}$$

$$Da - \rho^*Db = 0\,. \tag{3.9}$$

Equation (3.9) is a set of two partial differential equations involving the functions a, b and Θ which can be integrated once a parametrization of the constraint in (3.8) is given. With these functions being determined, one can represent the field equation (3.2) as the integrability condition (3.7) of a linear matrix equation for an $n \times n$ matrix $\Psi(\lambda, \rho, z)$, written as

$$D\Psi = -\Omega\Psi\,. \tag{3.10}$$

One can easily check that the matrix P is nothing but the value of the matrix function Ψ in the limit $\lambda \to 0$. The procedure of integration of the field equation under consideration assumes the knowledge of at least one particular solution. For this purpose let us introduce a particular solution P_0 of (3.2). For a given set of functions $\{\Theta, a, b\}$ one can construct the connection one-form Ω_0 and by solving Eq. (3.10) one can obtain the corresponding function Ψ_0. A transformation of the form

$$\Psi = \chi(\lambda, \rho, z)\Psi_0 \tag{3.11}$$

defines a new matrix $\Psi(\lambda, \rho, z)$ leading to a new solution $P(\rho, z)$. Substituting (3.11) into (3.10) we obtain the equation for the $n \times n$ matrix χ

$$D\chi = \chi\Omega_0 - \Omega\chi\,. \tag{3.12}$$

Thus the problem now consists of solving this system and determining the matrix χ.

In the general case the determination of χ reduces to solving the matrix Riemann problem of analytic function theory. Solutions which are determined by the analyticity properties of the matrix χ in the complex λ-plane are represented by the sum of a soliton part and a nonsoliton part. In this work we consider only the pure solitonic solutions. The existence of soliton-type solutions is due to the presence of pole singularities of the matrix χ in the complex λ-plane. Let us consider the general case in which the matrix χ has N such poles which we assume to be simple. It follows that $\chi(\lambda, \rho, z)$ has the form

$$\chi(\lambda) = I + \sum_{k=1}^{N} \frac{R_k}{\lambda - \mu_k}\,, \tag{3.13}$$

while inverse matrix $\chi^{-1}(\lambda, \rho, z)$ can be written as

$$\chi^{-1}(\lambda) = I + \sum_{k=1}^{N} \frac{S_k}{\lambda - \nu_k}\,, \tag{3.14}$$

where R_k, S_k, μ_k and ν_k are functions of ρ, z. We assume that the poles μ_k and ν_k are in general complex and there is no relation, for the moment, between them. The form of matrix $\chi(\lambda)$ is in agreement with $\lim_{\lambda\to\infty} \chi(\lambda, \rho, z) = I$ meaning that χ is an analytic function of λ at $\lambda \to \infty$.

We can now determine the matrices $R_k(\rho, z)$ and $S_k(\rho, z)$ explicitly. Since these matrices are independent of the complex parameter λ it is enough to consider Eq. (3.12) at the poles of χ. Using the form (3.13) in (3.12) we have that the right-hand side of this equation has only first order poles at $\lambda = \mu_k$ whereas the left-hand side may have second order poles. The requirement that the coefficients of the power $(\lambda - \mu_k)^{-2}$ on the left-hand side yields the following equation for the pole trajectories $\mu_k(\rho, z)$:

$$D(\lambda - \mu_k) = 0\,. \tag{3.15}$$

The solution of this equation corresponds to the roots of the algebraic equation

$$\Theta(\lambda, \rho, z)|_{\lambda=\mu_k} = -\omega_k\,, \tag{3.16}$$

where ω_k are some complex constants. On the other hand one can write the equation for χ^{-1} as

$$D\chi^{-1} = \chi^{-1}\Omega - \Omega_0\chi^{-1}. \tag{3.17}$$

Using the explicit form of χ^{-1} we see that in order to have first order poles only, we have the following equation for the pole trajectories $\nu_k(\rho, z)$:

$$D(\lambda - \nu_k) = 0\,, \tag{3.18}$$

where ν_k are the roots of the algebraic equation

$$\Theta(\lambda, \rho, z)|_{\lambda=\nu_k} = -\omega'_k\,. \tag{3.19}$$

Now, Eq. (3.12) can be written as

$$\Omega = (\chi\Omega_0 - D\chi)\chi^{-1}\,. \tag{3.20}$$

Left-hand side of this equation has no poles at $\lambda = \mu_k$, hence the residues of right-hand side should be zero at $\lambda = \mu_k$. This requirement leads to the following equation for the matrix $R_k(\rho, z)$:

$$\chi(DR_k - R_k\Omega_0)|_{\lambda=\mu_k} = 0\,. \tag{3.21}$$

By the same argument and considering (3.17) at the poles $\lambda = \nu_k$ we can write the following equation for the matrix $S_k(\rho, z)$:

$$\chi(DS_k + \Omega_0 S_k)|_{\lambda=\nu_k} = 0\,. \tag{3.22}$$

In order to guarantee that the expression in (3.14) is the inverse of (3.13) we must have $\chi\chi^{-1} = I$, which when written out in detail reads

$$\sum_{i=1}^{N}\frac{R_i}{\lambda - \mu_i} + \sum_{j=1}^{N}\frac{S_j}{\lambda - \nu_j} + \sum_{i=1}^{N}\sum_{j=1}^{N}\frac{R_i S_j}{(\lambda - \mu_i)(\lambda - \nu_j)} = 0\,. \tag{3.23}$$

