PROLONGATION ALGEBRA AND BÄCKLUND TRANSFORMATIONS OF DRINFELD–SOKOLOV SYSTEM OF EQUATIONS

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We show that the Drinfeld–Sokolov system of equations has a nontrivial prolongation structure. The closure process for prolongation algebra gives rise to the $sl(4, c)$ algebra which is used to derive the scattering problem for the system of equations under consideration. The nontrivial new Bäcklund transformations and some explicit solutions are given.

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1. Introduction

The systems of nonlinear partial differential equations are encountered in fundamental particle physics, plasma and fluid dynamics, statistical mechanics, many areas of solid state physics, protein dynamics, laser and fiber optics.[1] These equations are usually solved by the use of approximation techniques, but the range of applicability and usefulness of these solutions increase the interest on the exact solutions and methods of testing for complete integrability. During the past three decades the developments in mathematical physics show that the completely integrable systems of nonlinear partial differential equations have very rich mathematical structures such as the existence of Lax pair, bi-Hamiltonian structures, recursion operators and applicability of inverse scattering methods. One of the effective methods to test integrability is the prolongation structure technique of Wahlquist and Estabrook.[2] This geometrical method immediately proceeds with an attempt to construct the linear spectral problem or another device, such as Bäcklund transformations. If we are successful we have not only tested our system of equations for complete integrability but also constructed a device with which to integrate the system. Dodd and Fordy[3,4] made the method more algorithmic and put it in an algebraic, instead of differential geometric framework. The first step is to start with a differential equation and derive a set of generators and relations for

an incomplete Lie algebra. The second step is to complete this Lie algebra and find a finite matrix representation for the derived set of generators. Then, one expects that Bäcklund transformations, which are relations among solutions of the non-linear differential equations under consideration, may be obtained. For a review the reader is referred to an article by Harrison[5] and references quoted therein.

The integrable systems appear not one at a time, but in big families which are called hierarchies. First, the Korteweg–de Vries (KdV) hierarchy was invented and then infinitely many generalized KdV hierarchies were found. They were unified to a single one large Kadomtsev–Petviashvili (KP) hierarchy.[6] Very recently, it is shown by Gürses and Karasu[7] that the system of equations

$$u_t = -u_{xxx} + 6uu_x + 6v_x\,,$$
$$v_t = 2v_{xxx} - 6uv_x \tag{1.1}$$

admits a recursion operator and a bi-Hamiltonian structure, therefore it has constants of motion. The system (1.1) belongs to a type of equations which are called "quasi-polynomial flows." The Lax pair for this system was first given by Drinfeld and Sokolov[8] and later by Bogoyavlenskii.[9] Under the scale transformations, this system of equations reduces to a special case of the KP hierarchy which was given by Satsuma and Hirota.[10] They also gave the one-soliton solutions. Recently, auto-Bäcklund transformations and certain analytical solutions are obtained by Tian and Gao[11] via the Painléve analysis. In this work, we used the prolongation method to derive the linear scattering problem for the system (1.1). We obtain Bäcklund transformations by using pseudopotentials.

2. The Prolongation Structure

By introducing the variables $p = u_x, q = v_x, r = p_x, s = q_x$, the system of Eqs. (1.1) can be represented by the set of two-forms

$$\alpha_1 = du \wedge dt - p\,dx \wedge dt\,,$$
$$\alpha_2 = dp \wedge dt - r\,dx \wedge dt\,,$$
$$\alpha_3 = dv \wedge dt - q\,dx \wedge dt\,,$$
$$\alpha_4 = dq \wedge dt - s\,dx \wedge dt\,, \tag{2.1}$$
$$\alpha_5 = du \wedge dx - dr \wedge dt + 6(up + q)\,dx \wedge dt\,,$$
$$\alpha_6 = dv \wedge dx + 2ds \wedge dt - 6uq\,dx \wedge dt\,,$$

which constitutes a closed ideal I, such that $dI \subset I$.

