

Singularity Analysis of Spherical Kadomtsev–Petviashvili Equation

Ayşe KARASU-KALKANLI* and Sergei SAKOVICH[†]

Department of Physics, Middle East Technical University, 06531 Ankara, Turkey
[†]*Institute of Physics, National Academy of Sciences, 220072 Minsk, Belarus*

(Received October 10, 2004)

The (2 + 1)-dimensional spherical Kadomtsev–Petviashvili (SKP) equation of J.-K. Xue [Phys. Lett. A **314** (2003) 479] fails the Painlevé test for integrability at the highest resonance, where a nontrivial compatibility condition for recursion relations appears. This compatibility condition, however, is sufficiently weak and thus allows the SKP equation to possess an integrable (1 + 1)-dimensional reduction, which is found by the method of truncated singular expansion.

KEYWORDS: spherical Kadomtsev–Petviashvili equation, integrability, Painlevé analysis, truncated singular expansion

DOI: 10.1143/JPSJ.74.505

Recently, Xue¹⁾ deduced a spherical Kadomtsev–Petviashvili (SKP) equation for nonlinear dust acoustic waves in unmagnetized dusty plasmas with the consideration of the effects of spherical geometry and transverse perturbation. This SKP equation can be expressed in the form

$$u_{xxxx} - 12uu_{xx} - 12u_x^2 + u_{xt} + \frac{1}{t^2}u_{yy} + \frac{1}{t}u_x + \frac{1}{yt^2}u_y = 0 \quad (1)$$

by rescaling its dependent and independent variables.

Xue¹⁾ noted that the SKP equation [eq. (1)] permits two exact reductions to (1 + 1)-dimensional equations: first, the reduction $u = u(x, t)$ to the spherical Korteweg–de Vries (SKdV) equation $u_t + u_{xxx} - 12uu_x + t^{-1}u = 0$, which is believed to be a nonintegrable equation,²⁾ and second, the reduction $u = u(z, t)$ with

$$z = x - \frac{1}{4}y^2t \quad (2)$$

to the Korteweg–de Vries (KdV) equation $u_t + u_{zzz} - 12uu_z = 0$. Having used this reduction to the integrable KdV equation, Xue¹⁾ obtained an exact solitary wave solution of the SKP equation.

Owing to the reduction to the KdV equation, the SKP equation [eq. (1)] automatically possesses N -soliton solutions, for any N , derivable from those of the KdV equation through the variable z given by eq. (2). Of course, the existence of such (1 + 1)-dimensional solitons does not imply that the (2 + 1)-dimensional SKP equation is integrable. Moreover, the existence of the reduction to the SKdV equation suggests that the SKP equation is nonintegrable.

In the present Letter, we study the integrability of the SKP equation [eq. (1)] directly, not using its reductions. We show that the SKP equation does not pass the Painlevé test for integrability due to the nondominant logarithmic branching of its general solution. The singularity analysis indicates, however, that many special solutions of the SKP equation are free from this logarithmic branching. In order to select those single-valued special solutions, we use the method of truncated singular expansion, and in this way surprisingly obtain not a class of explicit special solutions but the exact reduction of the SKP equation to the KdV equation.

First, let us show that the SKP equation [eq. (1)] does not pass the Painlevé test for integrability. We follow the so-called Weiss–Kruskal algorithm of singularity analysis.^{3,4)} Note that the Painlevé analysis of wide classes of variable-coefficient Kadomtsev–Petviashvili equations was carried out in refs. 5 and 6, however the form of eq. (1) differs from the form of equations studied there.

Starting the singularity analysis, we substitute $u = u_0(y, t)\phi^\alpha + \dots + u_r(y, t)\phi^{r+\alpha} + \dots$ with $\partial_x\phi(x, y, t) = 1$ into the SKP equation [eq. (1)], and find that the singular behavior of a solution u corresponds to $\alpha = -2$ with $u_0 = 1$, the positions r of resonances being $r = -1, 4, 5, 6$. This is the generic branch representing the general solution.

Then, assuming that the singular behavior of u near a hypersurface $\phi(x, y, t) = 0$ with $\phi_x = 1$ is determined by the expansion

$$u = \sum_{n=0}^{\infty} u_n(y, t)\phi^{n-2}, \quad (3)$$

we obtain from eq. (1) the following recursion relation for the coefficients u_n :

$$\begin{aligned} &(n-2)(n-3)(n-4)(n-5)u_n \\ &+ (n-4)(n-5)\left(-6\sum_{i=0}^n u_i u_{n-i} + \left(\phi_t + \frac{1}{t^2}\phi_y^2\right)u_{n-2}\right) \\ &+ (n-5)\left(\left(\frac{1}{t} + \frac{1}{t^2}\phi_{yy} + \frac{1}{yt^2}\phi_y\right)u_{n-3}\right) \\ &+ \left(1 + \frac{2}{t^2}\phi_y\right)\partial_t u_{n-3} + \frac{1}{t^2}\partial_y^2 u_{n-4} + \frac{1}{yt^2}\partial_y u_{n-4} = 0, \\ &n = 0, 1, 2, \dots, \end{aligned} \quad (4)$$

where $u_{-4} = u_{-3} = u_{-2} = u_{-1} = 0$ formally.