The residues of this equation at $\lambda = \mu_i$ give

$$R_i + \sum_{j=1}^{N}\frac{R_i S_j}{\mu_i - \nu_j} = 0 \tag{3.24}$$

while the residues at $\lambda = \nu_j$ give

$$S_j + \sum_{i=1}^{N}\frac{R_i S_j}{\nu_j - \mu_i} = 0\,. \tag{3.25}$$

Equations (3.24) and (3.25) imply that R_k and S_k are both degenerate $n\times n$ matrices satisfying $R_k\chi^{-1}(\mu_k) = 0$, $\chi(\nu_k)S_k = 0$, and hence can be written in the form

$$R_k = \sum_{\alpha=1}^{n-1} m_k^\alpha n_k^{\alpha\dagger}\,, \qquad S_k = \sum_{\alpha=1}^{n-1} p_k^\alpha q_k^{\alpha\dagger}\,, \tag{3.26}$$

where m_k^α, n_k^α, p_k^α and q_k^α are n-component column vectors satisfying the relations

$$n_k^{\alpha\dagger}\chi^{-1}(\mu_k) = 0\,, \qquad \chi(\nu_k)p_k^\alpha = 0 \tag{3.27}$$

for each index α. Now, using (3.24) for the matrix R_k, Eq. (3.21) states that for a particular solution one can choose

$$Dn_k^{\alpha\dagger} - n_k^{\alpha\dagger}\Omega_0 = 0\,. \tag{3.28}$$

The solution for this equation can easily be expressed in terms of the given particular solution of (3.10)

$$D\Psi_0^{-1} - \Psi_0^{-1}\Omega_0 = 0 \tag{3.29}$$

hence we can choose

$$n_k^\alpha = \Psi_0^{-1\dagger}(\mu_k) n_{0k}^\alpha\,, \tag{3.30}$$

where n_{0k}^α are arbitrary complex constant vectors. By a similar argument we conclude that p_k^α satisfies the same differential equation as the particular solution Ψ_0, so

$$p_k^\alpha = \Psi_0(\nu_k) p_{0k}^\alpha\,, \tag{3.31}$$

where p_{0k}^α are complex constant vectors. There remains the task of determining the vectors m_k^α and q_k^α and thus the matrices R_k and S_k. This can be done by means of Eqs. (3.24) and (3.25). Substituting the expressions (3.26) into (3.24) and (3.25) and assuming that m_i^α and q_j^β are linearly independent for each α and β, we have

$$n_i^{\alpha\dagger} + \sum_{j=1}^{N}\sum_{\beta=1}^{n-1} M_{ij}^{\alpha\beta} q_j^{\beta\dagger} = 0\,, \qquad p_j^\beta - \sum_{i=1}^{N}\sum_{\alpha=1}^{n-1} m_i^\alpha M_{ij}^{\alpha\beta} = 0\,, \tag{3.32}$$

where

$$M_{ij}^{\alpha\beta} = \frac{n_i^{\alpha\dagger} p_j^\beta}{\mu_i - \nu_j}\,. \tag{3.33}$$

One can solve these linear algebraic equations for m_i^α and $q_j^{\beta\dagger}$ thus completing the determination of the matrices R_k and S_k. We then have

$$P(\rho, z) = \left[I - \sum_{k=1}^{N} \frac{R_k}{\mu_k} \right] P_0(\rho, z)\,, \tag{3.34}$$

as the new N-soliton solution of Eq. (3.2) generated from the known solution $P_0(\rho, z)$. As it stands the new solution contains $2Nn$ arbitrary parameters ω, ω', n_{0k}^α and p_{0k}^α.

In most cases the system of Eqs. (3.7) is too general to be useful in application to physics. Therefore the question of the possibility of imposing additional conditions on the connection one-form Ω arises. Such conditions are called *reductions*. The reductions reduce the number of equations contained in the system (3.7) and impose certain restrictions on the function $\chi(\lambda, \rho, z)$. Since the matrix G in (2.5) satisfies the chiral field equation (3.2), we can apply the inverse scattering technique and find a solution. But in order for this matrix G to represent the stationary axially symmetric vacuum Einstein fields, it has to be symmetric and furthermore has to satisfy Eq. (2.3).

(i) *The reduction problem for the matrix P to be symmetric*

By using the definitions of one-forms, W and $\Omega(\lambda)$, it is easy to obtain the following equation for $\tilde{\Omega}(\lambda)$:

$$\tilde{\Omega}(\lambda) = P^{-1}\Omega(\lambda)P \tag{3.35}$$

provided that $\tilde{P} = P$ where ˜ denotes the transposition. In order to find the restriction on $\chi(\lambda)$ let us use Eq. (3.17). Taking the transpose of both sides of (3.17), replacing λ by τ and multiplying both sides by P from the left and P_0^{-1} from the right of the resulting equation, we obtain

$$P[D\tilde{\chi}^{-1}(\tau)]P_0^{-1} = \Omega(\tau)P\tilde{\chi}^{-1}(\tau)P_0^{-1} - P\tilde{\chi}^{-1}(\tau)P_0^{-1}\Omega_0(\tau)\,, \tag{3.36}$$

where $\tau : \lambda \to \tau(\lambda, \rho, z)$ is a fractional linear transformation on the complex λ-plane with $\tau^2 = 1$. Under this transformation the connection one-form $\Omega(\lambda)$ transforms as

$$\Omega(\tau, \rho, z) = W - \Omega(\lambda, \rho, z)\,, \tag{3.37}$$

provided that

$$a(\tau, \rho, z) = 1 - a(\lambda, \rho, z)\,, \qquad b(\tau, \rho, z) = -b(\lambda, \rho, z)\,. \tag{3.38}$$

These put a further restriction on the function Θ as $\Theta(\lambda, \rho, z) = \Theta(\tau, \rho, z)$. Then Eq. (3.36) becomes

$$D[P\tilde{\chi}^{-1}(\tau)P_0^{-1}] = [P\tilde{\chi}^{-1}(\tau)P_0^{-1}]\Omega_0(\lambda) - \Omega(\lambda)[P\tilde{\chi}^{-1}(\tau)P_0^{-1}]\,. \tag{3.39}$$

