The Lie algebra $sl(2,R)$ and so-called Kepler-Ermakov systems

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Abstract

A recent paper by Karasu (Kalkanlı) and Yıldırım (*Journal of Nonlinear Mathematical Physics* **9** (2002) 475-482) presented a study of the Kepler-Ermakov system in the context of determining the form of an arbitrary function in the system which was compatible with the presence of the $sl(2,R)$ algebra characteristic of Ermakov systems and the existence of a Lagrangian for a subset of the systems. We supplement that analysis by correcting some results.

1 Introduction

Karasu and Yıldırım [5] recently discussed the Lie (point) symmetries of what is known as the Kepler-Ermakov system. Such a system was presented by Althorne [1]. A feature of this class of problems is that they maintain the property of being linearisable [1, 3, 4]. The system analysed is Ref [5] [equation (9)]

$$\begin{aligned} \ddot{x}+\omega^2(t)x &= -\frac{x}{r^3}H+\frac{1}{x^3}f\left(\frac{y}{x}\right) \\ \ddot{y}+\omega^2(t)y &= -\frac{y}{r^3}H+\frac{1}{y^3}g\left(\frac{y}{x}\right), \end{aligned} \tag{1.1}$$

where the overdot denotes differentiation with respect to the independent variable, t, $r^2 = x^2+y^2$, f and g arbitrary functions of their argument and H is a function of unspecified form of dependence upon x, y and r, in which the Kepler part is to be found in the first term of the right sides of the equations. The system (1.1) possesses Lie point symmetries of the form

$$\Gamma_\sigma = \sigma(t)\partial_t + \tfrac{1}{2}\dot{\sigma}(t)\left\{x\partial_x + y\partial_y\right\} \tag{1.2}$$

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provided the condition

$$(xH_x + yH_y + H)\,\frac{\dot\sigma}{r^3} + \dddot\sigma + 4\omega^2\dot\sigma + 4\omega\dot\omega\sigma = 0 \tag{1.3}$$

is satisfied. In (1.3) we observe a departure from the computations in Ref [5] [equation (18)] for there the equation is

$$(xH_x + yH_y + 2H)\,\frac{\dot\sigma}{r^3} + \dddot\sigma + 4\omega^2\dot\sigma + 4\omega\dot\omega\sigma = 0. \tag{1.4}$$

Equation (1.4) has a solution provided

$$\dddot\sigma + 4\omega^2\dot\sigma + 4\omega\dot\omega\sigma = C\dot\sigma, \tag{1.5}$$

where C is a constant. Equation (1.5) has the structure of a linear third-order ordinary differential equation of maximal symmetry [11] and as such has the solution [7]

$$\sigma = \alpha u^2 + \beta uv + \gamma v^2, \tag{1.6}$$

where α, β and γ are arbitrary constants and u and v are the two linearly independent solutions of the linear second-order ordinary differential equation

$$\ddot u + \left(\omega^2 - \tfrac{1}{4}C\right)u = 0. \tag{1.7}$$

Provided (1.5) applies, the solution of (1.4) is given by

$$H = -\frac{1}{y^2}h\left(\frac{x}{y}\right) + \tfrac{1}{5}Cr^3, \tag{1.8}$$

where h is an arbitrary function of its argument. Karasu and Yıldırım do not persist with (1.5) in its general form, but consider the somewhat simpler situation in which $\omega^2(t) = 0$. They show that the three symmetries obtained from the three solutions of (1.5) with $\omega = 0$ possess the Lie algebra $sl(2,R)$. Specifically the symmetries are [5] [equation (24)]

$$\begin{aligned} G_1 &= t^2\partial_t + t\,(x\partial_x + y\partial_y) \\ G_2 &= t\partial_t + \tfrac{1}{2}\,(x\partial_x + y\partial_y) \\ G_3 &= \partial_t \end{aligned} \tag{1.9}$$

in the case that $C = 0$ and [5] [equation (30)]

$$\begin{aligned} J_1 &= \mathrm{e}^{\beta t}\left[\partial_t + \tfrac{1}{2}\beta\,(x\partial_x + y\partial_y)\right] \\ J_2 &= \mathrm{e}^{-\beta t}\left[\partial_t - \tfrac{1}{2}\beta\,(x\partial_x + y\partial_y)\right] \\ J_3 &= -\beta^{-2}\partial_t, \end{aligned} \tag{1.10}$$

where $\beta = 2i\sqrt{C/5}$, in the case that $C \neq 0$. (One could write (1.10) in a more transparent form, but there is little point to it as source equation [5] [equation (28)] is incorrect.) Given (1.8), the Kepler-Ermakov system (1.1) with $\omega = 0$ has the form

$$\begin{aligned} \ddot x &= -x\left[\tfrac{1}{5}C - y^{-2}r^{-3}h\left(\frac{x}{y}\right)\right] + x^{-3}f\left(\frac{y}{x}\right) \\ \ddot y &= -y\left[\tfrac{1}{5}C - y^{-2}r^{-3}h\left(\frac{x}{y}\right)\right] + y^{-3}g\left(\frac{y}{x}\right). \end{aligned} \tag{1.11}$$

In the case that $C=0$ (1.11) is claimed to possess the symmetries (1.9), in particular the self-similar symmetry, G_2. This requires that the weights of all the terms in each of equations (1.11) be the same. This is clearly impossible for the terms on the right side (recall $C=0$) of each equation.