At $n = 0, 1, 2, 3$, the recursion relation [eq. (4)] gives us, respectively,

$$u_0 = 1, \quad (5)$$

$$u_1 = 0, \quad (6)$$

$$u_2 = \frac{1}{12}\left(\phi_t + \frac{1}{t^2}\phi_y^2\right), \quad (7)$$

$$u_3 = -\frac{1}{12}\left(\frac{1}{t} + \frac{1}{t^2}\phi_{yy} + \frac{1}{yt^2}\phi_y\right). \quad (8)$$

*E-mail: akarasu@metu.edu.tr

[†]E-mail: saks@tut.by

At the resonances $n = 4$ and $n = 5$, where the coefficients $u_4(y, t)$ and $u_5(y, t)$ are not determined, the recursion relation [eq. (4)] turns out to be compatible. However, at the highest resonance, $n = 6$, where the coefficient $u_6(y, t)$ is not determined, we obtain from eq. (4) the following nontrivial compatibility condition:

$$(\phi_y + \frac{1}{2}yt)(\phi_{yy} + \frac{1}{2}t) = 0, \quad (9)$$

which indicates that we should modify the expansion [eq. (3)] by introducing additional logarithmic terms, starting from the term that is proportional to $\phi^4 \log \phi$.

Consequently, the SKP equation [eq. (1)] does not pass the Painlevé test for integrability due to the nondominant logarithmic branching of its solutions. The observed analytic properties of the SKP equation suggest that it cannot possess any good Lax pair.

There is an interesting conjecture, formulated by Weiss,⁷⁾ that the differential constraints, which arise in the singularity analysis of nonintegrable equations, are always integrable themselves (see ref. 8 for further discussion on this conjecture). In the present case of the SKP equation [eq. (1)], we find that, in accordance with the Weiss conjecture, the compatibility condition [eq. (9)] with $\phi_x = 1$ can be solved exactly, the result being

$$\phi = x - \frac{1}{4}y^2t + yf(t) + g(t), \quad (10)$$

for any $f(t)$ and $g(t)$.

We can see from eq. (10) that the compatibility condition [eq. (9)] is not very restrictive. The class of single-valued solutions of the SKP equation [eq. (1)], which are free from the nondominant logarithmic branching, is very wide: it is determined by the Laurent-type expansion [eq. (3)] containing three arbitrary functions of two variables and two arbitrary functions of one variable, namely, $u_4(y, t)$, $u_5(y, t)$, $u_6(y, t)$, $f(t)$ and $g(t)$ (compare with the general solution which contains four arbitrary functions of two variables). For this reason, one can hope to find many special single-valued solutions of the SKP equation [eq. (1)] in a closed form, whereas the existing techniques provide no closed expressions for solutions possessing nondominant logarithmic singularities.

Now, let us apply the method of truncated singular expansion of Weiss⁹⁾ to the SKP equation [eq. (1)]. This method, which is able to produce Bäcklund transformations and Lax pairs for integrable nonlinear systems, is also very useful in nonintegrable cases for finding explicit special solutions.¹⁰⁾

Substituting the truncated singular expansion (note that the Kruskal's ansatz is not used from this point on)

$$u = \frac{u_0(x, y, t)}{\phi(x, y, t)^2} + \frac{u_1(x, y, t)}{\phi(x, y, t)} + u_2(x, y, t) \quad (11)$$

into the SKP equation [eq. (1)] and collecting terms with equal degrees of ϕ , we obtain the following:

$$u_0 = \phi_x^2, \quad (12)$$

$$u_1 = -\phi_{xx}, \quad (13)$$

$$u_2 = \frac{\phi_{xxx}}{3\phi_x} - \frac{\phi_{xx}^2}{4\phi_x^2} + \frac{\phi_y^2}{12t^2\phi_x^2} + \frac{\phi_t}{12\phi_x}, \quad (14)$$

$$\begin{aligned} \phi_{xxxx} - \frac{4\phi_{xx}\phi_{xxx}}{\phi_x} + \frac{3\phi_{xx}^3}{\phi_x^2} - \frac{\phi_y^2\phi_{xx}}{t^2\phi_x^2} \\ - \frac{\phi_t\phi_{xx}}{\phi_x} + \phi_{xt} + \frac{\phi_{yy}}{t^2} + \frac{\phi_x}{t} + \frac{\phi_y}{yt^2} = 0, \end{aligned} \quad (15)$$

$$(yt\phi_x + 2\phi_y)(\phi_y^2\phi_{xx} - 2\phi_x\phi_y\phi_{xy} + \phi_x^2\phi_{yy} + \frac{1}{2}t\phi_x^3) = 0. \quad (16)$$

We see that, for any solution $\phi(x, y, t)$ of the overdetermined nonlinear system [eqs. (15) and (16)], the truncated singular expansion [eq. (11)] with the coefficients given by eqs. (12)–(14) generates a solution $u(x, y, t)$ of the SKP equation [eq. (1)]. However, in order to use this fact, we need to solve the system expressed by eqs. (15) and (16).