This implies that $P\tilde{\chi}^{-1}(\tau)P_0^{-1}$ satisfies the same differential equation as $\chi(\lambda)$, so one can write

$$P\tilde{\chi}^{-1}(\tau)P_0^{-1} = \chi(\lambda)J\,, \tag{3.40}$$

where J is some matrix with $J\Omega_0 = \Omega_0 J$ and $DJ = 0$ meaning that $J = J(\Theta)$. In the limits $\lambda \to \infty$ ($\tau \to 0$), and $\lambda \to 0$ ($\tau \to \infty$) the conditions $\chi(\infty) = I$ and $\chi^{-1}(\infty) = I$ must be satisfied respectively. So it is convenient to choose the matrix J to be the identity matrix. Then the form

$$\chi(\lambda) = P\tilde{\chi}^{-1}(\tau)P_0^{-1} \tag{3.41}$$

guarantees the symmetry property of the matrix P. Substituting (3.13) and (3.14) into (3.40) and considering this equation at the poles $\lambda = \mu_k$, one can see that the residues of right-hand side are nonzero, so the residues of left-hand side are also nonzero if $\nu_k = \tau(\mu_k)$. Thus, if the matrix χ has N poles at $\lambda = \mu_k$, then χ^{-1} has also N poles at the points $\nu_k = -\frac{\rho^2}{\mu_k}$. Rewriting Eq. (3.40) in the form

$$P = \chi(\tau)P_0\tilde{\chi}(\lambda)\,, \tag{3.42}$$

and evaluating at the poles $\lambda = \nu_k$, we obtain the following algebraic system of equations for the matrices:

$$R_k P_0 \left[I + \sum_{l=1}^{N} \frac{\tilde{R}_l}{\nu_k - \mu_l} \right] = 0\,. \tag{3.43}$$

For the *reality* of P we have the requirements

$$\overline{\chi(\bar{\lambda})} = \chi(\lambda)\,, \quad \overline{\Psi(\bar{\lambda})} = \Psi(\lambda)\,. \tag{3.44}$$

If all functions $\mu_k(\rho, z)$ are real the matrix P is automatically real provided that we choose all the arbitrary constants appearing in the solution to be real. If the functions $\mu_k(\rho, z)$ are complex the condition (3.44) requires that all the complex poles appear as conjugate pairs.

In order to construct the N soliton solutions of stationary axially symmetric vacuum Einstein equations, one can follow the procedure given above for 2×2 real symmetric matrices. For this purpose a convenient choice for the set of functions $\{\Theta, a, b\}$ is the one given by Belinskii and Zakharov,

$$\begin{aligned} a(\lambda, \rho, z) &= \frac{\rho^2}{\lambda^2 + \rho^2}\,, \qquad & b(\lambda, \rho, z) &= -\frac{\lambda}{\lambda^2 + \rho^2}\,, \\ \Theta(\lambda, \rho, z) &= \frac{\rho^2}{2\lambda} - \frac{\lambda}{2} - z\,, \qquad & \tau &= -\frac{\rho^2}{\lambda}\,. \end{aligned} \tag{3.45}$$

The numerical functions $\mu_k(\rho, z)$ are determined from (3.16) as

$$\mu_k(\rho, z) = \omega_k - z \pm [(\omega_k - z)^2 + \rho^2]^{\frac{1}{2}}\,, \tag{3.46}$$

which satisfies the pair of differential equations in (3.15). It was emphasized by Belinskii and Zakharov that the number of solitons N must be even since an odd number would violate the physical signature of the metric. Thus the simplest case would be a two-soliton solution. Starting with a flat "seed" metric G_0 and F_0 they obtained the Kerr-NUT metric as two-soliton solution.

(ii) *The reduction problem for the symmetric spaces*

We have seen that for any symmetric space scalar field theory, the solution P of Eq. (3.2) must satisfy the conditions given in (2.11) for some constant matrix Γ with property $\Gamma^2 = I$. Hence we need additional restrictions on the form of the transformation matrix χ. In order to find these restrictions let us consider Eq. (3.3). Multiplying both sides by P and taking Hermitian conjugate, we have $WP - PW^\dagger = 0$, which implies that

$$\Omega(\lambda)P - P\Omega^\dagger(\bar{\lambda}) = 0\,, \tag{3.47}$$

provided that $a(\lambda)$ and $b(\lambda)$ are real on the real λ axis. On the other hand, conditions (2.11) imply that $P^{-1} = \Gamma P \Gamma$ or $\Gamma W + W^\dagger \Gamma = 0$. In terms of $\Omega(\lambda)$ we have

$$\Gamma\Omega(\lambda) + \Omega^\dagger(\bar{\lambda})\Gamma = 0\,. \tag{3.48}$$

By using this and condition (2.11), Eq. (3.47) can be written as

$$P\Gamma\Omega(\lambda)P\Gamma = -\Omega(\lambda)\,. \tag{3.49}$$

Multiplying both sides of (3.10) by ΓP^{-1} and using Eq. (3.37) we have

$$D[\Gamma P^{-1}\Psi(\lambda)] = \Gamma P^{-1}\Omega(\tau)\Psi(\lambda)\,. \tag{3.50}$$

Interchanging λ and τ in this equation and using (3.49) we obtain

$$D[\Gamma P^{-1}\Psi(\tau)] = -\Omega(\lambda)\Gamma P^{-1}\Psi(\tau)\,. \tag{3.51}$$

Hence, we see that the matrix $\Gamma P^{-1}\Psi(\tau)$ satisfies the same differential equation as $\Psi(\lambda)$. Thus we may write

$$\Gamma P^{-1}\Psi(\tau) = \Psi(\lambda)J\,, \tag{3.52}$$

where J is a matrix with the properties $DJ = 0$ and $\det J \neq 0$. Considering this equation in the limits $\lambda \to \infty$ ($\tau \to 0$), and $\lambda \to 0$ ($\tau \to \infty$) we have the relations $\Gamma = \Psi(\infty)J(\infty)$ and $\Gamma P^{-1}\Psi(\infty) = PJ(\infty)$ respectively, implying $J^2(\infty) = I$. Now, reconsider (3.10) in the form

$$D[\Gamma\Psi(\lambda)] = -\Gamma\Omega(\lambda)\Psi(\lambda)\,. \tag{3.53}$$

Taking Hermitian conjugate of both sides of this equation, replacing λ by $\bar{\lambda}$ and using (3.48) we have

$$D[\Psi^\dagger(\bar{\lambda})\Gamma] = \Psi^\dagger(\bar{\lambda})\Gamma\Omega(\lambda)\,, \tag{3.54}$$