We extend the ideal I by adding to it the system of one-forms

$$w^A = dy^A + F^A\,dx + G^A\,dt\,, \qquad A = 1,\dots,N\,, \tag{2.2}$$

where F^A and G^A are functions of (u, v, p, q, r, s, y^A), which are assumed in the form $F^A = F^A_B y^B$, $G^A = G^A_B y^B$. The extended ideal must be closed under exterior

differentiations. This requirement gives the set of partial differential equations for F^A and G^A. Dropping the indices for simplicity, we have

$$F_p = F_q = F_r = F_s = 0\,, \quad G_s = 2F_v\,, \quad G_r = -F_u\,,$$

$$pG_u + qG_v + rG_p + sG_q - 6(up+q)F_u + 6uqF_v - [F,G] = 0\,, \quad (2.3)$$

where

$$[F,G] = F^B \frac{\partial G}{\partial y^B} - G^B \frac{\partial F}{\partial y^B}\,. \quad (2.4)$$

Next, we integrate Eq. (2.3). In reaching the results we equate the coefficients of quadratic terms in F to zero, since these coefficients are in the centre of prolongation algebra.[4] The final result is

$$F = X_1 + X_2 u + X_3 v\,,$$

$$G = X_0 + (-r + 3u^2 + 6v)X_2 + 2(s - 3uv)X_3 - pX_4 - uX_5$$

$$- \frac{u^2}{2}X_6 + 2qX_7 + 2vX_8 + v^2 X_9 + 2uvX_{10}\,, \quad (2.5)$$

where X_0, X_1, X_2, X_3 are constants of integration, depending on y^A only. The remaining elements are

$$X_4 = [X_1, X_2]\,, \quad X_5 = [X_1, X_4]\,,$$

$$X_6 = [X_2, X_4]\,, \quad X_7 = [X_1, X_3]\,, \quad (2.6)$$

$$X_8 = [X_1, X_7]\,, \quad X_9 = [X_3, X_7]\,, \quad X_{10} = [X_2, X_7]\,.$$

The integrability conditions impose following restrictions on X_i $(i = 1, \ldots, 10)$,

$$[X_2, X_3] = 0\,, \quad [X_2, X_6] = 0\,, \quad [X_3, X_9] = 0\,, \quad [X_1, X_0] = 0\,,$$

$$[X_3, X_4] = 6X_3 - 2X_{10}\,, \quad [X_2, X_9] + 2[X_3, X_{10}] = 0\,,$$

$$[X_2, X_{10}] - \frac{1}{4}[X_3, X_6] = 0\,,$$

$$[X_1, X_6] + 2[X_2, X_5] - 6X_4 = 0\,,$$

$$[X_1, X_9] + 2[X_3, X_8] = 0\,,$$

$$[X_1, X_5] - [X_2, X_0] = 0\,,$$

$$2[X_1, X_8] + [X_3, X_0] + 6X_4 = 0\,,$$

$$[X_1, X_{10}] + [X_2, X_8] - \frac{1}{2}[X_3, X_5] - 3X_7 = 0\,. \quad (2.7)$$

Using the Jacobi identities we obtain further relations:

$$[X_1, X_9] = [X_3, X_8] = 0,$$
$$[X_2, X_9] = [X_3, X_{10}] = 0,$$
$$[X_2, X_{10}] = [X_3, X_6] = 0,$$
$$[X_2, X_7] = [X_3, X_4],$$
$$[X_1, X_{10}] - [X_4, X_7] - [X_2, X_8] = 0, \qquad (2.8)$$
$$[X_1, X_6] = [X_2, X_5] = 2X_4,$$
$$[X_4, X_6] = -2X_6,$$
$$[X_3, X_5] + [X_7, X_4] + 2[X_1, X_{10}] = 6X_7.$$

In order to find the Lie algebra generated by F and matrix representations of the generators $\{X_i\}_0^{10}$, we follow the strategy of Dodd–Fordy.[4] It can be summarized as follows:

(1) Locate elements of the center of the algebra. Assuming the algebra to be (semi-)simple, equate these elements with zero.
(2) Locate a nilpotent and semi-simple element.
(3) Embed these elements in a simple Lie algebra g.
(4) Express the remaining generators of the prolongation algebra as linear combinations of a suitable basis of g.
(5) Use the fundamental representation of g to generate a linear scattering problem.

