ABOUT GEOMETRIZATION OF THE DYNAMICS *)

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The connection between Killing tensors and Lax operators are presented. The Toda lattice case and the Rindler system are analyzed in details.

1 Introduction

It is well known that, many of the classical dynamical theories are completely integrable [1–2]. These are both finite as well as infinite dimensional theories which describe physical systems of importance. For discrete finite systems, it is known that the zero Nijenhuis tensor [3] condition can be used to construct conserved quantities in involution [4]. Let us note that for a given $(1,1)$ tensor such as S_μ^ν we can define the Nijenhuis torsion tensor as [3]

$$N^\mu_{\alpha\beta} = -N^\mu_{\beta\alpha} = S^\lambda_\alpha \partial_\lambda S^\mu_\beta - S^\lambda_\beta \partial_\lambda S^\mu_\alpha - S^\mu_\lambda(\partial_\alpha S^\lambda_\beta - \partial_\beta S^\lambda_\alpha). \tag{1}$$

We can construct the conserved quantities

$$K_n = \left(\frac{1}{n}\right) \operatorname{Tr} S^n, \; n = \pm 1, \ldots, \; K_0 = \log|\det S|. \tag{2}$$

In [4] it was shown that for a theory with a dual Poisson bracket structure, the sufficient condition for integrability is the vanishing of the Nijenhuis tensor. Another method to construct integrals of motion, for the finite systems, is to use the Jacobi geometry and to analyze the existence of Killing tensor in the case of geodesic motion. The geodesic equations are

$$\frac{\mathrm{d}^2 x^l}{\mathrm{d}s^2} + \Gamma^l_{jk} \frac{\mathrm{d}s^j}{\mathrm{d}s} \frac{\mathrm{d}x^k}{\mathrm{d}s} = 0. \tag{3}$$

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If each integral of the equation (3) satisfies the condition

$$k_{r_1\cdots r_m}\frac{\mathrm{d}x^{r_1}}{\mathrm{d}s}\cdots\frac{\mathrm{d}x^{r_m}}{\mathrm{d}s}=\text{const.},\tag{4}$$

the equations (3) are said to admit a first integral of the m-th order. Without loss of generality we can consider the tensor $k_{r_1\cdots r_n}$ as a symmetric tensor and it is easy to show that it satisfies the following equation

$$k_{(r_1\cdots r_n;r_\lambda)}=0,\tag{5}$$

where the parenthesis denotes the full symmetrization [5]. The equation (5) is the definition of the Killing tensor of order m. Killing tensors are indispensable tools in the guest for exact solutions in many branches of general relativity as well as classical mechanics [6–8]. Killing tensors are important for solving the equations of motion in particular space-times. The notable example here is the Kerr metric which admits a second rank Killing tensor [7]. In order to describe the first integral of motion of order m in the case of geodesic motion we can analyze the existence of Killing-Yano tensors [6]. A Killing-Yano tensor of order m is an antisymmetric tensor $f_{\mu_1\cdots\mu_r}$ which satisfies the following equations

$$f_{\mu_1\cdots\mu_r;\mu_\lambda}+f_{\mu_\lambda\cdots\mu_r;\mu_1}=0.\tag{6}$$

If a Killing-Yano tensor exists then using (6) we can construct immediately a Killing tensor, of order two [7]. It was a big success of Gibbons et all. [8] to have been able to show that the Killing-Yano tensor, which had long been known as a rather mysterious structure, can be understood as an object generating "a non-generic" supersymmetry, i.e a supersymmetry appearing only in specific space-times [8, 9]. Another method to obtaining a Killing tensor on a given manifold $g_{\mu\nu}$ is to investigate the existence of the Lax tensors $L^{\mu}_{\nu\lambda}$ and $A^{\mu}_{\nu\lambda}$ [10]. In some specific cases, the Lax tensors $L^{\mu}_{\nu\lambda}$ are Killing-Yano tensors of order three [10].

The main aim of this paper is to analyze the connection between Killing tensors and Lax tensors. We investigate the Lax tensors for the three dimensional open Toda's lattice and the Rindler system.

