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LETTER TO THE EDITOR

Colliding Abelian gauge plane waves

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Abstract. The characteristic initial-value problem of the gravitational and N-Maxwell plane wave collision is solved exactly.

One of the basic problems in gravitational theory is to find the exact solution of colliding electromagnetic waves. This problem was first formulated by Bell and Szekeres (1974) who gave an exact solution representing the collision of impulsive gravitational waves coupled with electromagnetic shock waves in a conformally flat spacetime. In a recent work Chandrasekhar and Xanthopoulos (1985, 1987) have obtained a new exact solution which is, in some sense, a generalisation of the Bell–Szekeres metric. In obtaining the solution, they extended the relationship between the solutions describing stationary axially symmetric spacetimes and solutions describing colliding plane waves, which admit two spacelike Killing vectors, to the Einstein–Maxwell equations. On the other hand it was shown by Gürses and Xanthopoulos (1982) that there exists an analogy between static axially symmetric self-dual SU(3) Yang–Mills and axially symmetric stationary Einstein–Maxwell field equations. This analogy is further extended by Gürses (1984). It was shown that a restricted class of static axially symmetric self-dual SU($n+1$) Yang–Mills field equations are equivalent to the stationary axially symmetric Einstein-($n-1$)-Maxwell field equations.

In this work we consider the collision of N-Abelian gauge plane waves by adopting the Gürses equations for the spacetimes admitting two spacelike Killing vector fields.

In order to discuss the collision problem we consider the spacetime where u and v are null coordinates and x and y are spacelike coordinates. We split the spacetime into four distinct regions labelled by I ($u<0$, $v<0$), II ($u>0$, $v<0$), III ($u<0$, $v>0$) and IV ($u>0$, $v>0$).

In region I, the spacetime is flat Minkowskian space.

In region II, the spacetime metric is given by

$$ds^2 = 2\,e^{-M(u)}\,du\,dv - g_{ij}(u)\,dx^i\,dx^j \qquad i=1,2. \tag{1}$$

The Abelian gauge plane wave is defined by the Newman–Penrose (NP) spinors $\phi_2^A(u)$ and $\phi_1^A = \phi_0^A = 0$, where $A = 1, \ldots, K$.

Similarly in region III, the metric is

$$ds^2 = 2\,e^{-M(v)}\,du\,dv - g_{ij}(v)\,dx^i\,dx^j \tag{2}$$

and the wave is defined by the NP spinors $\phi_0^{A'}(v)$ and $\phi_2^{A'} = \phi_1^{A'} = 0$, where $A' = K+1, \ldots, N$.

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In the interaction region IV, the spacetime metric is given by Bell–Szekeres

$$ds^2 = 2\,e^{-M}\,du\,dv - e^{-U}(e^{V}\cosh W\,dx^2 + e^{-V}\cosh W\,dy^2 - 2\sinh W\,dx\,dy) \tag{3}$$

where the metric functions M, U, V and W depend on the coordinates u and v only and the NP spinors satisfy

$$\phi_0^a(u,v)\neq 0 \qquad \phi_2^a(u,v)\neq 0 \qquad \phi_1^a = 0 \qquad (a=1,\ldots,N).$$

We shall assume that the potential 1-form A^a has two components

$$A^a = A_x^a\,dx + A_y^a\,dy \qquad a=1,\ldots,N \tag{4}$$

where A_x^a and A_y^a are functions of u and v. The Einstein-N-Maxwell field equations are

$$U_{uv} - U_u U_v = 0 \tag{5}$$

$$2U_{uu} - U_u^2 + 2U_u M_u = W_u^2 + V_u^2\cosh^2 W - 4k\phi_2^a\bar\phi_2^a \tag{6}$$

$$2U_{vv} - U_v^2 + 2U_v M_v = W_v^2 + V_v^2\cosh^2 W - 4k\phi_0^a\bar\phi_0^a \tag{7}$$

$$2M_{uv} + U_u U_v - W_u W_v = V_u V_v\cosh^2 W \tag{8}$$

$$2W_{uv} - W_u U_v - W_v U_u = 2V_u V_v \sinh W\cosh W + 2ik(\phi_2^a\bar\phi_0^a - \bar\phi_2^a\phi_0^a) \tag{9}$$

$$V_{uv} - V_u U_v - V_v U_u = -2(V_u W_v + V_v W_u)\tanh W - \frac{2k}{\cosh W}(\phi_2^a\bar\phi_0^a + \bar\phi_2^a\phi_0^a) \tag{10}$$