First we reduce the number of elements. By using Eqs. (2.6)–(2.8), we get

$$X_9 = 0, \qquad X_6 = 2X_2, \qquad X_{10} = 2X_3. \qquad (2.9)$$

Next, we locate nilpotent and neutral elements. Equations (2.6) and (2.9) give that X_2 is nilpotent and X_4 is neutral element. Let us note that the system of equations in (1.1) has the following scale symmetry:

$$x \to \lambda^{-1}x, \quad t \to \lambda^{-3}t, \quad u \to \lambda^2 u, \quad v \to \lambda^4 v, \qquad (2.10)$$

which implies that the elements X_i must satisfy

$$X_0 \to \lambda^3 X_0, \quad X_1 \to \lambda X_1, \quad X_2 \to \lambda^{-1}X_2, \quad X_3 \to \lambda^{-3}X_3,$$
$$X_4 \to X_4, \quad X_5 \to \lambda X_5, \quad X_7 \to \lambda^{-2}X_7, \quad X_8 \to \lambda^{-1}X_8. \qquad (2.11)$$

By using the basis elements, we try to embed the prolongation algebra into $\mathrm{sl}(n + 1, c)$. Starting from the cases $n = 1, 2$, we found that $\mathrm{sl}(2, c)$ and $\mathrm{sl}(3, c)$ cannot be the whole algebra. The simplest nontrivial closure is in terms of $\mathrm{sl}(4, c)$. Without

giving the details here we present the results:

$$
X_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{\lambda^4}{2} & 0 & 0 & 0 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},
$$

$$
X_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\dfrac{1}{2} & 0 & 0 & 0 \end{pmatrix}, \qquad X_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},
$$

$$
X_5 = \begin{pmatrix} 0 & -2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -2 \\ \lambda^4 & 0 & 0 & 0 \end{pmatrix}, \quad X_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\dfrac{1}{2} & 0 & 0 & 0 \\ 0 & \dfrac{1}{2} & 0 & 0 \end{pmatrix},
$$

$$
X_8 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad X_0 = \begin{pmatrix} 0 & 0 & 0 & -8 \\ -4\lambda^4 & 0 & 0 & 0 \\ 0 & -2\lambda^4 & 0 & 0 \\ 0 & 0 & -4\lambda^4 & 0 \end{pmatrix}.
$$

(2.12)

These generators satisfy the following commutation relations:

$$
\begin{aligned}
&[X_2, X_4] = 2X_2, && [X_2, X_3] = 0, && [X_3, X_4] = 2X_3, \\
&[X_1, X_2] = X_4, && [X_1, X_3] = X_7, && [X_4, X_7] = 0, \\
&[X_3, X_7] = 0, && [X_1, X_7] = X_8, && [X_3, X_8] = 0, \\
&[X_2, X_7] = 2X_3, && [X_1, X_4] = X_5, && [X_2, X_5] = 2X_4, \\
&[X_3, X_5] = 2X_7, && [X_2, X_8] = 2X_7, && [X_0, X_1] = 0, \\
&[X_2, X_0] = [X_1, X_5], && [X_3, X_0] = -6X_4 - 2[X_1, X_8].
\end{aligned}
$$

(2.13)

By substituting the matrix representations of the generators into Eq. (2.5) we form the matrices F and G. Using equations $y_x = -Fy$, $y_t = -Gy$, the linear scattering problem can be obtained as

$$
\begin{pmatrix} y^1 \\ y^2 \\ y^3 \\ y^4 \end{pmatrix}_x = \begin{pmatrix} 0 & -u & 0 & -\dfrac{\lambda^4}{2} + \dfrac{v}{2} \\ -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & -u \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \\ y^3 \\ y^4 \end{pmatrix},
$$