which implies that $\Psi^\dagger(\bar{\lambda})\Gamma$ satisfies the same differential equation as $\Psi^{-1}(\lambda)$, hence we have

$$\Psi^\dagger(\bar{\lambda})\Gamma = J_0\Psi^{-1}(\lambda)\,, \tag{3.55}$$

with $\det J_0 \neq 0$. Considering this equation in the limit $\lambda \to 0$ and requiring that the equation $P^{-1} = \Gamma P\Gamma$ must be satisfied, the matrix J_0 becomes equal to Γ, hence we have

$$\Psi^{-1}(\lambda) = \Gamma\Psi^\dagger(\bar{\lambda})\Gamma\,. \tag{3.56}$$

The expressions (3.52) and (3.56) mean that the matrix $\chi(\lambda)$ satisfies the following conditions:

$$\chi^{-1}(\lambda) = \Gamma\chi^\dagger(\bar{\lambda})\Gamma\,, \tag{3.57}$$

$$P = \chi(\lambda)P_0\chi^\dagger(\bar{\tau})\,, \tag{3.58}$$

provided that $P_0 = P_0^\dagger$. These relations represent reduction conditions which must be satisfied by the matrix χ for the preservation of the symmetric space property.

Now substituting (3.13) and (3.14) into (3.57) and considering at the poles $\lambda = \nu_k$ one can conclude that the residues of right-hand side are nonzero. The residues of left-hand side are also nonzero if $\nu_k = \bar{\mu}_k$. In this case the matrices R_k and S_k are related to each other by the relation

$$S_k = \Gamma R_k^\dagger\Gamma\,. \tag{3.59}$$

Using these relations we see that Eq. (3.24) is same as (3.25). So it is enough to use one of them:

$$R_k + \sum_{i=1}^{N}\frac{R_k\Gamma R_i^\dagger\Gamma}{\mu_k - \bar{\mu}_i} = 0\,. \tag{3.60}$$

On the other hand, Eq. (3.58) evaluated at the poles $\lambda = \mu_k$ reads

$$R_k P_0 \left[I + \sum_{i=1}^{N} \frac{R_i^\dagger}{\tau(\mu_k) - \bar{\mu}_i} \right] = 0 \,. \tag{3.61}$$

4. Construction of Soliton Solutions

In this section we shall integrate Eq. (3.2) and construct the two-soliton solution for the matrix P of (2.10) and (2.15) separately. We will use the same set of functions $\{\Theta, a, b\}$ and the function $\mu(\rho, z)$ which are given in (3.45) and (3.46). The two relations (3.60) and (3.61) must consistently be satisfied by the matrix R_k so that the new solution P of the field equation obey the symmetric space property (2.11). Indeed, these two linear equations are equivalent if the poles of the matrix χ are even in number and are related pairwise

$$\mu_{N+k} = \tau(\mu_k) \,, \quad k = 1, 2, \ldots, N \,, \tag{4.1}$$

with the corresponding vectors satisfying

$$n^\alpha_{N+k} = \Gamma P_0 n^\alpha_k \,. \tag{4.2}$$

This implies that two-soliton solution corresponds to two poles for the matrix χ, such that one pole is at $\lambda = \mu$ and the other pole is at $\lambda = \tau(\mu)$. The matrix χ and and its inverse χ^{-1} can then be written in the form

$$\begin{aligned} \chi(\lambda) &= I + \frac{R_1}{\lambda - \mu} + \frac{R_2}{\lambda - \tau(\mu)} \,, \\ \chi^{-1}(\lambda) &= I + \frac{\Gamma R_1^\dagger \Gamma}{\lambda - \bar{\mu}} + \frac{\Gamma R_2^\dagger \Gamma}{\lambda - \tau(\bar{\mu})} \,. \end{aligned} \tag{4.3}$$

Following from (3.11) the required matrix P is

$$P(\rho, z) = \chi(0, \rho, z) P_0(\rho, z) \,, \tag{4.4}$$

where P_0 is the given background solution of (3.2). So the calculation of the matrices R_1 and R_2 will complete the determination of the two-*soliton* solution for the matrix P.

Two-soliton solution for SAS-vacuum field equations

Since the 2×2 matrices R_1 and R_2 are degenerate, by Eqs. (3.26) and (4.2), they can be written in the form

$$R_1 = m_1 n_1^\dagger \,, \qquad R_2 = m_2 n_1^\dagger P_0 \Gamma \,, \tag{4.5}$$

where

$$n_1^\dagger = n_{01}^\dagger \Psi_0^{-1}(\mu) \,, \tag{4.6}$$

and $n_{01}^\dagger$ is an arbitrary complex constant vector while $\Psi_0^{-1}(\mu)$ is the matrix satisfying the condition given in (3.56). We can calculate the vectors m_1 and m_2 by means of Eq. (3.61) which can be written in the form

$$\sum_{i=1}^{2N} M_{ki} m_i = \frac{P_0}{\bar{\mu}_k} n_k \,, \tag{4.7}$$

where the matrix M_{ki} is Hermitian with elements given by

$$M_{ki} = \frac{\overline{(n_k^\dagger P_0 n_i)}}{\rho^2 + \bar{\mu}_k \mu_i} \,. \tag{4.8}$$

For the case $N = 1$ we can find the inverse matrix L_{ki} such that

$$\sum_{l=1}^{2} L_{kl} M_{li} = \delta_{ki} \,. \tag{4.9}$$

Using (4.7), the vectors m_1 and m_2 can be written as

$$\begin{aligned} m_1 &= \xi^{-1}\left[\frac{A}{\bar{\mu}(\rho^2+\mu\bar{\mu})} P_0 n_1 + \frac{B}{\rho^2(\mu-\bar{\mu})}\Gamma n_1\right], \\ m_2 &= \xi^{-1}\left[\frac{B}{\mu\bar{\mu}(\mu-\bar{\mu})} P_0 n_1 - \frac{A}{\mu(\rho^2+\mu\bar{\mu})}\Gamma n_1\right], \end{aligned} \tag{4.10}$$

where

$$\xi = \frac{A^2}{(\rho^2+\mu\bar{\mu})^2} + \frac{B^2}{\rho^2(\mu-\bar{\mu})^2} \,, \quad A = n_1^\dagger P_0 n_1 \,, \quad B = n_1^\dagger \Gamma n_1 \,. \tag{4.11}$$