2 The correct Kepler-Ermakov system

The reason for the incorrectness of the results in Ref [5] doubtless lies in the presence of the 2 in (1.4) (their equation (18)) rather than the more orthodox (1.3). We further examine (1.3).

If one assumes the same separation (1.5), the function H satisfies

$$xH_x+yH_y+H+Cr^3=0 \tag{2.12}$$

the associated Lagrange's system of which is

$$\frac{\mathrm{d}x}{x}=\frac{\mathrm{d}y}{y}=-\frac{\mathrm{d}H}{(H+Cr^3)}. \tag{2.13}$$

The invariants of (2.13) are

$$\zeta_1=\frac{x}{y}\quad\text{and}\quad\zeta_2=xH+\tfrac{1}{4}Cxr^3 \tag{2.14}$$

so that

$$H=-\tfrac{1}{4}Cr^3-\frac{1}{x}h\left(\frac{x}{y}\right) \tag{2.15}$$

and the Kepler-Ermakov system, (1.1), has the form

$$\begin{aligned}\ddot{x}+\left(\omega^2-\tfrac{1}{4}C\right)x &= \frac{1}{r^3}h\left(\frac{x}{y}\right)+\frac{1}{x^3}f\left(\frac{y}{x}\right)\\ \ddot{y}+\left(\omega^2-\tfrac{1}{4}C\right)y &= \frac{y}{xr^3}h\left(\frac{x}{y}\right)+\frac{1}{y^3}g\left(\frac{y}{x}\right).\end{aligned} \tag{2.16}$$

The terms in the right side of the equations in (2.16) are manifestly of equal weights in the dependent variables.

The effect of C is to shift the time-dependent frequency function as one would anticipate from (1.7). In the Conclusion of Ref [5] [p 481] the claim is made that the frequency function – recall that they concentrated on the autonomous case $\omega(t)=0$ – depends upon the dynamical variables (sic). We must emphasise that C is a constant just as the constant arising in the separation of variables of, say, the heat equation is a constant.

3 Symmetries of the Kepler-Ermakov system

In (2.16) we have the correct form of the Kepler-Ermakov system invariant under three symmetries of the form (1.2) with σ given by the solution of (1.6). It remains to demonstrate explicitly that the algebra is actually $sl(2,R)$ although one would expect that to be the case given the relationship between (1.7) and (1.5).

We write the symmetries in the more compact forms

$$\begin{aligned}\Gamma_1 &= \sigma_1\partial_t + \tfrac{1}{2}\dot\sigma_1 r\partial_r\\ \Gamma_2 &= \sigma_2\partial_t + \tfrac{1}{2}\dot\sigma_2 r\partial_r\\ \Gamma_3 &= \sigma_3\partial_t + \tfrac{1}{2}\dot\sigma_3 r\partial_r,\end{aligned} \tag{3.17}$$

where we take

$$\sigma_1 = W^{-1}u^2, \quad \sigma_2 = W^{-1}uv, \quad \sigma_3 = W^{-1}v^2 \tag{3.18}$$

with $W = u\dot v - \dot u v$ being the Wronskian of the two linearly independent solutions, $u(t)$ and $v(t)$, of (1.7). Then we have

$$[\Gamma_1, \Gamma_2]_{LB} = (\sigma_1\dot\sigma_2 - \dot\sigma_1\sigma_2)\,\partial_t + \tfrac{1}{2}\,(\sigma_1\dot\sigma_2 - \dot\sigma_1\sigma_2)^{\cdot}\, r\partial_r \tag{3.19}$$

and $(\sigma_1\dot\sigma_2 - \dot\sigma_1\sigma_2) = W^{-2}u^2\,(u\dot v - \dot u v) = W^{-1}u^2 = \sigma_1$. With similar calculations for the other Lie Brackets we confirm that

$$[\Gamma_1, \Gamma_2]_{LB} = \Gamma_1, \quad [\Gamma_2, \Gamma_3]_{LB} = \Gamma_3, \quad [\Gamma_3, \Gamma_1]_{LB} = -2\Gamma_2 \tag{3.20}$$

and the algebra is indeed $sl(2, R)$.

Consequently (2.16) does indeed represent the Kepler-Ermakov system which maintains the algebra $sl(2, R)$.

It is a simple matter to show that (2.16) possesses the Ermakov-Lewis invariant [2, 8, 9, 10]

$$I = \tfrac{1}{2}\,(x\dot y - \dot x y)^2 + \int^{y/x} \left[\eta f(\eta) - \eta^{-3}g(\eta)\right] \mathrm{d}\eta \tag{3.21}$$

which in this case is actually a first integral since it is autonomous.