First, we solve eq. (16), which is equivalent to

$$\phi_y^2\phi_{xx} - 2\phi_x\phi_y\phi_{xy} + \phi_x^2\phi_{yy} + \frac{1}{2}t\phi_x^3 = 0 \quad (17)$$

because any solution of $yt\phi_x + 2\phi_y = 0$ satisfies eq. (17). Setting x to be the new dependent variable,

$$x = \psi(\phi, y, t), \quad (18)$$

we rewrite eq. (17) in the equivalent linear form

$$\psi_{yy} = \frac{1}{2}t. \quad (19)$$

The general solution of eq. (19) and the expression eq. (18) give us the following implicit general solution of eq. (16):

$$x = \frac{1}{4}y^2t + a(\phi, t)y + b(\phi, t), \quad (20)$$

where a and b are arbitrary functions.

Now, having solved eq. (16), we use eq. (20) and find that eq. (15) is equivalent to

$$a = 0, \quad (21)$$

$$b_{\phi t} + \frac{b_{\phi\phi\phi\phi}}{b_\phi^3} - \frac{6b_{\phi\phi}b_{\phi\phi\phi}}{b_\phi^4} + \frac{6b_\phi^3}{b_\phi^5} = 0. \quad (22)$$

Owing to the condition expressed by eq. (21), we determine using eq. (20) that ϕ is in fact a function of two variables

$$\phi(x, y, t) = \omega(z, t), \quad (23)$$

where z is given by eq. (2). Then the condition expressed by eq. (22) is equivalent to

$$\left(\frac{\omega_t}{\omega_z} + \frac{\omega_{zzz}}{\omega_z} - \frac{3\omega_{zz}^2}{2\omega_z^2} \right)_z = 0. \quad (24)$$

We have solved the overdetermined nonlinear system expressed by eqs. (15) and (16): its general solution is eq. (23), where z is given by eq. (2) and ω is any solution of eq. (24). Using this, we find from eq. (11) with eqs. (12)–(14) that the most general solution u , obtainable for the SKP equation [eq. (1)] by the method of truncated singular expansion, is

$$u(x, y, t) = v(z, t), \quad (25)$$

with z given by eq. (2) and any function v satisfying the equation

$$(v_t + v_{zzz} - 12vv_z)_z = 0. \quad (26)$$

Consequently, the Weiss method,⁹⁾ being applied to the SKP equation, rediscovers the reduction of this (2 + 1)-dimensional nonintegrable equation to the (1 + 1)-dimensional integrable KdV equation.

In conclusion, let us summarize the obtained results. The

discovered analytic properties of the SKP equation [eq. (1)] suggest that this equation cannot possess any good Lax pair but can possess many single-valued solutions. The attempt of selecting those single-valued solutions by the method of truncated singular expansion leads not to a class of explicit special solutions, as one could expect, but to an exact reduction of the studied nonintegrable equation to a lower dimensional integrable equation (as far as we know, this is a new phenomenon that has not been described in the literature as yet). It is very likely, however, that the truncated singular expansion represents not all single-valued solutions of the SKP equation [eq. (1)]. Indeed, positions $\phi = 0$ of singularities of a generic single-valued solution are determined using eq. (10) with arbitrary $f(t)$ and $g(t)$, whereas the positions of singularities of eqs. (25) and (26), obtained by the truncation method, are restricted by the condition $f(t) = 0$. Therefore it is an interesting problem for future study to attempt different methods for finding any closed form solution of the SKP equation [eq. (1)] without

this restriction $f(t) = 0$ imposed on the positions of its singularities.

Acknowledgment

The authors are grateful to the Scientific and Technical Research Council of Turkey (TÜBİTAK) for support.

- 1) J.-K. Xue: Phys. Lett. A **314** (2003) 479.
- 2) F. Calogero and A. Degasperis: *Spectral Transform and Solitons: Tools to Solve and Investigate Nonlinear Evolution Equations* (North-Holland, Amsterdam, 1982) p. 51.
- 3) J. Weiss, M. Tabor and G. Carnevale: J. Math. Phys. **24** (1983) 522.
- 4) M. Jimbo, M. D. Kruskal and T. Miwa: Phys. Lett. A **92** (1982) 59.
- 5) W. Oevel and W.-H. Steeb: Phys. Lett. A **103** (1984) 239.
- 6) T. Brugarino and A. M. Greco: J. Math. Phys. **32** (1991) 69.
- 7) J. Weiss: in *Solitons in Physics, Mathematics and Nonlinear Optics*, ed. P. J. Olver and D. H. Sattinger (Springer, New York, 1990) p. 175.
- 8) S. Yu. Sakovich: J. Nonlinear Math. Phys. **9** (2002) 21.
- 9) J. Weiss: J. Math. Phys. **24** (1983) 1405.
- 10) F. Cariello and M. Tabor: Physica D **39** (1989) 77.