Substituting (4.10) into (4.5) we obtain

$$\begin{aligned} R_1 &= \xi^{-1}\left[\frac{A}{\bar{\mu}(\rho^2+\mu\bar{\mu})} P_0 n_1 n_1^\dagger + \frac{B}{\rho^2(\mu-\bar{\mu})}\Gamma n_1 n_1^\dagger\right], \\ R_2 &= \xi^{-1}\left[\frac{B}{\mu\bar{\mu}(\mu-\bar{\mu})} P_0 n_1 n_1^\dagger P_0 \Gamma - \frac{A}{\mu(\rho^2+\mu\bar{\mu})}\Gamma n_1 n_1^\dagger P_0 \Gamma\right]. \end{aligned} \tag{4.12}$$

Using (4.3), (4.4) and (4.12) the two-soliton solution for the matrix P is written in the form

$$\begin{aligned} P = P_0 &- \frac{(\mu-\bar{\mu})(\rho^2+\mu\bar{\mu})}{\mu\bar{\mu}[A^2\rho^2(\mu-\bar{\mu})^2 + B^2(\rho^2+\mu\bar{\mu})^2]}\,[A\rho^2(\mu-\bar{\mu})P_0 n_1 n_1^\dagger P_0 \\ &+ B\bar{\mu}(\rho^2+\mu\bar{\mu})\Gamma n_1 n_1^\dagger \Gamma - B\mu(\rho^2+\mu\bar{\mu})P_0 n_1 n_1^\dagger \Gamma + A\mu\bar{\mu}(\mu-\bar{\mu})\Gamma n_1 n_1^\dagger \Gamma]\,. \end{aligned} \tag{4.13}$$

One can easily verify that, the matrix P satisfies the symmetric space property $(\Gamma P)^2 = I$ if $(\Gamma P_0)^2 = I$.

Next we have to calculate the metric coefficient $e^{2\gamma}$ by solving Eq. (2.14) which can also be written in the form

$$\begin{aligned} (\ln e^{2\gamma})_{,\rho} &= \frac{1}{4\rho}\,\mathrm{Tr}[U^2 - V^2]\,, \\ (\ln e^{2\gamma})_{,z} &= \frac{1}{2\rho}\,\mathrm{Tr}(UV)\,, \end{aligned} \tag{4.14}$$

where U and V are defined as

$$U = \rho P_{,\rho} P^{-1}, \quad V = \rho P_{,z} P^{-1}. \tag{4.15}$$

Using these definitions and explicit form of Ω given in (3.4), Eq. (3.21) can be written as

$$\begin{aligned} D_\rho \chi &= \frac{(\rho U + \lambda V)}{\lambda^2+\rho^2}\chi - \chi\frac{(\rho U_0 + \lambda V_0)}{\lambda^2+\rho^2}, \\ D_z \chi &= \frac{(\rho V - \lambda U)}{\lambda^2+\rho^2}\chi - \chi\frac{(\rho V_0 - \lambda U_0)}{\lambda^2+\rho^2}, \end{aligned} \tag{4.16}$$

where χ is given by (4.3) and

$$D_\rho = \partial_\rho + \frac{2\lambda\rho}{\lambda^2+\rho^2}\partial_\lambda, \qquad D_z = \partial_z - \frac{2\lambda^2}{\lambda^2+\rho^2}\partial_\lambda, \tag{4.17}$$

following from (3.6). It is clear from (4.14) that to calculate γ we must determine the matrices U and V. This can be done by evaluating the residues of left and right-hand sides of Eq. (4.16) at the poles $\lambda = -i\rho$ and $\lambda = i\rho$. For the case $\mu_1 = \mu$ and $\mu_2 = -\rho^2/\mu$ we have

$$\begin{aligned} U &= \frac{i}{\rho}\left\{\left[\frac{\rho^2 R_1 - \mu^2 R_2}{(i\rho+\mu)^2}\right]\chi^{-1}(-i\rho) - \left[\frac{\rho^2 R_1 - \mu^2 R_2}{(i\rho-\mu)^2}\chi^{-1}(i\rho)\right]\right\} \\ &\quad + \frac{1}{2}\chi(-i\rho)(U_0 - iV_0)\chi^{-1}(-i\rho) + \frac{1}{2}\chi(i\rho)(U_0 + iV_0)\chi^{-1}(i\rho), \\ V &= -\frac{1}{\rho}\left\{\left[\frac{\rho^2 R_1 - \mu^2 R_2}{(i\rho+\mu)^2}\right]\chi^{-1}(-i\rho) + \left[\frac{\rho^2 R_1 - \mu^2 R_2}{(i\rho-\mu)^2}\chi^{-1}(i\rho)\right]\right\} \\ &\quad + \frac{1}{2i}\chi(i\rho)(U_0 + iV_0)\chi^{-1}(i\rho) - \frac{1}{2i}\chi(-i\rho)(U_0 - iV_0)\chi^{-1}(-i\rho). \end{aligned} \tag{4.18}$$

Calculating the traces of $(U^2 - V^2)$ and (UV) and substituting them in (4.14) we find that

$$(\ln e^{2\gamma})_{,\rho} = \left\{\ln\left[\frac{A^2\rho^2(\mu-\bar{\mu})^2 + B^2(\rho^2+\mu\bar{\mu})^2}{(\bar{\mu}^2+\rho^2)(\mu^2+\rho^2)}\right]\right\}_{,\rho} + \{\ln e^{2\gamma_0}\}_{,\rho} \tag{4.19}$$

which can be integrated easily,

$$\ln e^{2\gamma} = \ln\left[\frac{A^2\rho^2(\mu-\bar{\mu})^2 + B^2(\rho^2+\mu\bar{\mu})^2}{(\bar{\mu}^2+\rho^2)(\mu^2+\rho^2)}\right] + \ln e^{2\gamma_0} + \ln C(z). \tag{4.20}$$