where

$$\begin{aligned}\phi_2^a = \frac{1}{\sqrt2}\exp[\tfrac12(U-V)]\Bigg[&\left(e^{V}\sinh\frac{W}{2}\frac{\partial A_y^a}{\partial u} + \cosh\frac{W}{2}\frac{\partial A_x^a}{\partial u}\right)\\ &- i\left(e^{V}\cosh\frac{W}{2}\frac{\partial A_y^a}{\partial u} + \sinh\frac{W}{2}\frac{\partial A_x^a}{\partial u}\right)\Bigg]\end{aligned} \tag{11}$$

$$\begin{aligned}\phi_0^a = -\frac{1}{\sqrt2}\exp[\tfrac12(U-V)]\Bigg[&\left(e^{V}\sinh\frac{W}{2}\frac{\partial A_y^a}{\partial v} + \cosh\frac{W}{2}\frac{\partial A_x^a}{\partial v}\right)\\ &+ i\left(e^{V}\cosh\frac{W}{2}\frac{\partial A_y^a}{\partial v} + \sinh\frac{W}{2}\frac{\partial A_x^a}{\partial v}\right)\Bigg].\end{aligned} \tag{12}$$

The non-trivial Maxwell equations can be written in terms of ϕ_2^a and ϕ_0^a as

$$\phi_{2,v}^a = -\tfrac12(V_u\cosh W + iW_u)\phi_0^a + \tfrac12(U_v + iV_v\sinh W)\phi_2^a \tag{13}$$

$$\phi_{0,u}^a = -\tfrac12(V_v\cosh W - iW_v)\phi_2^a + \tfrac12(U_u - iV_u\sinh W)\phi_0^a. \tag{14}$$

For $N=2$, we have $\phi_2^1(u)\neq 0$ in region II and $\phi_0^2(v)\neq 0$ in region III which constitute the initial data to determine U, V, W, M, ϕ_0^2 and ϕ_2^1 in region IV. Rewriting Maxwell equations (13) and (14) for $a=1$ and $a=2$, and using the initial data we find

$$W=0 \qquad V=\text{constant} \qquad \phi_2^1 = e^{U/2}f(u) \qquad \phi_0^2 = e^{U/2}g(v). \tag{15}$$

The solution of equation (5) is given as

$$e^{-U} = a(u) + b(v). \tag{16}$$

Substituting (15) and (16) into equations (6) and (7) we have

$$2a_{uu}+a_u\gamma_u=4kf\bar{f} \tag{17}$$

$$2b_{vv}+b_v\gamma_v=4kg\bar{g} \tag{18}$$

where $\gamma=U+2M$.

In region II we have

$$M=0 \qquad e^{-U}=\tfrac{1}{2}+a(u) \qquad \text{and} \qquad 2a_{uu}-\frac{a_u^2}{(\frac{1}{2}+a)}=4kf\bar{f} \tag{19}$$

which determines $f(u)$ in terms of $a(u)$.

In region III we have

$$M=0 \qquad e^{-U}=\tfrac{1}{2}+b(v) \qquad \text{and} \qquad 2b_{vv}-\frac{b_v^2}{(\frac{1}{2}+b)}=4kg\bar{g} \tag{20}$$

which determines $g(v)$ in terms of $b(v)$. Here we have chosen $a(0)=\frac{1}{2}$, $b(0)=\frac{1}{2}$ for convenience.

Using $f(u)$ and $g(v)$ in the equations (17) and (18) we find the solution for the metric function M,

$$M=\tfrac{1}{2}\log\frac{a+b}{(\frac{1}{2}+a)(\frac{1}{2}+b)}. \tag{21}$$

Thus the line element becomes

$$ds^2=2\left(\frac{(\frac{1}{2}+a)(\frac{1}{2}+b)}{(a+b)}\right)^{1/2}du\,dv-(a+b)(dx^2+dy^2). \tag{22}$$

The components of potential 1-forms A^1 and A^2 can be calculated from equations (11) and (12),

$$f(u)=\frac{1}{\sqrt{2}}\frac{\partial}{\partial u}(A_x^1-iA_y^1) \tag{23}$$

$$g(v)=\frac{1}{\sqrt{2}}\frac{\partial}{\partial v}(A_x^2+iA_y^2) \tag{24}$$

which can be integrated for any given $f(u)$ and $g(v)$ (or $a(u)$ and $b(v)$).