2 Lax pair tensors

Let us consider a Riemannian or pseudo-Riemannian geometry with the metric

$$\mathrm{d}s^2=g_{\mu\nu}\mathrm{d}q^{\mu}\mathrm{d}q^{\nu}.\tag{7}$$

The geodesic equation can be represented by the Hamiltonian

$$H=\tfrac{1}{2}\,g^{\mu\nu}p_{\mu}p_{\nu}\tag{8}$$

together with the natural Poisson bracket on the cotangent bundle. The geodesic system has the form

$$\dot{q}^{\alpha}=g^{\alpha\mu}p_{\mu},\dot{p}_{\alpha}=\Gamma^{\mu\nu}_{\alpha}p_{\mu}p_{\nu}\,.\tag{9}$$

The complete integrability of this system can be shown with the help of a pair of matrices L and A with entries defined on the phase space and satisfying the Lax pair equation [11]

$$\dot{L} = \{L, H\} = [L, A]. \tag{10}$$

It follows from (10) that the quantities $I_k = (1/k)\,\mathrm{Tr}\,L^k$ are all constants of motion. If in addition they commute with each other $\{I_k, I_j\} = 0$ then it is possible to integrate the system completely at least in principle. We know that the Lax pair equation is invariant under a transformation of the form

$$\tilde{L} = ULU^{-1}, \tilde{A} = UAU^{-1} - \dot{U}U^{-1}. \tag{11}$$

We see that L transforms as a tensor while A transforms as a connection. Typically, the Lax matrices are linear in the momenta and in the geometric setting they may also be assumed to be homogeneous. This motivates the introduction of two third rank geometrical objects $L_\beta^{\alpha\gamma}$ and $A_\beta^{\alpha\gamma}$ such that the Lax matrices can be written as

$$L = (L_\beta^\alpha) = (L_\beta^{\alpha\mu} p_\mu), \quad A = (A_\beta^\alpha) = (A_\beta^{\alpha\mu} p_\mu). \tag{12}$$

We will refer to $L_\beta^{\alpha\gamma}$ and $A_\beta^{\alpha\gamma}$ as the Lax tensor and the Lax connection, respectively. Defining

$$B = (B_\beta^\alpha) = (B_\beta^{\alpha\mu} p_\mu) = A - \Gamma, \tag{13}$$

where $\Gamma = (\Gamma_\beta^\alpha) = (\Gamma_\beta^{\alpha\mu} p_\mu)$ is the Levi-Civita connection with respect to $g_{\alpha\beta}$, it then follows that the Lax pair equation takes the covariant form

$$L^\alpha_{\mu(\gamma;\delta)} = L^\alpha_{\mu(\gamma} B^{|\mu|}_{\beta\delta)} - B^\alpha_{\mu(\gamma} L^{|\mu|}_{\beta\delta)}, \tag{14}$$

where $L^\alpha_{\beta\gamma}$ and $B^\alpha_{\beta\gamma}$ are tensorial objects. Let us suppose that a manifold $g_{\mu\nu}$ admits Lax pair tensors $L_{\alpha\beta\gamma}, A_{\alpha\beta\gamma}$ in such a way that

$$L_{\alpha\beta\gamma;\delta} + L_{\alpha\beta\delta;\gamma} = 0. \tag{15}$$

The symmetric part of $L_{\alpha\beta\gamma}$ is a Killing tensor of rank three because $L_{(\alpha\beta\gamma;\delta)} = 0$. Here the parentheses represent the full symmetrization. Any Killing tensor $L_{\alpha\gamma\delta}$ with covariant derivative zero is a Lax tensor of order three because (15) is satisfies. $L_{\alpha\beta\gamma}$ generates an infinite number of Killing tensors on a given manifold. Of course not all Killing tensors generated by Lax tensors are independent and some of them are trivial Killing tensors. Another important observation is that in the case when we have $g_{\alpha\beta} = L^\mu_{\nu\alpha} L^\nu_{\mu\beta}$ we can identify the invariant I_2 with the geodesic Hamiltonian. $A^\alpha_{\beta\gamma}$ plays an important role because it can be related to the torsion of the manifold.