Calculating $(\ln e^{2\gamma})_{,z}$ and comparing with (4.14) we find that $C(z)$ must be a constant. Hence the result becomes

$$e^{2\gamma} = C_0 e^{2\gamma_0}\frac{A^2\rho^2(\mu-\bar{\mu})^2 + B^2(\rho^2+\mu\bar{\mu})^2}{(\bar{\mu}^2+\rho^2)(\mu^2+\rho^2)}, \tag{4.21}$$

where C_0 is a complex constant and γ_0 is the corresponding metric coefficient for the given background solution. The explicit parametrization (2.10) of P enables one to identify from (4.13) the metric function f and the potential ψ. The remaining task is to integrate Eq. (2.9) for the metric function w and determine the metric completely. For this purpose let us specialize to flat background. In this case the matrix P_0 and the metric function γ_0 are given by

$$P_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad e^{2\gamma_0} = 1\,, \tag{4.22}$$

implying that $f_0 = 1$ and $\psi_0 = 0$. The corresponding solution Ψ_0 of $D\Psi_0 = 0$ satisfying the reduction conditions can be calculated as

$$\Psi_0(\lambda, \rho, z) = \begin{pmatrix} 1 - \dfrac{1}{\Theta} & -\dfrac{1}{\Theta} \\ \dfrac{1}{\Theta} & 1 + \dfrac{1}{\Theta} \end{pmatrix}. \tag{4.23}$$

Because of (3.16) the value of Ψ_0 at $\lambda = \mu$ will be a constant matrix. It follows from (4.6) that the column vector n_1 will be a constant vector. Hence we can choose

$$n_1 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \tag{4.24}$$

Now, let us rename the following combination of the complex constants α and β,

$$a = \alpha\bar{\alpha} + \beta\bar{\beta}\,, \quad -m = \beta\bar{\alpha} + \alpha\bar{\beta}\,, \quad -b = \alpha\bar{\alpha} - \beta\bar{\beta}\,, \quad i\sigma = \beta\bar{\alpha} - \alpha\bar{\beta}\,, \tag{4.25}$$

where σ is the imaginary part of ω_1. Clearly we have

$$a^2 - b^2 - m^2 = \sigma^2\,. \tag{4.26}$$

With these definitions we have $A = a$ and $B = -\sigma$. After some algebra we obtain the following results:

$$\begin{aligned}
f &= \frac{-4\mu\bar{\mu}[a^2\rho^2(\mu-\bar{\mu})^2 + \sigma^2(\rho^2+\mu\bar{\mu})^2]}{[a(\mu\bar{\mu}-\rho^2)+b(\mu\bar{\mu}+\rho^2)]^2(\mu-\bar{\mu})^2+[m(\mu-\bar{\mu})+i\sigma(\mu+\bar{\mu})]^2(\mu\bar{\mu}+\rho^2)^2}\,,\\
\psi &= \frac{2(\mu-\bar{\mu})(\mu\bar{\mu}+\rho^2)[ma(\mu-\bar{\mu})(\mu\bar{\mu}-\rho^2)-ib\sigma(\mu+\bar{\mu})(\mu\bar{\mu}+\rho^2)]}{[a(\mu\bar{\mu}-\rho^2)+b(\mu\bar{\mu}+\rho^2)]^2(\mu-\bar{\mu})^2+[m(\mu-\bar{\mu})+i\sigma(\mu+\bar{\mu})]^2(\mu\bar{\mu}+\rho^2)^2}\,,\\
w &= 2i\sigma\frac{a\rho^2[m(\mu^2-\bar{\mu}^2)+i\sigma(\mu-\bar{\mu})^2]+i\sigma[a(\mu\bar{\mu}+\rho^2)^2+b(\mu^2\bar{\mu}^2-\rho^4)]}{a^2\rho^2(\mu-\bar{\mu})^2+\sigma^2(\mu\bar{\mu}+\rho^2)^2}\,,\\
e^{2\gamma} &= C_0\frac{a^2\rho^2(\mu-\bar{\mu})^2+\sigma^2(\mu\bar{\mu}+\rho^2)^2}{(\bar{\mu}^2+\rho^2)(\mu^2+\rho^2)}\,.
\end{aligned} \tag{4.27}$$

Introducing the Boyer–Lindquist coordinates r, θ by the transformation

$$\rho = [(r-m)^2+\sigma^2]^{\frac{1}{2}}\sin\theta\,, \quad z = (r-m)\cos\theta \tag{4.28}$$

and choosing the plus sign for the function $\mu(\rho, z)$ we get

$$\mu(\rho, z) = [(r-m)+i\sigma](1-\cos\theta)\,. \tag{4.29}$$

In these coordinates the metric functions become

$$
\begin{aligned}
f &= \frac{(r-m)^2+\sigma^2-a^2\sin^2\theta}{(b-a\cos\theta)^2+r^2}\,,\\
\psi &= -\frac{2[(r-m)b+ma\cos\theta]}{(b-a\cos\theta)^2+r^2}\,,\\
w &= -2\frac{a\sin^2\theta[m(r-m)-\sigma^2]+(a-b\cos\theta)[(r-m)^2+\sigma^2]}{(r-m)^2+\sigma^2-a^2\sin^2\theta}\,,\\
e^{2\gamma} &= \sigma^2 C_0\frac{(r-m)^2+\sigma^2-a^2\sin^2\theta}{(r-m)^2+\sigma^2\cos^2\theta}\,.
\end{aligned}
\tag{4.30}
$$

Substituting these functions in (2.6), the final form of the solution becomes

$$
\begin{aligned}
dS^2 = \Sigma\Delta^{-1}dr^2 + \Sigma d\theta^2 - \Sigma^{-1}\{&(\Delta - a^2\sin^2\theta)d\tau^2\\
&- [4\cos\theta - 4a\sin^2\theta(mr+b^2)]d\tau\, d\phi\\
&+ [\Delta(a\sin^2\theta + 2b\cos\theta)^2 - \sin^2\theta(r^2+b^2+a^2)^2]d\phi^2\}\,,
\end{aligned}
\tag{4.31}
$$

where

$$
\tau = t + 2a\phi\,,\quad \Sigma = r^2 + (b - a\cos\theta)^2\,,\quad \Delta = r^2 - 2mr + a^2 - b^2\,. \tag{4.32}
$$

This is nothing but the well-known Kerr–NUT solution with m, a and b standing for the mass, angular momentum per unit mass and the NUT parameter respectively. The relation (4.26) between these parameters implies that we have the solution without horizon. Starting with a different background solution P_0 one can find, in principle, new solutions for SAS-vacuum field equations by means of (4.13) and (4.21).