(2.14)

$$\begin{pmatrix} y^1 \\ y^2 \\ y^3 \\ y^4 \end{pmatrix}_t = \begin{pmatrix} p & 4\lambda^4 - 4v + r - 2u^2 & q & \lambda^4 u + s - uv \\ -2u & -p & 2\lambda^4 - 2v & -q \\ 0 & 4u & p & 4\lambda^4 - 4v + r - 2u^2 \\ 8 & 0 & -2u & -p \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \\ y^3 \\ y^4 \end{pmatrix} \quad (2.15)$$

which is equivalent to the scalar Lax equation

$$L\psi = (\partial^4 - 2u\partial^2 - 2u_x\partial - u_{xx} + u^2 + v)\psi = \lambda^4\psi\,, \quad (2.16)$$

where $\psi = y^4$ and $\lambda = \text{constant}$. The corresponding time evolution of ψ is

$$\psi_t = (-4\partial^3 + 6u\partial + 3u_x)\psi\,. \quad (2.17)$$

3. Bäcklund Transformations

Within the prolongation scheme, Bäcklund transformations can be derived by assuming the new solution variables as functions of old ones and the ratios of pseudopotentials.[5] For this purpose, let us define new variables

$$\alpha = \frac{y^1}{y^4}\,, \qquad \beta = \frac{y^2}{y^4}\,, \qquad \gamma = \frac{y^3}{y^4}\,. \quad (3.1)$$

By using Eqs. (2.14) and (2.15) we can find the equations satisfied by α, β and γ,

$$\begin{aligned} \alpha_x &= \alpha\gamma - u\beta + \frac{1}{2}(v - \lambda^4)\,, \\ \beta_x &= -\alpha + \beta\gamma\,, \\ \gamma_x &= \gamma^2 - 2\beta - u\,, \\ \alpha_t &= -8\alpha^2 + 2(p + 2u\gamma)\alpha + (4\lambda^4 - 4v + r - 2u^2)\beta \\ &\quad + q\gamma + (\lambda^4 u + s - uv)\,, \\ \beta_t &= -2u\alpha + 2(u\gamma - 4\alpha)\beta + 2(\lambda^4 - v)\gamma - q\,, \\ \gamma_t &= 2u\gamma^2 + 2(p - 4\alpha)\gamma + 4u\beta + (4\lambda^4 - 4v + r - 2u^2)\,. \end{aligned} \quad (3.2)$$

The compatibility conditions $\alpha_{xt} = \alpha_{tx}$ and $\gamma_{xt} = \gamma_{tx}$ hold if u and v satisfy Eq. (1.1), while $\beta_{xt} = \beta_{tx}$ holds automatically. One can easily check that the function β satisfies the following equation:

$$\beta_t - 2\beta_{xxx} + 6(u + 2\gamma_x)\beta_x = 0\,. \quad (3.3)$$

This means that

$$\begin{aligned} \tilde{u} &= u + 2\gamma_x\,, \\ \tilde{v} &= c_1\beta + c_2 \end{aligned} \quad (3.4)$$

are the new solutions of Eqs. (1.1) if

$$(4\beta^2 + c_1\beta)_x = 0\,, \quad (3.5)$$

where c_1 and c_2 are constants. We note that the same results for $\tilde{u}$ and $\tilde{v}$ were obtained when we followed the step (3) of Ref. 5. By seting $y_4 = \psi$ and using Eqs. (2.14), (2.15) and (3.1) we obtain

$$\tilde{u} = u - 2\frac{\psi_{xx}}{\psi} + 2\frac{\psi_x^2}{\psi^2},$$

$$\tilde{v} = c_1\left(-\frac{u}{2} + \frac{\psi_{xx}}{\psi}\right) + c_2.$$

(3.6)

Here, u is a known solution of Eqs. (1.1) and ψ is the solution of Eqs. (2.16) and (2.17) satisfying the condition

$$\left\{\frac{1}{2\psi^2}[\psi_{xx} - u\psi][2(\psi_{xx} - u\psi) + c_1\psi]\right\}_x = 0,$$

(3.7)

which is equivalent to (3.5).