3 Examples

A. The Hamiltonian corresponding to the three-particle open Toda lattice has the form [12]

$$H = \tfrac{1}{2}\,(p_1^2 + p_2^2 + p_3^2) + \mathrm{e}^{2(q^1 - q^2)} + \mathrm{e}^{2(q^2 - q^3)}. \tag{16}$$

The standard symmetric Lax representation is

$$L = \begin{pmatrix} p_1 & a_1 & 0 \\ a_1 & p_2 & a_2 \\ 0 & a_2 & p_3 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & a_1 & 0 \\ -a_1 & 0 & a_2 \\ 0 & -a_2 & 0 \end{pmatrix}, \tag{17}$$

where

$$a_1 = \mathrm{e}^{(q^1 - q^2)}, \quad a_2 = \mathrm{e}^{(q^2 - q^3)}. \tag{18}$$

The Hamiltonian (16) admits the linear invariant $I_1 = \mathrm{Tr}\, L = p_1 + p_2 + p_3$. The Lax representation also gives rise to the two invariants $I_2 = \frac{1}{2}\mathrm{Tr}\, L^2 = H$ and $I_3 = \frac{1}{3}\mathrm{Tr}\, L^3$. Because L and A from (17) are not yet linear and homogeneous in the momenta we will perform a canonical transformation

$$\hat{q}^1 = q^1 + \ln p_1, \; \hat{q}^2 = q^2, \; \hat{q}^3 = q^3 - \ln p_3, \; \hat{p}_1 = p_1, \; \hat{p}_2 = p_2, \; \hat{p}_3 = p_3. \tag{19}$$

The resulting Lax pair matrices are [10]

$$L = \begin{pmatrix} p_1 & a_1 p_1 & 0 \\ a_1 p_1 & p_2 & a_2 p_3 \\ 0 & a_2 p_3 & p_3 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & a_1 p_1 & 0 \\ -a_1 p_1 & 0 & a_2 p_3 \\ 0 & -a_2 p_3 & 0 \end{pmatrix}, \tag{20}$$

where

$$a_1 = \mathrm{e}^{(q^1 - q^2)}, \quad a_2 = \mathrm{e}^{(q^2 - q^3)}. \tag{21}$$

The Hamiltonian is now purely kinetic

$$H = \tfrac{1}{2}\left[(1 + 2a_1^2)p_1^2 + p_2^2 + (1 + 2a_2^2)p_3^2\right]. \tag{22}$$

Using (22) we identify the metric

$$\mathrm{d}s^2 = g_{11}(\mathrm{d}q^1)^2 + (\mathrm{d}q^2)^2 + g_{33}(\mathrm{d}q^3)^2, \tag{23}$$

where

$$g_{11} = (1 + 2a_1^2)^{-1}, \quad g_{33} = (1 + 2a_2^2)^{-1}. \tag{24}$$

The non-zero Levi-Civita connection coefficients, $\Gamma^{\alpha}_{\beta\gamma}$ of this metric are

$$\begin{aligned} &\Gamma^1_{11} = -2a_1^2 g_{11}, \; \Gamma^1_{12} = 2a_1^2 g_{11}, \; \Gamma^2_{11} = -2a_1^2 (g_{11})^2, \; \Gamma^2_{33} = 2a_2^2 (g_{33})^2, \\ &\Gamma^3_{23} = -2a_2^2 g_{33}, \; \Gamma^3_{33} = 2a_2^2 g_{33}. \end{aligned} \tag{25}$$

After a similarity transformation we found that final forms of the Lax matrices are

$$\hat{L} = \begin{pmatrix} g_{11} p_1 & a_1 p_1 \sqrt{g_{11}} & 0 \\ a_1 \sqrt{g_{11}} p_1 & p_2 & a_2 p_3 \sqrt{g_{33}} \\ 0 & a_2 p_3 \sqrt{g_{33}} & g_{33} p_3 \end{pmatrix}, \tag{26}$$

$$\hat{A} = \begin{pmatrix} \Gamma^1_{11} p_1 + \Gamma^2_{11} p_2 & a_1 \sqrt{g_{11}} p_1 & 0 \\ -a_1 \sqrt{g_{11}} p_1 & 0 & a_2 \sqrt{g_{33}} p_3 \\ 0 & -a_2 \sqrt{33} p_3 & \Gamma^2_{33} p_2 + \Gamma^3_{33} p_3 \end{pmatrix}. \tag{27}$$