This result can be generalised for arbitrary N. In this case we have

$$\phi_2^A=e^{U/2}f^A(u) \qquad \phi_0^{A'}=e^{U/2}g^{A'}(v) \tag{25}$$

where $A=1,\ldots,K$, $A'=K+1,\ldots,N$.

Equations (17) and (18) can be written as

$$2a_{uu}+a_u\gamma_u=4kf^A\bar{f}^A \tag{26}$$

$$2b_{vv}+b_v\gamma_v=4kg^{A'}\bar{g}^{A'} \tag{27}$$

and equations (23) and (24) become

$$f^A(u)=\frac{1}{\sqrt{2}}\frac{\partial}{\partial u}(A_x^A-iA_y^A) \tag{28}$$

$$g^{A'}(v)=-\frac{1}{\sqrt{2}}\frac{\partial}{\partial v}(A_x^{A'}+iA_y^{A'}). \tag{29}$$

The metric functions remain the same as (22) for $N=2$. As a summary we have the following.

Data.

I Flat spacetime

II ($v=0$, or $b=\frac{1}{2}$), $\phi_2^A(u)=(\frac{1}{2}+a)^{-1/2}f^A(u)$, $A=1,\ldots,K$

$$ds^2=2\,du\,dv-(\tfrac{1}{2}+a)(dx^2+dy^2)$$

III ($u=0$ or $a=\frac{1}{2}$), $\phi_0^{A'}(v)=(\frac{1}{2}+b)^{-1/2}g^{A'}(v)$, $A'=K+1,\ldots,N$

$$ds^2=2\,du\,dv-(\tfrac{1}{2}+b)(dx^2+dy^2).$$

Solution.

IV ($u>0$, $v>0$)

$$\phi_2^A(u,v)=(a+b)^{-1/2}f^A(u) \tag{30}$$

$$\phi_0^{A'}(u,v)=(a+b)^{1/2}g^{A'}(v) \tag{31}$$

$$ds^2=2\left(\frac{(\frac{1}{2}+a)(\frac{1}{2}+b)}{a+b}\right)^{1/2}du\,dv-(a+b)(dx^2+dy^2) \tag{32}$$

where by an appropriate choice $f^A(0)=0$ and $g^{A'}(0)=0$. Hence for a given datum $[a(u),\ b(v)]$ we have an exact solution (30, 31, 32).

Choosing the null tetrad basis

$$\begin{aligned} 1&=e^{-M/2}\,du \qquad n=e^{-M/2}\,dv\\ m&=2^{-1/2}[e^{(V-U)/2}(-\cosh W/2+i\sinh W/2)\,dx\\ &\qquad+e^{-(U+V)/2}(\sinh W/2-i\cosh W/2)\,dy] \end{aligned} \tag{33}$$

the non-vanishing NP spin coefficients are found to be

$$\rho=\tfrac{1}{2}e^{M/2}U_v \qquad \varepsilon=-\tfrac{1}{4}e^{M/2}M_v \qquad \mu=-\tfrac{1}{2}e^{M/2}U_u \qquad \gamma=\tfrac{1}{4}e^{M/2}M_u \tag{34}$$

where U and M are given in equations (16) and (21). The non-vanishing Ricci spinors and Weyl spinors are found to be

$$\Phi_{00}=\frac{-kg^{A'}\bar{g}^{A'}}{[(\frac{1}{2}+a)(\frac{1}{2}+b)(a+b)]^{1/2}} \tag{35}$$

$$\Phi_{11}=\frac{-kf^A\bar{f}^A}{[(\frac{1}{2}+a)(\frac{1}{2}+b)(a+b)]^{1/2}} \tag{36}$$

$$\psi_2=-\frac{1}{4}\,\frac{a_ub_v}{(a+b)[(\frac{1}{2}+a)(\frac{1}{2}+b)(a+b)]^{1/2}}. \tag{37}$$

We conclude that the solution given in equation (22) for region IV has type-D character. The spacetimes describing the initial plane waves are conformally flat. The metric becomes singular on $a(u)+b(v)=0$ which is an essential spacetime singularity. Choosing $a(u)=\frac{1}{2}-\sin^2 a_1u$, $b(v)=\frac{1}{2}-\sin^2 b_1v$ the metric (22) reduces to the one given by Griffiths (1976) which represents a collision between an electromagnetic wave and a neutrino field.

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