Next, we consider the simple case $u = v = 0$ as the known solution of (1.1) and find a new solution. With this choice, we find that

$$\psi = d_1 e^{-\lambda(4\lambda^2 t - x)} + d_2 e^{\lambda(4\lambda^2 t - x)} + d_3 e^{i\lambda(4\lambda^2 t + x)} + d_4 e^{-i\lambda(4\lambda^2 t + x)}$$

(3.8)

is a solution of Eqs. (2.16) and (2.17) where d_1, d_2, d_3 and d_4 are constants. Substituting (3.8) into condition (3.7) we obtain two sets of solutions for ψ whereas $\{d_1 = d_2 = 0, c_1 = 4\lambda^2\}$ and $\{d_3 = d_4 = 0, c_1 = -4\lambda^2\}$.

The respective solutions for $\tilde{u}$ are

$$\tilde{u_1} = \frac{8d_3 d_4 \lambda^2 e^{2i\lambda(4\lambda^2 t + x)}}{[d_4 + d_3 e^{2i\lambda(4\lambda^2 t + x)}]^2},$$

$$\tilde{u_2} = \frac{-8d_1 d_2 \lambda^2 e^{2\lambda(4\lambda^2 t - x)}}{[d_1 + d_2 e^{2\lambda(4\lambda^2 t - x)}]^2}.$$

(3.9)

where in both cases $\tilde{v} = c_2 - 2\lambda^4 = $ constant. Thus, starting from trivial background, we obtained the one soliton solution of KdV equation with $\tilde{v} = $ constant.

In order to find the more general Bäcklund transformations for system (1.1), we assume that new solutions $\tilde{U}$ and $\tilde{V}$ are functions of old variables u, v, p, q, r, s and α, β, γ which are the ratios of prolongation variables satisfying Eq. (3.2). After some straightforward but long calculations we obtain the following results:

$$\tilde{U} = \frac{\Omega}{2} - \Delta - \frac{2}{\gamma}(\alpha + 2\beta\gamma),$$

$$\tilde{V} = \Theta_{xx} - \frac{\Delta}{\gamma}\Theta_x - \Theta^2,$$

(3.10)

where

$$\Omega = -\frac{1}{\gamma^2}(p\gamma + u^2 + 4\beta u + 4\beta^2 - 4\alpha\gamma),$$

$$\Delta = 2\beta - 2\gamma^2 + u,$$

$$\Theta = \frac{1}{4\gamma^2}[-u^2 + 2(\gamma^2 - 2\beta)u - 4(\alpha\gamma + \beta^2)],$$

(3.11)

and Θ must satisfy the condition

$$\Theta_t = 2\Theta_{xxx} + 3\Omega\Theta_x \,. \tag{3.12}$$

Thus, if any solutions u and v to the Drinfeld–Sokolov system of equations are known and if α, β, γ are solutions of (3.2) satisfying the condition (3.12), then $\tilde{U}$ and $\tilde{V}$ are the new solutions of (1.1).

If we take the trivial seed solutions $u = v = 0$, we get

$$\tilde{U} = -\frac{2\beta^2}{\gamma^2} + 2(\gamma^2 - 2\beta)\,,$$

$$\tilde{V} = -\frac{1}{\gamma^4}(17\beta^4 + 8\beta^3\gamma^2 - 10\beta^2\alpha\gamma + \alpha^2\gamma^2) + \left(\frac{\lambda^4}{2} + \frac{c_0}{18}\right)\,, \tag{3.13}$$

with the condition

$$\beta[-56\beta^4 + 48\beta^2\gamma\alpha - 3\beta\gamma^2\lambda^4 - 8\gamma^2\alpha^2 + \gamma^4\lambda^4] = 0\,. \tag{3.14}$$