Note that the upper triangular part of L and A coincide. This property is peculiar to open Toda lattice. The corresponding connection matrix Γ is given by

$$\hat{\Gamma} = \begin{pmatrix} \Gamma_{11}^1 p_1 + \Gamma_{11}^2 p_2 & 2a_1^2 g_{11} p_1 & 0 \\ -2a_1^2 g_{11} p_1 & 0 & 2a_2^2 g_{33} p_3 \\ 0 & -2a_2^2 g_{33} p_3 & \Gamma_{33}^2 p_2 + \Gamma_{33}^3 p_3 \end{pmatrix}. \tag{28}$$

We observe that the off-diagonal part of the matrix Γ is antisymmetric like that of $\hat{A}$ and furthermore that their off-diagonal components are related by the simple relation $\Gamma_{\alpha\beta}^{\gamma} = 2(L_{\alpha\beta}^{\gamma})^2$ for $\alpha \prec \beta$. Using the relation $\hat{A} = \hat{\Gamma} + \hat{B}$, we can find after some calculations that $\hat{B}_{\beta\gamma}^{\alpha} = -\hat{B}_{\gamma\beta}^{\alpha}$ [10].

B. The Rindler system [13] is conventionally denoted by τ and r:

$$t = r\sinh\tau,\;\; x = r\cos\tau,\;\; 0 < r < \infty,\;\; -\infty < \tau < \infty, \tag{29}$$

with coordinates curves (timelike hyperbolas and spacelike straight lines) given by

$$x^2 - t^2 = r^2, \quad \frac{t}{x} = \tanh\tau, \tag{30}$$

the metric

$$\mathrm{d}s^2 = r^2\mathrm{d}\tau^2 - \mathrm{d}r^2, \tag{31}$$

and the associated Killing tensor

$$k^{ik} = \begin{pmatrix} 1 - \frac{c}{r^2} & 0 \\ 0 & c \end{pmatrix}. \tag{32}$$

Here c is a constant. The non-zero Christoffel symbols are $\Gamma_{11}^2 = r$, $\Gamma_{12}^1 = 1/r$. We have four independent symmetric components of $L_{\alpha\beta\gamma}$ and eight independent equations:

$$\begin{aligned} &L_{111};\tau = 0, \quad L_{111};r + L_{112};\tau = 0, \quad L_{112};\tau + L_{111};r = 0,\\ &L_{112};r = 0, \quad L_{122};\tau + L_{121};r = 0, \quad L_{122};\tau + L_{121};r = 0,\\ &L_{122};r = 0, \quad L_{222};\tau + L_{221};r = 0. \end{aligned} \tag{33}$$

Here semicolon denotes the covariant derivative. We found that a solution of (33) has the following form:

$$\begin{aligned} &L_{122} = (C_1\mathrm{e}^{-\tau} + C_2\mathrm{e}^{3\tau})r, && L_{112} = (C_1\mathrm{e}^{-\tau} + C_2\mathrm{e}^{3\tau})r^2,\\ &L_{111} = -(3C_1\mathrm{e}^{-\tau} - C_2\mathrm{e}^{3\tau})r^3, && L_{222} = -3C_1\mathrm{e}^{-\tau} + C_2\mathrm{e}^{3\tau}, \end{aligned} \tag{34}$$

where C_1, C_2 are constants.

4 Conclusions

In this paper the connection between Killing tensors and Lax tensors was investigated. We found that a covariant constant third rank Killing tensor is a Lax tensor. The solution of Lax tensors for three dimensional open Toda lattice was presented. Solving (33) we found, for the Rindler system, a nontrivial solution for Lax tensors. It would be an interesting problem to analyze the solution of the Lax tensors equation in the presence of torsion and it will be given in a separate paper.

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