Two-soliton solution for SAS–Einstein–Maxwell field equations

Now, we shall find the solution for the 3×3 matrix P of (2.15) satisfying the symmetric space property (2.11) where the matrix Γ was given in (2.16). Then, we shall integrate (2.14) for the function γ of SAS–Einstein–Maxwell fields. Since the 3×3 matrices R_1 and R_2 are degenerate, again by Eqs (3.26) and (4.2), they can be written as

$$
R_1 = m_1 n_1^\dagger + r_1 s_1^\dagger\,,\quad R_2 = m_2 n_1^\dagger P_0\Gamma + r_2 s_1^\dagger P_0\Gamma\,, \tag{4.33}
$$

where m_1, m_2, n_1, r_1, r_2, s_1 are three components column vectors. The vectors n_1 and s_1 can be solved easily,

$$
n_1^\dagger = n_{01}^\dagger\Psi_0^{-1}(\mu)\,,\quad s_1^\dagger = s_{01}^\dagger\Psi_0^{-1}(\mu)\,, \tag{4.34}
$$

and m_1, m_2 and r_1, r_2 can be found by solving the matrix equation:

$$
\begin{pmatrix} m_i \\ r_i \end{pmatrix} = \begin{pmatrix} \delta_{ki} & \alpha_{ki} \\ \beta_{ki} & \sigma_{ki} \end{pmatrix}^{-1} \begin{pmatrix} \dfrac{P_0}{\bar{\mu}_k} n_k \\ \dfrac{P_0}{\bar{\mu}_k} s_k \end{pmatrix}, \tag{4.35}
$$

where $(i, k = 1, 2)$, $n_2 = \Gamma P_0 n_1$, $s_2 = \Gamma P_0 s_1$, $\mu_1 = \mu$, $\mu_2 = -\frac{\rho^2}{\mu}$ and

$$\begin{aligned} \delta_{ki} &= \frac{\overline{n_k^\dagger P_0 n_i}}{\mu_i \bar{\mu}_k + \rho^2}\,, \quad \beta_{ki} = \frac{\overline{s_k^\dagger P_0 n_i}}{\mu_i \bar{\mu}_k + \rho^2}\,, \\ \alpha_{ki} &= \frac{\overline{n_k^\dagger P_0 s_i}}{\mu_i \bar{\mu}_k + \rho^2}\,, \quad \sigma_{ki} = \frac{\overline{s_k^\dagger P_0 s_i}}{\mu_i \bar{\mu}_k + \rho^2}\,. \end{aligned} \tag{4.36}$$

This completes the determination of matrices R_1 and R_2 and thus the matrix $P = \chi(0)P_0$. Starting with an arbitrary background solution P_0 and γ_0 we find that

$$P = \left[I + \frac{1}{4i(\omega_1 - \bar{\omega}_1)} \sum_{i=1}^{2} \sum_{k=1}^{2} \frac{\mu_k \bar{\mu}_i + \rho^2}{\mu_k^2 \bar{\mu}_i} \Gamma R_i^\dagger \Gamma R_k \right] P_0 \tag{4.37}$$

and

$$e^{2\gamma} = C_0 e^{2\gamma_0} \frac{\rho^8 (\mu - \bar{\mu})^4 (\rho^2 + \mu\bar{\mu})^4}{(\bar{\mu}^2 + \rho^2)^2 (\mu^2 + \rho^2)^2 \mu^2 \bar{\mu}^2} (\det M)\,, \tag{4.38}$$

where M is the 4×4 matrix of Eq. (4.35). Now, we are ready for calculation of the metric coefficients and the potential functions. By choosing the known solution of (3.2) as $P_0 = I$ and $\gamma_0 = 0$, with complex constant vectors

$$n_1 = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \qquad s_1 = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}, \tag{4.39}$$

we obtain the following results:

$$\begin{aligned} f &= \frac{-4\mu\bar{\mu}[a^2\rho^2(\mu - \bar{\mu})^2 + \sigma^2(\rho^2 + \mu\bar{\mu})^2]}{[a(\mu\bar{\mu} - \rho^2) + b(\mu\bar{\mu} + \rho^2)]^2(\mu - \bar{\mu})^2 - [\sigma(\mu + \bar{\mu}) - im(\mu - \bar{\mu})]^2(\mu\bar{\mu} + \rho^2)^2}\,, \\ \psi &= -\frac{2(\mu - \bar{\mu})(\mu\bar{\mu} + \rho^2)[ma(\mu - \bar{\mu})(\mu\bar{\mu} - \rho^2) - ib\sigma(\mu + \bar{\mu})(\mu\bar{\mu} + \rho^2)]}{[a(\mu\bar{\mu} - \rho^2) + b(\mu\bar{\mu} + \rho^2)]^2(\mu - \bar{\mu})^2 - [\sigma(\mu + \bar{\mu}) - im(\mu - \bar{\mu})]^2(\mu\bar{\mu} + \rho^2)^2}\,, \\ w &= \frac{2i\sigma m a\rho^2(\mu^2 - \bar{\mu}^2) + a\rho^2(\mu - \bar{\mu})^2[2(a^2 - \sigma^2) - q\bar{q}] - 2\sigma^2 b(\mu^2\bar{\mu}^2 - \rho^4)}{a^2\rho^2(\mu - \bar{\mu})^2 + \sigma^2(\mu\bar{\mu} + \rho^2)^2}\,, \\ e^{2\gamma} &= C_0 \frac{a^2\rho^2(\mu - \bar{\mu})^2 + \sigma^2(\mu\bar{\mu} + \rho^2)^2}{(\bar{\mu}^2 + \rho^2)(\mu^2 + \rho^2)}\,, \\ \Phi &= -iq\frac{(\mu - \bar{\mu})^2[a(\mu^2\bar{\mu}^2 - \rho^4) + b(\mu\bar{\mu} + \rho^2)^2] - (\mu\bar{\mu} + \rho^2)^2[\sigma(\mu^2 - \bar{\mu}^2) - im(\mu - \bar{\mu})^2]}{[a(\mu\bar{\mu} - \rho^2) + b(\mu\bar{\mu} + \rho^2)]^2(\mu - \bar{\mu})^2 - [\sigma(\mu + \bar{\mu}) - im(\mu - \bar{\mu})]^2(\mu\bar{\mu} + \rho^2)^2}\,, \\ A_\phi &= i\epsilon\frac{[\sigma(\mu^2 - \bar{\mu}^2) - im(\mu - \bar{\mu})^2][4a\mu\bar{\mu}\rho^2 - 2b(\mu^2\bar{\mu}^2 - \rho^4)]}{[a(\mu\bar{\mu} - \rho^2) + b(\mu\bar{\mu} + \rho^2)]^2(\mu - \bar{\mu})^2 - [\sigma(\mu + \bar{\mu}) - im(\mu - \bar{\mu})]^2(\mu\bar{\mu} + \rho^2)^2} \\ &\quad + \varrho\frac{[(\mu - \bar{\mu})^2[(a^2 - b^2)(\mu^2\bar{\mu}^2 - \rho^4) + 4ab\mu\bar{\mu}\rho^2] - (\mu^2\bar{\mu}^2 - \rho^4)[\sigma(\mu + \bar{\mu}) - im(\mu - \bar{\mu})]^2]}{[a(\mu\bar{\mu} - \rho^2) + b(\mu\bar{\mu} + \rho^2)]^2(\mu - \bar{\mu})^2 - [\sigma(\mu + \bar{\mu}) - im(\mu - \bar{\mu})]^2(\mu\bar{\mu} + \rho^2)^2}\,, \end{aligned} \tag{4.40}$$

where

$$
\begin{aligned}
\sigma &= \eta_2\bar{\eta}_2 + i(\eta_1\bar{\eta}_3 - \eta_3\bar{\eta}_1)\,, \\
a &= \eta_1\bar{\eta}_1 + \eta_2\bar{\eta}_2 + \eta_3\bar{\eta}_3\,, \\
-m &= \eta_1\bar{\eta}_3 + \eta_3\bar{\eta}_1\,, \\
b &= \eta_3\bar{\eta}_3 - \eta_1\bar{\eta}_1\,, \\
q &= \epsilon + i\varrho = \sqrt{2}\eta_2(\eta_1 + i\eta_3)\,, \\
q\bar{q} &= a^2 - \sigma^2 - m^2 - b^2\,,
\end{aligned}
\tag{4.41}
$$

and

$$
\eta_1 = \alpha_1\beta_2 - \alpha_2\beta_1\,, \quad \eta_2 = \alpha_1\beta_3 - \alpha_3\beta_1\,, \quad \eta_3 = \alpha_2\beta_3 - \alpha_3\beta_2\,, \tag{4.42}
$$

are complex constants. Using (4.28) and (4.29) the solutions in terms of Boyer–Lindquist coordinates can be written as

$$
\begin{aligned}
f &= \frac{(r-m)^2 + \sigma^2 - a^2\sin^2\theta}{(b - a\cos\theta)^2 + r^2}\,, \\
\psi &= \frac{2[(r-m)b + ma\cos\theta]}{(b - a\cos\theta)^2 + r^2}\,, \\
w &= -\frac{a\sin^2\theta[2(mr + b^2) + q\bar{q}] - 2b\cos\theta[(r-m)^2 + \sigma^2]}{(r-m)^2 + \sigma^2 - a^2\sin^2\theta}\,, \\
e^{2\gamma} &= \sigma^2 C_0 \frac{(r-m)^2 + \sigma^2 - a^2\sin^2\theta}{(r-m)^2 + \sigma^2\cos^2\theta}\,, \\
\Phi &= iq\frac{(b - a\cos\theta) + ir}{(b - a\cos\theta)^2 + r^2}\,, \\
A_t &= \frac{\epsilon r + \varrho(b - a\cos\theta)}{(b - a\cos\theta)^2 + r^2}\,, \\
A'_\phi &= \frac{\varrho r - \epsilon(b - a\cos\theta)}{(b - a\cos\theta)^2 + r^2}\,, \\
A_\phi &= \frac{\epsilon[(a\sin^2\theta + 2b\cos\theta)r] + \varrho[(b - a\cos\theta)(a + b\cos\theta) - r^2\cos\theta]}{(b - a\cos\theta)^2 + r^2}\,.
\end{aligned}
\tag{4.43}
$$

This solution is a Kerr–Newman type of solution describing the external electrovacuum field of a rotating charged body. The parameters m, a, b, ϵ and ϱ stand for the mass, angular momentum per unit mass, NUT-parameter, electric charge and the magnetic charge.[17,18]

5. Conclusion

In the work of Belinskii and Zakharov, the well-known Kerr–NUT metric was generated from flat space with two real poles in the matrix χ. In this paper, we obtained the same solution for stationary axially symmetric vacuum fields written in terms of

Ernst potentials, from flat space with two complex poles in the scattering matrix. We applied the technique to the stationary axially symmetric Einstein–Maxwell problem which was formulated as a nonlinear sigma model on symmetric space. We obtained a Kerr–Newman-like solution including NUT-parameter. In both cases the matrix χ has complex poles at $\lambda = \mu$ and $\lambda = -\rho^2/\mu$ which are complex conjugated to the poles of χ^{-1}. Considering the same pole structure, very different solutions can be constructed by using alternative background solutions. Since the dimension of the matrices is arbitrary, the technique can be applied, with minor modifications, to every field theory which is formulated as a nonlinear sigma model.

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