4 Normal form of the Kepler-Ermakov system

As we have had occasion to remark in a previous paper [6] in the case of generalised Ermakov systems, the presence of the time-dependent frequency function $\omega^2(t) - \frac{1}{4}C$ in (2.16) presents a spurious generality. The transformation of the three symmetries of (3.17) to the standard form

$$\begin{aligned}\Delta_1 &= \partial_t\\ \Delta_2 &= t\partial_t + \tfrac{1}{2}r\partial_r\\ \Delta_3 &= t^2\partial_t + tr\partial_r\end{aligned} \tag{4.22}$$

is achieved by the transformation to new time and rescaled radial distance

$$T = \int \frac{\mathrm{d}t}{u^2} \qquad R = \frac{r}{u}, \ \left(X = \frac{x}{u},\ Y = \frac{y}{u}\right) \tag{4.23}$$

(equally one could use the 'other' linearly independent solution of (1.7), *videlicet* $v(t)$). Under this transformation the system (2.16) becomes

$$\begin{aligned} \frac{\mathrm{d}^2X}{\mathrm{d}T^2} &= \frac{1}{R^3}h\left(\frac{X}{Y}\right) + \frac{1}{X^3}f\left(\frac{Y}{X}\right) \\ \frac{\mathrm{d}^2Y}{\mathrm{d}T^2} &= \frac{Y}{XR^3}h\left(\frac{X}{Y}\right) + \frac{1}{Y^3}g\left(\frac{Y}{X}\right) \end{aligned} \tag{4.24}$$

and we note that not only does the transformation remove the time-dependent $\omega^2(t)$ but also the constant C introduced in the separation of (1.4).

In a discussion of Kepler-Ermakov systems it suffices to study simply the system (4.24) or its polar equivalent

$$\begin{aligned} \ddot{r} - r\dot{\theta}^2 &= \frac{1}{r^3\cos\theta}h(\cot\theta) + \frac{1}{r^3}\left\{\sec^2\theta f(\tan\theta) + \mathrm{cosec}^2\theta g(\tan\theta)\right\} \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} &= -\frac{1}{r^3}\left\{\sec^2\theta\tan\theta f(\tan\theta) - \mathrm{cosec}^2\theta\cot\theta g(\tan\theta)\right\} \end{aligned} \tag{4.25}$$

in which we have reverted to lower case variables.

5 Condition for the existence of a Lagrangian

The existence of a Lagrangian for (4.24), equally (4.25), imposes a constraint on the hitherto arbitrary functions f, g and h. A potential for the right sides of (4.25) exists provided

$$\frac{\partial}{\partial\theta}\left(\frac{1}{\cos\theta}h(\cot\theta)\right) = 0 \quad \Longrightarrow \quad h(\cot\theta) = \mu\cos\theta, \tag{5.26}$$

where μ is a constant, and

$$\begin{aligned} &\frac{\partial}{\partial\theta}\left\{\{\sec^2\theta f(\tan\theta) + \mathrm{cosec}^2\theta g(\tan\theta)\right\} \\ &= -r^3\frac{\partial}{\partial r}\left\{\frac{1}{r^2}\left[\sec^2\theta\tan\theta f(\tan\theta) - \mathrm{cosec}^2\theta\cot\theta g(\tan\theta)\right]\right\} \\ &\Longrightarrow \quad \sin^2\theta f'(\tan\theta) + \cos^2\theta g'(\tan\theta) = 0. \end{aligned} \tag{5.27}$$

The Lagrangian is then

$$L = \tfrac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{\mu}{2r^2} - \frac{G(\theta)}{2r^2}, \tag{5.28}$$

where

$$G(\theta) = \sec^2\theta f(\tan\theta) + \mathrm{cosec}^2\theta g(\tan\theta) \tag{5.29}$$

subject to the constraint on f and g in (5.27). This differs from the expression given in Ref [5] [(39)].

From (5.28) it is evident that the Hamiltonian, *videlicet*

$$H = \tfrac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + \tfrac{1}{2}\frac{\mu}{r^2} + \tfrac{1}{2}\frac{G(\theta)}{r^2}, \tag{5.30}$$

is also a conserved quantity in addition to the Ermakov-Lewis invariant (3.21) which in polar coordinates is

$$I = \tfrac{1}{2}p_\theta^2 + F(\theta), \tag{5.31}$$

where

$$F'(\theta) = \sec^2\theta\tan\theta f(\tan\theta) - \mathrm{cosec}^2\theta\cot\theta g(\tan\theta). \tag{5.32}$$

Since I is both autonomous and a conserved quantity, it has zero Poisson Bracket with H. The two integrals are in involution and so the system is integrable in the sense of Liouville [12] [p 323].

6 Conclusion

As a final comment we note that the term $\frac{1}{2}\mu/r^2$ in (5.30) which is characterised 'as [a] perturbation(s) of the classical Kepler problem' [5] [p 475] is difficult to reconcile with a Kepler potential. On the other hand it is readily recognised as a Newton-Cotes potential [12] [p 83]. The form of (5.30) is more that of the Hamiltonian of a free particle moving in an angle-dependent Newton-Cotes potential. This interpretation provides a new insight into the nature of Ermakov systems which are expressible in Hamiltonian form and takes the origin of this type of system back to the early days of Newtonian Mechanics at the beginning of the eighteenth century.

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