By setting $y^4 = \psi$, one can write these expressions in terms of the solutions of Lax equations (2.16), (2.17) and see that our results are more general than the given by Tian and Gao.[11]

As an example, we consider the simple case, $u = v = 0$ as the known solutions of (1.1) and we obtain the following two sets of explicit solutions:

$$\tilde{U} = \frac{2c_1^2}{(c_1 x + c_2)^2}\,, \qquad \tilde{V} = \frac{c_0}{18} \tag{3.15}$$

and

$$\tilde{U} = \frac{30(c_1 + x^2)\{4x[5c_1[2(24t - x^3) - 3c_1 x] - (c_2 + 3x^5)] + 15(x^2 + c_1)^3]\}}{\{5c_1[2(24t - x^3) - 3c_1 x] - (c_2 + 3x^5)\}^2}\,,$$

$$\tilde{V} = -\frac{120x[c_2 + 3x^5 + 15c_1^2 x - 10c_1(24t - x^3)]^3}{\{5c_1[2(24t - x^3) - 3c_1 x] - (c_2 + 3x^5)\}^4}\,, \tag{3.16}$$

where c_0, c_1, c_2 are constants.

As we observed that there are no solitary wave solutions belonging to the class (3.13). Very recently, all the special solutions of (1.1) are obtained by Karasu and Sakovich.[12]

4. Conclusion

In this work we have rederived the linear scattering problem for Drinfeld–Sokolov system of equations by using the prolongation algorithm. We found the auto-Bäcklund transformations and some exact solutions of these equations. The system can be integrated by the method of inverse scattering problem associated with the fourth order Lax operator L, which was developed by Iwasaki.[13] It is known that the most general Bäcklund transformation would be the one which utilizes the infinite dimensional algebra and not all of finite algebras give rise to Bäcklund transformations. Without searching if the incomplete algebra given in (2.7) is finite

or infinite dimensional we used a finite dimensional representation of prolongation algebra and derived nontrivial Bäcklund transformations. Thus, the methods given in Refs. 4 and 5 are quite useful from the practical point of view, for the systems of nonlinear partial differential equations. Finally, we note that a close connection between some stationary flows associated with fourth-order Lax operators and generalisations of some integrable Hamiltonian systems with quartic potentials is known.[14] Equations (2.16) and (2.17) can be considered in this context.

Acknowledgments

References

1. A. P. Fordy, *Soliton Theory*: *A Survey of Results* (Manchester University Press, 1990).
2. H. D. Wahlquist and F. B. Estabrook, *J. Math. Phys.* **16**, 1 (1975).
3. R. Dodd and A. Fordy, *Phys. Lett.* **A89**, 168 (1982).
4. R. Dodd and A. Fordy, *Proc. R. Soc. London* **A385**, 389 (1983).
5. K. Harrison, *J. Nonlinear Math. Phys.* **2**, 201 (1995).
6. E. Date, M. Jimbo, M. Kashiwara and T. Miwa, in *Nonlinear Integrable Systems, Classical and Quantum Theory — Proc. RIMS Symp.*, eds. M. Jimbo and T. Miwa (World Scientific, Singapore, 1983).
7. M. Gürses and A. Karasu, *Phys. Lett.* **A251**, 347 (1999).
8. V. G. Drinfeld and V. V. Sokolov, *Procs. S. L. Sobolev Seminar*, Novosibirsk, **2**, 5 (1981) (in Russian).
9. O. I. Bogoyavlenskii, *Russian Math. Surveys* **45**, 1 (1990).
10. J. Satsuma and R. Hirota, *J. Phys. Soc. J.* **51**, 3390 (1982).
11. B. Tian and Y. Gao, *Phys. Lett.* **A208**, 193 (1995).
12. A. Karasu (Kalkanlı) and S. Yu. Sakovich, *J. Phys.* **A34**, 7355 (2001).
13. K. Iwasaki, *Japan J. Math.* **14**, 1 (1988).
14. S. Baker, V. Z. Enolskii and A. P. Fordy, *Phys. Lett.* **A201**, 167 (1995).